Comparison theorems in optimal transport and beyond

Aram Karakhanyan

April 12, 2017

1 The rolling ball Theorem of Blaschke

- Let $\mathscr{M}$ and $\mathscr{M}'$ be two hypersurfaces in $\mathbb{R}^d$. We say that $\mathscr{M}$ and $\mathscr{M}'$ are *internally tangent at* $x \in \mathscr{M}$ if they are tangent at x and have the same outward normal.

- Denote by $\mathrm{II}_x\mathscr{M}$ the second fundamental form of $\mathscr{M}$ at x and let $n(x)$ be the outward unit normal at x. Then we have

Theorem 1.1:

Suppose $\mathscr{M}$ and $\mathscr{M}'$ are smooth convex surfaces with strictly positive scalar curvature such that $\mathrm{II}_x\mathscr{M} \geq \mathrm{II}_{x'}\mathscr{M}'$ for all $x \in \mathscr{M}, x' \in \mathscr{M}'$ such that $n(x) = n'(x')$. If $\mathscr{M}$ and $\mathscr{M}'$ are internally tangent at one point then $\mathscr{M}$ is contained in the convex region bounded by $\mathscr{M}'$.

- W. Blaschke proved Theorem 1.1 in 1918 for closed curves in $\mathbb{R}^2$.
- D. Koutroufiotis generalized Blaschke's theorem for complete curves in $\mathbb{R}^2$ and complete surfaces in $\mathbb{R}^3$ (Arch. Math 1972).
- J. Rauch for compact surfaces in $\mathbb{R}^d$ (JDG 1974)
- J.A. Delgado for complete surfaces (JDG 1979)
- J.N. Brooks and J.B. Strantzen generalized Blaschke's theorem for non-smooth convex sets showing that the local inclusion implies global inclusion (Mem. AMS 1989)

- Observe that if $\mathscr{M}$ and $\mathscr{M}'$ are internally tangent at x, then a necessary condition for $\mathscr{M}$ to be inside $\mathscr{M}'$ near x is

$$\mathrm{II}_x(v) \geq \mathrm{II}'_{x'}(v) \quad \text{for all } v \in \mathbb{T}_x\mathscr{M} \cong \mathbb{T}_{x'}\mathscr{M}'. \tag{1.1}$$

The tangent planes are parallel because $\mathscr{M}$ and $\mathscr{M}'$ are internally tangent at x.

- Therefore Theorem 1.1 says that if for all $x \in \mathscr{M}, x' \in \mathscr{M}', x \neq x'$ with coinciding normals $n'(x') = n(x)$ such that after translating $\mathscr{M}$ by $x - x'$ we have that the translated surface $\widetilde{\mathscr{M}}$ is locally inside $\mathscr{M}'$ then $\mathscr{M}$ is globally inside $\mathscr{M}'$. In other words,

the local inclusion implies global inclusion or $\mathscr{M}$ rolls freely inside $\mathscr{M}'$.

1.1 Blaschke's proof in $\mathbb{R}^2$

- Support function $h(t)$
- Support line $x\cos t + y\sin t - h(t) = 0$ and $-x\sin t + y\cos t - h'(t) = 0$ from where

$$x = h\cos t - h'\sin t$$
$$y = h\sin t - h'\cos t$$

- Radius of curvature $\rho(t) = h''(t) + h(t)$
- In our case

$$h(0) = 0, \quad h'(0) = 0$$
$$h(0) = 0, \quad h'(0) = 0$$

- From periodicity we get $\int_{-\pi}^{\pi} \rho(s)\cos s ds = 0, \quad \int_{-\pi}^{\pi} \rho(s)\sin s ds = 0$

$$h(t) = \int_0^t \rho(s)\sin(t-s)ds.$$

1.2 Shape operator

- If $\mathscr{M}$ is a surface with positive sectional curvature then by Sacksteder's theorem (AJM 1960) $\mathscr{M}$ is convex.

- For $x \in \mathscr{M}$, let $n(x)$ be the unit outward normal at x ($n(x)$ points outside of the convex body bounded by $\mathscr{M}$). The Gauss map $x \to n(x)$ is a diffeomorphism of $\mathscr{M}$ onto $\mathbb{S}^d$ (H.Wu, JDG, 1974), where $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^d$. The inverse map n^{-1} gives a parametrization of $\mathscr{M}$ by $\mathbb{S}^d$.

- If $\mathscr{M}'$ is another smooth convex surface, and $w \in \mathbb{S}^d$, then $n^{-1}(w)$ and $(n')^{-1}(w)$ are the points on $\mathscr{M}$ and $\mathscr{M}'$ with equal outward normals.

