Do electromagnetic waves with fixed OAM exist?

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The significance of angular momentum

Angular momentum is a member of
a very important family of physical quantities:
the generators of symmetry transformations

Angular momentum is the generator of rotations

Due to the vectorial character of the electric and magnetic fields the angular momentum has two parts:
The first part (orbital angular momentum) generates the rotation of fields as if they were scalars disregarding their vectorial character
The second part (spin) rotates the field vectors

Rotation of vector fields

The classical generator of a rotation around the k-th axis of a vector $\boldsymbol{V}$ field is

$$G_k V_i(\boldsymbol{r}) = \epsilon_{klm} x_l \nabla_m V_i(\boldsymbol{r}) + \epsilon_{kij} V_j(\boldsymbol{r})$$

This formula is more familiar in its quantum version

$$\boldsymbol{J}\boldsymbol{V}(\boldsymbol{r}) = \textbf{orbital part} + \textbf{spin part} = \boldsymbol{L}\boldsymbol{V}(\boldsymbol{r}) + \boldsymbol{S}\boldsymbol{V}(\boldsymbol{r})$$

$$\boldsymbol{L} = \frac{\hbar}{i}\boldsymbol{r} \times \boldsymbol{\nabla} \qquad \boldsymbol{S} = \hbar\boldsymbol{\Sigma} \qquad \Sigma_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \text{ etc.}$$

In what follows I will use the quantum version

Riemann-Silberstein vector

The use of the quantum-mechanical description in electromagnetism is natural when the electric and magnetic vectors are combined into a complex vector which I named in 1996 the Riemann-Silberstein vector

$$\boldsymbol{F} = \frac{\boldsymbol{D}}{\sqrt{2\epsilon}} + i\frac{\boldsymbol{B}}{\sqrt{2\mu}}$$

In terms of the RS vector the Maxwell equations take the form of the Weyl equation for massless neutrinos

$$\text{Maxwell}: \; i\hbar\partial_t \boldsymbol{F} = c\,(\boldsymbol{\Sigma}\cdot\boldsymbol{p})\,\boldsymbol{F} \qquad \text{Weyl}: \; i\hbar\partial_t \boldsymbol{\psi} = c\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\boldsymbol{\psi}$$

The RS vector is a perfect tool when we use quantum concepts in the classical domain

There are no solutions with fixed l

Let us suppose that a solution of Maxwell equations F is an eigenfunction of the operator L_z i.e. it has a fixed value $\hbar l$ of the z component of the angular momentum

$$L_z \boldsymbol{F} = \hbar l \boldsymbol{F}$$

The divergence of this equation gives $\partial_x F_y - \partial_y F_x = 0$ Maxwell equations give $[L_z, (\boldsymbol{\Sigma} \cdot \boldsymbol{p})]\boldsymbol{F} = 0$ i.e.

$$\partial_x F_z = 0 \quad \partial_y F_z = 0 \quad \partial_z F_z = 0 \quad \partial_x F_x + \partial_y F_y = 0$$

There is no bounded $\boldsymbol{F}$ satisfying all these conditions

Electromagnetic fields in momentum space

General solution of Maxwell equations

$$\boldsymbol{F}(\boldsymbol{r},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \boldsymbol{e}(\boldsymbol{k}) \left[f_L(\boldsymbol{k}) e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{r}} + f_R^*(\boldsymbol{k}) e^{i\omega_{\boldsymbol{k}} t - i\boldsymbol{k}\cdot\boldsymbol{r}} \right]$$

Polarization vector $\boldsymbol{e}(\boldsymbol{k})$ obeys the equation

$$-i\omega\, \boldsymbol{e}(\boldsymbol{k}) = c\boldsymbol{k} \times \boldsymbol{e}(\boldsymbol{k}) \qquad \boldsymbol{e}^*\cdot\boldsymbol{e} = 1$$

The amplitudes $f_L(\boldsymbol{k})$ and $f_R(\boldsymbol{k})$ are the wave functions in momentum space

Generators of rotations in momentum space

Generators acting on $\boldsymbol{F}$ have counterparts in $\boldsymbol{k}$ space
For the generator of rotations we obtain