- Let $F:\Omega \to \mathbb{R}^m$ be a smooth map on a set $\Omega \subset \mathbb{R}^d$ and $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$ then

$$\partial_v F(y) = \sum_{i=1}^{d} v_i \frac{\partial F(y)}{\partial y_i}, y \in \Omega$$

  is the directional derivative operator.

- We view the tangent space as a linear subspace of $\mathbb{R}^d$ consisting of tangential directions. Then the tangent space $\mathbb{T}_x\mathscr{M}$ is the set of vectors perpendicular to $n(x)$.

- The Weingarten map $W_x : \mathbb{T}_x \to \mathbb{T}_x$ is defined by $W_x(v) = \partial_v n(x)$. $\mathbb{T}_x$ is an inner product space (induced by the inner product in $\mathbb{R}^d$). Then W_x is self-adjoint operator on $\mathbb{T}_x$ and the eigenvalues of W_x are the principal curvatures at x.

- Since W_x is self-adjoint and $\mathbb{T}_x$ is finite dimensional then there exists an orthonormal basis of $\mathbb{T}_x$ consisting of eigenvectors of W_x.

Definition 1.1

The second fundamental form is defined as $\mathrm{II}_x(v,w) = W_x(v)\cdot w$. When $v = w$ we denote $\mathrm{II}_x(v)$.

From definition it follows that if $\mathscr{M}$ is parametrized by $r = r(u)$ and $x = r(u_0)$ then

$$\mathrm{II}_x(v) = -\partial_v^2 r \cdot n(x), \quad v \in \mathbb{T}_x \tag{1.2}$$

which readily follows from the differentiation of $n \cdot \partial_v r = 0$.

Definition 1.2:

Let $c:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ be a cost function such that $c\in C^4(\mathbb{R}^d\times\mathbb{R}^d)$ and $\mathscr{U},\mathscr{V}\subset\mathbb{R}^d$.

- Let $u:\mathscr{U}\to\mathbb{R}$ be a continuous function. A $c-$support function of u at $x_0\in\mathscr{U}$ is $\varphi_{x_0}(x)=c(x,y_0)+a_0, y_0\in\mathbb{R}^d$ such that the following two conditions hold

$$\begin{aligned} u(x_0) &= \varphi_{x_0}(x_0),\\ u(x) &\geq \varphi_{x_0}(x), x\in\mathscr{U}. \end{aligned}$$

- If u has $c-$support at every $x_0\in\mathscr{U}$ then we say that u is $c-$convex in $\mathscr{U}$.
- $c-$segment with respect to a point $y_0\in\mathbb{R}^d$ is the set

$$\{x\in\mathbb{R}^d \ s.t.\ c_y(x,y_0)=\text{line segment}\}.$$

  One may take in the above definition $\{x\in\mathbb{R}^d \ s.t.\ c_y(x,y_0)=tp_1+(1-t)p_0\}$ with $t\in[0,1]$ and p_0,p_1 being two points in $\mathbb{R}^d$.
- We say that $\mathscr{U}$ is $c-$convex with respect to $\mathscr{V}\subset\mathbb{R}^d$ if the image of the set $\mathscr{U}$ under the mapping $c_y(\cdot,y)$ denoted by $c_y(\mathscr{U},y)$ is convex set for all $y\in\mathscr{V}$. Equivalently, $\mathscr{U}$ is $c-$convex with respect to $\mathscr{V}$ if for any pair of points $x_1,x_2\in\mathscr{U}$ there is $y_0\in\mathscr{V}$ such that there is a $c-$segment with respect to y_0 joining x_1 with x_2 and lying in $\mathscr{U}$.

1.3 Sub-level sets

Definition 1.3

Let u be a $c-$convex function then the sub-level set of u at $x_0 \in \mathcal{U}$ is

$$S_{h,u}(x_0) = \{x \in \mathbb{R}^d \;\; s.t. \;\; u(x) < c(x, y_0) + [u(x_0) - c(x_0, y_0)] + h\} \tag{1.3}$$

for some constant h.

- Equivalently, $S_{h,u}(x_0) = \{x \in \mathcal{U} \;\; s.t. \;\; u(x) < \varphi_{x_0}(x) + h\}$ where φ_{x_0} is the $c-$support function of u at $x_0 \in \mathcal{U}$.
- Observe that in the previous definition on may take $u(x) = c(x, y_1)$ for some fixed $y_1 \neq y_0$.

- We recall Kantorovich's formulation of optimal transport problem: Let $f:\mathscr{U}\to\mathbb{R}, g:\mathscr{V}\to\mathbb{R}$ be two nonnegative integrable functions satisfying the mass balance condition

$$\int_{\mathscr{U}} f(x)dx = \int_{\mathscr{V}} g(y)dy.$$

Then one wishes to minimize

$$\int_{\mathscr{U}} u(x)f(x) + \int_{\mathscr{V}} v(y)g(y)dy \to \min$$

among all pairs of functions $u:\mathscr{U}\to\mathbb{R}, v:\mathscr{V}\to\mathbb{R}$ such that $u(x)+v(y)\geq c(x,y)$. It is well-known that a minimizing pair (u,v) exists and formally the potential u solves the equation

$$\det(u_{ij}-A_{ij}(x,Du)) = |\det c_{x_i,y_j}|\frac{f(x)}{(g\circ y)(x)}.$$

Here $A_{ij}(x,p)=c_{x_ix_j}(x,y(x,p))$ where $y(x,p)$ is determined from $D_x(c(x,y(x,p)))=p$.

Assume that c satisfies the following conditions:

A1 For all $x, p \in \mathbb{R}^d$ there is unique $y = y(x,p) \in \mathbb{R}^d$ such that $\partial_x c(x,y) = p$ and for any $y, q \in \mathbb{R}^d$ there is unique $x = x(y,q)$ such that $\partial_y c(x,y) = q$.

A2 For all $x, y \in \mathbb{R}^d$ $\det c_{x_i,y_j}(x,y) \neq 0$.

A3 For $x, p \in \mathbb{R}^d$ there is a positive constant $c_0 > 0$ such that

$$A_{ij,kl}(x,p)\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2 \quad \forall \xi, \eta \in \mathbb{R}^d, \xi \perp \eta. \tag{1.4}$$

- **A3** is the Ma-Trudinger-Wang condition.
- J.Liu proved that if **A1**-**A3** hold then $S_{h,u}(x_0)$ is $c-$convex with respect to y_0.
- There are cost functions satisfying the weak **A3**

$$A_{ij,kl}(x,p)\xi_i\xi_j\eta_k\eta_l \ge 0 \quad \forall \xi,\eta \in \mathbb{R}^d, \xi \perp \eta. \tag{1.5}$$

  i.e. when $c_0 = 0$ in (1.4), such that the corresponding sub-level sets are convex in classical sense.

- We also remark that the condition **A3** is equivalent to

$$\frac{d^2}{dt^2} c_{ij}(x, y(x, p_t))\xi_i\xi_j \ge c_0|p_1 - p_0|^2 \tag{1.6}$$

  where x is fixed, $c_x(x, y(x, p_t)) = tp_1 + (1-t)p_0, t \in [0,1]$ $c_x(x,y) = p_1, c_x(x, y_0) = p_0$ (this determines the so-called c^*-segment with respect to fixed x).

2 Main result

Theorem 2.1

et $y_1, y_2 \in \overline{\mathcal{V}}$ and $\mathcal{N}(y_1, y_2, a) = \{x \in \mathbb{R}^d : c(x, y_0) = c(x, y_1) + a\}$ for some $a \in \mathbb{R}$ where c satisfies **A1**,**A2** and weak **A3**. Assume that $\mathcal{N}$ is convex for all y_1, y_2, a and $\mathcal{U}$ is convex domain with smooth boundary such that $\mathcal{U}$ is $c-$convex with respect to $\mathcal{V}$. If $\mathcal{N}$ and $\partial\mathcal{U}$ are internally tangent at some point z_0 then $\mathcal{U}$ is inside $\mathcal{N}$.

Using the terminology of Blaschke's theorem it follows that under the conditions of Theorem 2.1 $\mathcal{U}$ rolls freely inside $\mathcal{N}$. Observe that the $c-$convexity of sub-level sets is known under stronger condition **A3** (Liu). In the next section we give an example of cost function c satisfying weaker form of **A3** (1.5) but such that $\mathcal{N}$ is convex for all y_1, y_2, a. Proof to follow is inspired in Trudinger-Wang paper (ARMA 2009).

Proof. **Step 1:** (*Parametrizations*)

To apply Theorem 1.1 we take $\mathscr{M} = \mathscr{U}$ and $\mathscr{M}' = \mathscr{N}$ and assume that $\mathscr{U}$ and $\mathscr{N}$ are internally tangent at z_0. Assume that at $x_0' \in \mathscr{N}$ and $x_0 \in \partial\mathscr{U}$ $\mathscr{N}$ and $\partial\mathscr{U}$ have the same outward normal, see Figure 1.