Total angular momentum: $\tilde{\boldsymbol{J}} = -i\hbar\boldsymbol{k} \times \mathcal{D}_{\boldsymbol{k}} + \hat{\chi}\hbar\boldsymbol{k}/k$

Helicity operator $\hat{\chi}$ has $\pm$ signs for f_L and f_R

$$\mathcal{D}_{\boldsymbol{k}} = \boldsymbol{\nabla}_k - i\hat{\chi}\boldsymbol{\alpha}(\boldsymbol{k}) \qquad \boldsymbol{\alpha}(\boldsymbol{k}) = \frac{k_z}{k(k_x^2 + k_y^2)}\{-k_y, k_x, 0\}$$

Covariant derivative $\mathcal{D}_{\boldsymbol{k}}$ appears here due to the dependence of the polarization vector $\boldsymbol{e}(\boldsymbol{k})$ on $\boldsymbol{k}$

Check: $\langle -i\hbar\boldsymbol{k} \times \mathcal{D}_{\boldsymbol{k}} + \hat{\chi}\hbar\boldsymbol{k}/k\rangle = \int d^3r\, [\boldsymbol{r} \times (\boldsymbol{D} \times \boldsymbol{B})]$

Nonexistence of the eigenstates of $\tilde{L}_z$

The operator $\tilde{\boldsymbol{J}}$ splits into orbital and spin parts
Orbital part $\tilde{\boldsymbol{L}}$ is perpendicular to momentum
Spin (helicity) part $\tilde{\boldsymbol{S}}$ is parallel to momentum

The eigenequation: $\tilde{L}_z f_L(\boldsymbol{k}) = \hbar l f_L(\boldsymbol{k})$ **gives**

$$\left[-i(k_x\partial_{k_y} - k_y\partial_{k_x}) - k_z/k\right] f_L(\boldsymbol{k}) = l f_L(\boldsymbol{k})$$

The solution in cylindrical coordinates is

$$f_L = a e^{i(l+k_z/k)\varphi}$$

It is unacceptable since it is not periodic in φ

Optical beams carrying angular momentum

There are many examples of optical beams endowed with angular momentum: Bessel, Laguerre-Gauss, Exponential beams, etc. They are described by exact solutions of Maxwell equations and can be chosen to be eigenfunctions of J_z

A simple explanation of the connection between the orbital and the total angular momentum is obtained with the help of the Whittaker construction

Whittaker construction

Whittaker in 1904 has shown that solutions of Maxwell equations can be constructed from derivatives of two real solutions of the wave equation We found that it is convenient to construct the RS vector from one complex solution of the wave equation

$$\boldsymbol{F}(\boldsymbol{r},t) = (i/c\,\partial_t + \boldsymbol{\nabla}\times)\,\boldsymbol{\nabla}\times\boldsymbol{m}\,\Phi(\boldsymbol{r},t)$$

$\Phi(\boldsymbol{r},t)$ is a solution of the wave equation and $\boldsymbol{m}$ is a constant vector Time derivative does not change l while space derivatives lower it by one unit creating a mismatch

Vorticity/Circulation of velocity

Nonexistence of eigenstates of orbital momentum does not mean that circulation of velocity Γ vanishes

$$\Gamma = \oint d\boldsymbol{l}\cdot\boldsymbol{v}$$

Velocity for beams of light is defined locally as the ratio of momentum density to energy density