In what follows we use the following radial parametrizations:

$$\begin{array}{ll} \partial\mathscr{U} & R(\zeta), \quad \zeta \in D_{\mathscr{U}}, \\ \mathscr{N} & \mathscr{X}(\omega), \quad \omega \in D_{\mathscr{N}}, \\ \partial(c_y(\partial\mathscr{U}, y_0)) & \rho(\zeta) = c_y(R(\zeta), y_0). \end{array}$$

Here $D_{\mathscr{U}}$ and $D_{\mathscr{N}}$ are the domains of corresponding parameters. Moreover, there are $\bar\omega \in D_{\mathscr{N}}$ and $\bar\zeta \in D_{\mathscr{U}}$ such that

$$x_0' := \mathscr{X}(\bar\omega) \in \mathscr{N} \quad \text{and} \quad x_0 := R(\bar\zeta) \in \partial\mathscr{U}. \tag{2.1}$$

From now on $\bar{\zeta}$ and $\bar{\omega}$ are fixed. Let $\bar{n}(\bar{\zeta})$ denote the outward normal of the image $c_y(\mathcal{U}, y_0)$ at the point $\rho(\bar{\zeta})$. We have

$$\bar{n}^m(\bar{\zeta}) = c_{y_m, x_i}(R(\bar{\zeta}), y_0) n^i(\bar{\zeta}). \tag{2.2}$$

Observe that by assumption the constant matrix $\mu = [c_{y_m, x_i}(R(\bar{\zeta}), y_0)]^{-1}$ has non-trivial determinant, by **A2**. Furthermore, the set $\mu c_y(\mathcal{U}, y_0) = \{\mu x \ s.t. \ x \in c_y(\mathcal{U}, y_0)\}$ is again convex because for any two points $q_1 = \mu z_1, q_2 = \mu z_2$ such that $q_1, q_2 \in \mu c_y(\mathcal{U}, y_0)$ and $z_1, z_2 \in c_y(\mathcal{U}, y_0)$ we have

$$\mu c_y(\mathcal{U}, y_0) \ni \mu(\theta z_1 + (1-\theta) z_2) = \theta \mu z_1 + (1-\theta)\mu z_2 = \theta q_1 + (1-\theta) q_2$$

for all $\theta \in [0,1]$.

Step 2: (*Computing the second fundamental form of* $\mathscr{X}$)

Next, we introduce the vectorfield $r = r(\zeta), \zeta \in D_{\mathcal{U}}$ such that

$$r(\zeta) = \mu\rho(\zeta) = \mu c_y(R(\zeta), y_0). \tag{2.3}$$

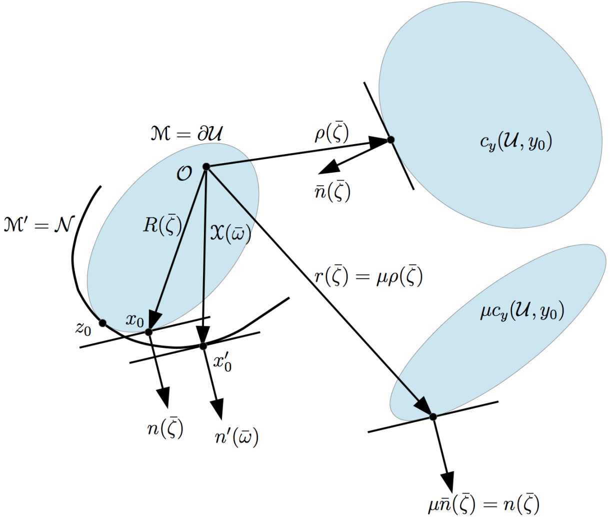


Figure 1: Schematic view to parametrizations of $\partial\mathscr{U}, \mathscr{N}, \partial(c_y(\mathscr{U}, y_0))$ and $\mu\partial(c_y(\mathscr{U}, y_0))$.

We compute the first and second derivatives

$$r^m_{\zeta_s} := r^m_s = \mu_{\alpha\beta} c_{y_\beta, x_i} R^i_s, \tag{2.4}$$

$$r^m_{st} = \mu_{\alpha\beta} \left[c_{y_\beta, x_i x_j} R^i_s R^j_t + c_{y_\beta, x_i} R^i_{st} \right]. \tag{2.5}$$

From (2.4) and (2.2) we see that at $r(\bar{\zeta})$ the normal is

$$n(\bar{\zeta}) = \mu \bar{n}(\bar{\zeta}). \tag{2.6}$$

Take $p_t = (1-t)p_0 + tp_1, t \in [0,1]$ and

$$p_t = c_x(x_0, y(x'_0, p_t)), \tag{2.7}$$

then $y_t := y(x_0', p_t)$ defines the $c-$segment joining y_0 and y_1, by **A2** and inverse mapping. In particular, one has

$$\begin{aligned} p_1^i - p_0^i &= c_{x_i, y_m}(x_0, y(x_0', p_t)) \frac{d}{dt} y^m(x_0', p_t) \\ &= \left[\frac{d}{dt} y^m(x_0', p_t)\right] c_{y_m, x_i}(x_0, y(x_0', p_t)) = \left[\frac{d}{dt} y^m(x_0', p_t)\right] \mu_{m,i}^{-1}. \end{aligned} \tag{2.8}$$

Let $\mathscr{X}^t(\omega)$ be the parametrization of $\mathscr{N}(t) = \{x \in \mathscr{U} : c(x, y_0) = c(x, y_t) + a\}$ (recall that $\mathscr{N}(t)$ is convex as the boundary of sub-level set). We can choose $a = a(t)$ so that all $\mathscr{N}(t)$ pass through the point x_0', in other words there is $\bar{\omega}^t$ such that $\mathscr{X}^t(\bar{\omega}^t) = x_0'$. Moreover, by (2.7) it follows that

$$\begin{aligned} c_{x_i}(\mathscr{X}^t(\bar{\omega}^t), y_0) - c_{x_i}(\mathscr{X}^t(\bar{\omega}^t), y_t) &= c_{x_i}(x_0', y_0) - c_{x_i}(x_0', y_t) \\ &= p_0^i - p_t^i \\ &= t(p_0^i - p_1^i). \end{aligned} \tag{2.9}$$

After fixing t and differentiating the identity $c(\mathscr{X}^t(\omega), y_0) = c(\mathscr{X}^t(\omega), y_t) + a(t)$ in ω we get

$$\left[c_{x_i}(\mathscr{X}^t, y_0) - c_{x_i}(\mathscr{X}^t, y_t)\right] \mathscr{X}^{i,t}_{\omega_k} = 0,$$

$$\left[c_{x_i x_j}(\mathscr{X}^t, y_0) - c_{x_i x_j}(\mathscr{X}^t, y_t)\right] \mathscr{X}^{j,t}_{\omega_l} \mathscr{X}^{i,t}_{\omega_k} + \left[c_{x_i}(\mathscr{X}^t, y_0) - c_{x_i}(\mathscr{X}^t, y_t)\right] \mathscr{X}^{i,t}_{\omega_k \omega_l} = 0. \tag{2.10}$$

Thus the normals of $\mathscr{N}(t)$ at x_0' are collinear to $p_1 - p_0$ for all $t \in [0,1]$, that is

$$n(x_0) = n'(x_0') = \frac{p_1 - p_0}{|p_1 - p_0|}, \quad \mu \bar{n} = n \quad (\text{recall} \quad n(\bar{\zeta}) = \mu \bar{n}(\bar{\zeta})). \tag{2.11}$$

Hence we can rewrite (2.10) as follows

$$\left[(c_{x_i x_j}(\mathscr{X}^t, y_0) - c_{x_i x_j}(\mathscr{X}^t, y_t)\right] \mathscr{X}^{j,t}_{\omega_l} \mathscr{X}^{i,t}_{\omega_k} = -t(p_0^i - p_1^i) \mathscr{X}^{i,t}_{\omega_k \omega_l}. \tag{2.12}$$

Keeping $\mathscr{X}^t(\bar{\omega}^t) = x_0'$ fixed for all $t \in [0,1]$, dividing both sides of the last identity by t and then sending $t \to 0$ we obtain

$$-\left[y'(x_0', p_0) c_{y, x_i x_j}(x_0', y_0)\right] \mathscr{X}^{j,t=0}_{\omega_l} \mathscr{X}^{i,t=0}_{\omega_k} = -(p_0^i - p_1^i) \mathscr{X}^{i,t=0}_{\omega_k \omega_l}. \tag{2.13}$$

On the other hand from (2.8) we see that $\frac{d}{dt}y(x_0', p_t)\big|_{t=0} = (p_1 - p_0)\mu$. Thus substituting this into the last equality we obtain

$$\begin{aligned} \left[(p_1 - p_0)\mu c_{y,x_i x_j}(x_0', y_0)\right] \mathscr{X}_{\omega_l}^{j,t=0} \mathscr{X}_{\omega_k}^{i,t=0} &= (p_0^i - p_1^i)\mathscr{X}_{\omega_k \omega_l}^{i,t=0} \\ &= -(p_1^i - p_0^i)\mathscr{X}_{\omega_k \omega_l}^{i,t=0} \end{aligned} \tag{2.14}$$

or equivalently

$$\left[n^\alpha \mu_{\alpha\beta} c_{y_\beta, x_i x_j}(x_0', y_0)\right] \mathscr{X}_{\omega_l}^{j,t=0} \mathscr{X}_{\omega_k}^{i,t=0} = -n^i \mathscr{X}_{\omega_k \omega_l}^{i,t=0} \tag{2.15}$$

if we utilize (2.11).