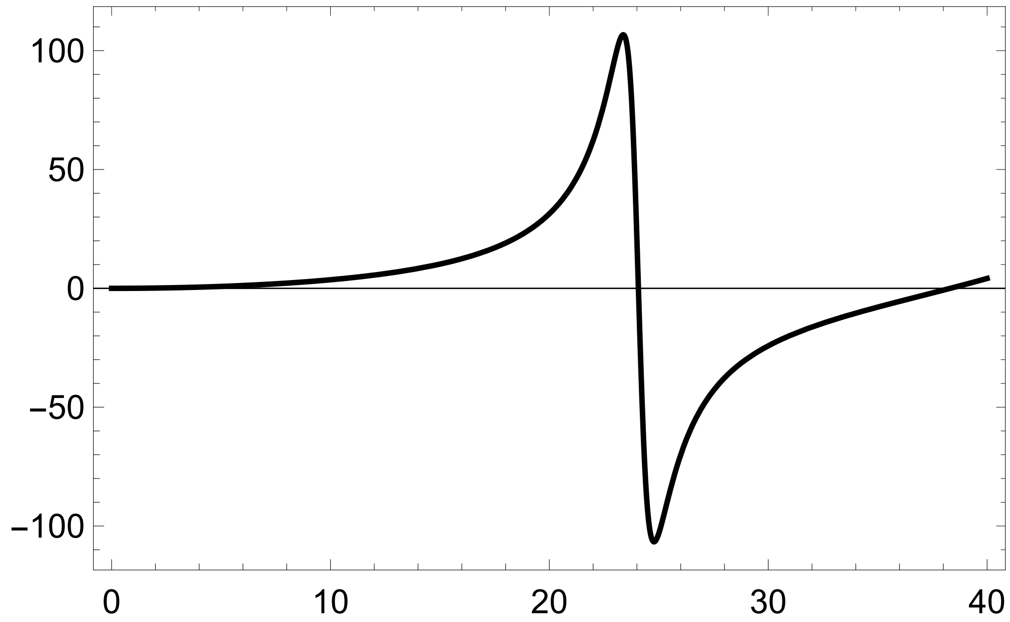
$$\boldsymbol{v} = \frac{c}{i}\,\frac{\boldsymbol{F}^* \times \boldsymbol{F}}{\boldsymbol{F}^*\cdot\boldsymbol{F}} \leq c$$

Circulation of velocity for Bessel beams

Circulation drops to zero at the beam center

There is no singular vortex line

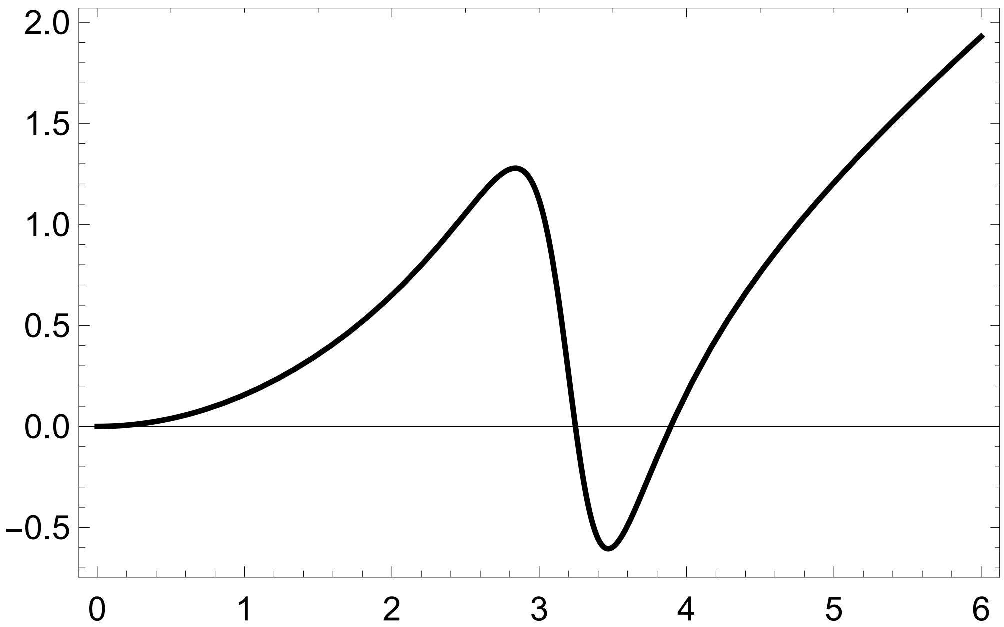
Vorticity is spread over the whole space



Circulation reversal

Reversal of circulation seems to be a common property of structured light

This is the circulation for exponential beams



Summary

Nonexistence of eigenstates of the orbital momentum is proven in the coordinate representation and in the momentum representation

Scalar solutions of the wave equation with given l can be easily found

However, the vectorial nature of the electromagnetic field necessarily requires a mixture of parts with different values of l

Beams of light with fixed l do not exist