Step 3: (*Monotone bending*)

Recall that by assumption $\mathbb{T}_{x_0}\partial\mathscr{U}$ and $\mathbb{T}_{x_0'}\mathscr{N}(t=0)$ have the same local coordinate system (by reparametrizing $\mathscr{N}(t=0)$ if necessary). From convexity of $\mu c_y(\mathscr{U}, y_0)$ boundary of which is parametrized by r we have

$$
\begin{aligned}
0 &\geq r^\alpha_{st} n^\alpha = \mu\rho_{st} n = \mu_{\alpha\beta}(c_{y_\beta, x_i x_j} R^i_s R^j_t + c_{y_\beta, x_i} R^i_{st}) n^\alpha \\
&= \mu_{\alpha\beta} c_{y_\beta, x_i x_j} R^i_s R^j_t n^\alpha + R^i_{st} n^i \\
&\overset{(2.15)}{=} -n^i \mathscr{X}^{i,t=0}_{\omega_k\omega_l} + R^i_{st} n^i.
\end{aligned}
\tag{2.16}
$$

Now **A3** yields that at x_0'

$$
n^i \mathscr{X}^{i,t}_{\omega_k\omega_l} \geq n^i \mathscr{X}^{i,t=0}_{\omega_k\omega_l} \overset{(4.8)}{\geq} R^i_{st} n^i. \tag{2.17}
$$

Recalling $\mathrm{II} = -n\partial^2 r$ we finally obtain the required inequality

$$
\mathrm{II}_{x_0'} \mathscr{N} \leq \mathrm{II}_{x_0} \partial \mathscr{U}.
$$

The proof is now complete. □

Remark 2.1

Note that weak **A3** (i.e. when $c_0 = 0$ in (1.6)) is enough for the monotonicity to conclude the inequality $n^i \mathscr{X}^{i,t}_{\omega_k\omega_l} \geq n^i \mathscr{X}^{i,t=0}_{\omega_k\omega_l}$.

- There is a wide class of cost functions for which the set $\mathscr{N}$ is convex. Observe that $c(x,y) = \frac{1}{p}|x-y|^p$ satisfies **A3** for $-2 < p < 1$ and weak **A3** if $p = \pm 2$ (MTW).

- It is useful to note that if $\Omega_\psi = \{x \in \mathbb{R}^d \ s.t. \ \psi(x) < 0\}$ for some smooth function $\psi : \mathbb{R}^d \to \mathbb{R}$ such that $\Omega_\psi \neq \emptyset$ then

$$\partial^2 \psi(x)\tau(x) \cdot \tau(x) \geq 0, \quad \forall \tau(x) \in \mathbb{T}_x \tag{2.18}$$

is a necessary and sufficient condition for Ω_ψ to be convex provided that $\frac{\nabla\psi}{|\nabla\psi|}$ is directed towards positive ψ.

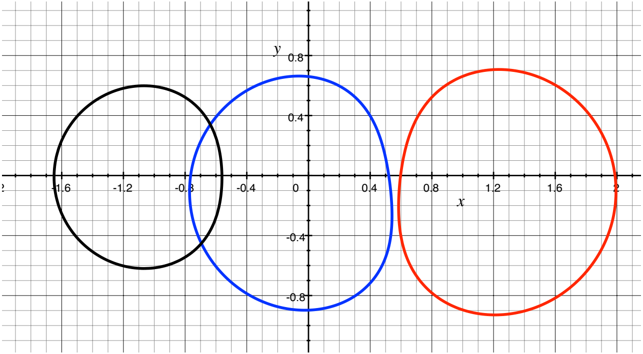


Figure 2: From left to right: $a = -2, y_1 = (-10^{-3}, 0), y_2 = (-1, -10^{-2})$; $a = 1, y_1 = (-10^{-1}, -10^{-1}), y_2 = (1, 10^{-2})$; $a = -1, y_1 = (-10^{-4}, 0), y_2 = (1.1, -10^{-1})$.

3 Antenna design problems

- In parallel reflector problem one deals with the paraboloids of revolution

$$P(x,\sigma,Z) = \frac{\sigma}{2} + Z^{n+1} - \frac{1}{2\sigma}|x-z|^2 \tag{3.1}$$

  which play the role of support functions.

- Here the point $Z = (z, Z^{n+1}) \in \mathbb{R}^{n+1}$ is the focus of the paraboloid such that $\psi(z, Z^{n+1}) = 0$ for some smooth function ψ satisfying some structural conditions and σ is a constant.

- If P_1 is internally tangent to P_2 at z_0 and $\mathrm{II}_{z_0}P_1 \geq \mathrm{II}_{z_0}P_2$ then P_1 is inside P_2, see Lemma 8.1 (K, 2014). This again follows from Blaschke's theorem. Indeed, we have that at the points x and x' corresponding to coinciding outward normals

$$\mathrm{II}_x P_1 = \frac{1}{\sqrt{1+|DP_1(x)|^2}}\frac{1}{\sigma_1}\delta_{ij}$$

and

$$\mathrm{II}_{x'} P_2 = \frac{1}{\sqrt{1+|DP_2(x')|^2}}\frac{1}{\sigma_2}\delta_{ij}.$$

Furthermore $DP_1(x) = DP_2(x')$ and hence

$$\sqrt{1+|DP_1(x)|^2} = \sqrt{1+|DP_2(x')|^2}. \tag{3.2}$$

From $\mathrm{II}_{z_0}P_1 \geq \mathrm{II}_{z_0}P_2$ we infer that

$$\frac{1}{\sigma_1} \geq \frac{1}{\sigma_2}. \tag{3.3}$$

Consequently (3.3) and (3.2) imply that

$$\mathrm{II}_x P_1 \geq \mathrm{II}_{x'} P_2.$$

4 Near-field Refractor

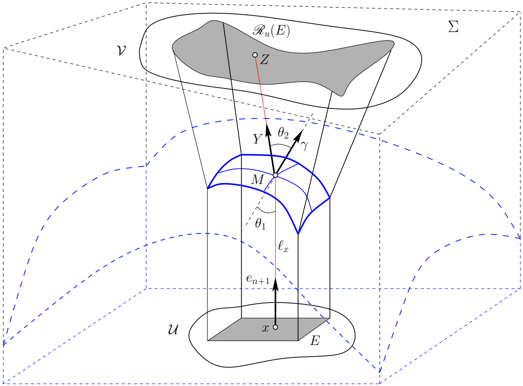


Figure 3: The blue doted lines confine the boundary of media I.

If n_1 and n_2 are the refractive indices of media I and II respectively then

$$\varepsilon = \frac{n_1}{n_2} = \frac{\sin\theta_2}{\sin\theta_1} = \begin{cases} \frac{\sqrt{a^2-b^2}}{a} < 1 & \text{for ellipsoids,} \\ \frac{\sqrt{a^2+b^2}}{a} > 1 & \text{for hyperboloids.} \end{cases} \tag{4.1}$$

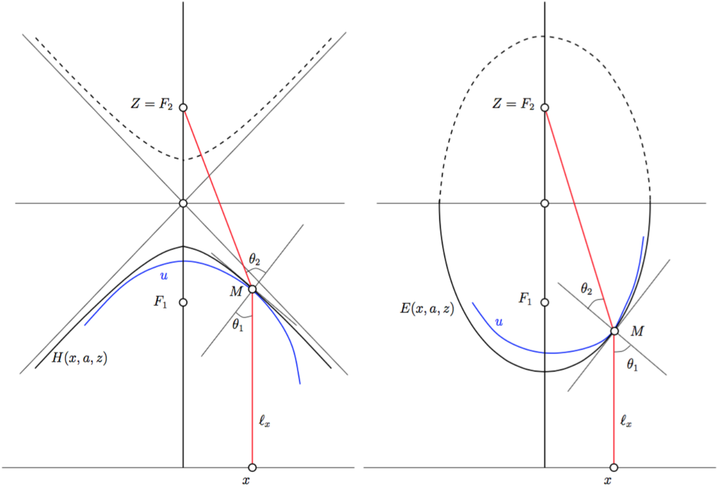
Here ε is the eccentricity. Since ε is fixed we can drop the dependence of E and H from $b = a\sqrt{|\varepsilon^2 - 1|}$ and take

$$E(x,a,Z) = Z^{n+1} - a\varepsilon - a\sqrt{1 - \frac{(x-z)^2}{a^2(1-\varepsilon^2)}}, \qquad \text{if } \varepsilon < 1, \tag{4.2}$$

$$H(x,a,Z) = Z^{n+1} - a\varepsilon - a\sqrt{1 + \frac{(x-z)^2}{a^2(\varepsilon^2-1)}}, \qquad \text{if } \varepsilon > 1. \tag{4.3}$$

We also define the constant

$$\kappa = \frac{\varepsilon^2 - 1}{\varepsilon^2}. \tag{4.4}$$


$Z = F_2$
θ_2
u
M
F_1
θ_1
$H(x,a,z)$
ℓ_x
x
$Z = F_2$
θ_2
F_1
$E(x,a,z)$
u
M
θ_1
ℓ_x
x

Theorem 4.1

Let $u \in C^2(\mathscr{U})$ be a solution. Then

1° $Y = \varepsilon\left(\frac{\kappa Du}{1+q}, 1 - \frac{\kappa}{1+q}\right)$ is the unit direction of refracted ray,

2° u solves the equation

$$\left|\det\left[\frac{q+1}{t\varepsilon\kappa}\{\mathrm{Id} - \kappa\varepsilon^2 Du \otimes Du\} + D^2 u\right]\right| = \left|-\varepsilon q \left[\frac{q+1}{t\varepsilon\kappa}\right]^n \frac{\nabla\psi \cdot Y}{|\nabla\psi|}\frac{f}{g}\right|, \tag{4.5}$$

where

$$q(x) = \sqrt{1 - \kappa(1 + |Du|^2)}, \quad \kappa = \frac{\varepsilon^2 - 1}{\varepsilon^2} \tag{4.6}$$

and t is the stretch function defined via an implicit relation $\psi(x + e_{n+1}u(x) + Yt) = 0$.

$$\nabla\psi(Z)\cdot(X-Z)>0 \quad \forall X\in(\mathscr{U}\times[0,m_0]), \forall Z\in\Sigma \text{ and for some large constant } m_0>0,$$

$$\text{dist}(\mathscr{U},\mathscr{V})>0,$$

$$\mathscr{V} \text{ is } R-\text{convex with respect to } \mathscr{U},$$

$$f,g>0,$$

$$\frac{1}{t}\left[\frac{t\varepsilon\kappa}{q+1}\right]^2 \mathrm{II}+\frac{\kappa}{q}\frac{\psi_{n+1}}{|\nabla\psi|}\left(\mathrm{Id}+\kappa\frac{p\otimes p}{q^2}\right)<0, \quad \text{if } \kappa>0.$$

- Let $H_i(x)=H(x,a_i,Z_i), i=1,2$ be two global supporting hyperboloids of u at x_0 such that the contact set $\Lambda\neq\{x_0\}$. Thus u is not differentiable at x_0. To fix the ideas take $x_0=0$.

- If γ_i is the normal of the graph of $H_i, i=1,2$ at x_0 then for any $\theta\in(0,1)$ there is $Z_\theta\in\Sigma\cap\mathscr{C}_{0,\gamma_1,\gamma_2}$ and $a_\theta>0$ such that $H(x)=H(x,a_\theta,Z_\theta)$ is a local supporting hyperboloid of u at 0 and

$$DH_\theta(0)=\theta DH_1(0)+(1-\theta)DH_2(0). \tag{4.7}$$

- Observe that the correspondence $\theta \mapsto Z_\theta$ is one-to-one thanks to our assumptions. By choosing a suitable coordinate system we can assume that $DH_1(0) - DH_2(0) = (0, \dots, 0, \alpha)$ for some $\alpha > 0$. Then we have that for all $0 < \theta < 1$ (Loeper type argument)

$$
\begin{aligned}
\min[H_1(x), H_2(x)] &\leq \theta H_1(x) + (1-\theta) H_2(x) \\
&= u(0) + [DH_2(0) + \alpha\theta]\, x_n + \frac{1}{2}\left[\theta D^2 H_1(0) + (1-\theta) D^2 H_2(0)\right] x \otimes x \\
&\quad + o(|x|^2)
\end{aligned}
$$

where the last line follows from Taylor's expansion.

Then

$$
D^2 H_\theta(0) = -\frac{G(x_0, u(0), p_1 + \theta(p_2 - p_1))}{\varepsilon\kappa}.
$$

where we set $p_i = DH_i(0), i = 1,2$ and used (4.7). For all unit vectors τ perpendicular to x_n axis we have

$$\begin{aligned}
\frac{d^2}{d\theta^2} D^2_{\tau\tau} H_\theta(0) &= -\frac{d^2}{d\theta^2} \frac{G^{ij}(0, u(0), p_1 + \theta(p_2 - p_1))\tau_i \tau_j}{\varepsilon\kappa} \\
&= -\alpha^2 \frac{\partial^2}{\partial p_n^2} \frac{G^{jj}(0, u(0), p_1 + \theta(p_2 - p_1))\tau_i \tau_j}{\varepsilon\kappa} \\
&\leq -\alpha^2 c_0
\end{aligned} \tag{4.8}$$

where the last line follows from **(A3)** with $c_0 > 0$.

5 More inclusion principles

- There are various inclusion principles in geometry, we want to mention the following elementary one due to J. Nitsche : Each continuous closed curve of length L in Euclidean 3-space is contained in a closed ball of radius $R < L/4$. Equality holds only for a "needle", i.e., a segment of length $L/2$ gone through twice, in opposite directions.

- Later J. Spruck generalized this result for compact Riemannian manifold $\mathscr{M}$ of dimension $n \geq 3$ as follows: if the sectional curvatures $K(\sigma) \geq 1/c^2$ for all tangent plane sections σ then $\mathscr{M}$ is contained in a ball of radius $R < \frac{1}{2}\pi c$, and this bound is best possible.

- We remark here that there is a smooth surface $S \subset \mathbb{R}^3$ such that the mean curvature $H \geq 1$ and the Gauss curvature $K \geq 1$ then the unit ball cannot be fit inside S, (Spruck, JDG 1973). Notice that K is an intrinsic quantity and $H \geq 1$ implies that $K \geq 1$.

Thank You