Generalised Finite Difference Methods for Monge-Ampère Equations

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Generated Jacobian Equations: from Geometric Optics to Economics
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April 10, 2017

Monge-Ampère Equations

- Generalised Monge-Ampère equations

$$\begin{cases} \det(A(x,\nabla u(x)) + D^2u(x)) = F(x,\nabla u(x)) \\ A(x,\nabla u(x)) + D^2u(x) > 0 \end{cases}$$

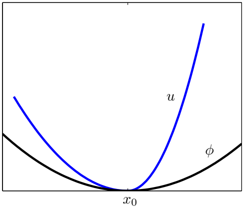
- Dirichlet problem for prescribed Gaussian curvature

$$\begin{cases} \det(D^2u(x)) = \kappa(x)(1+|\nabla u(x)|^2)^{(d+2)/2} \\ u \text{ is convex} \\ u(x) = g(x), \qquad x \in \partial\Omega \end{cases}$$

- Optimal transport with quadratic cost

$$\begin{cases} g(\nabla u(x)) \det(D^2u(x)) = f(x) \\ u \text{ is convex} \\ \nabla u(X) \subset Y \end{cases}$$

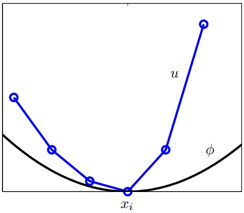
Viscosity Solutions



If $F(D^2 u) = 0$ in viscosity sense then for smooth ϕ:

- $\phi(x_0) = u(x_0)$
- $\phi(x) \leq (\geq) u(x)$

$\Rightarrow F(D^2 \phi) \geq (\leq) 0$



For smooth ϕ if:

- $F(D^2 u_i) \approx F^h(u_j - u_i)\big|_{j \in \mathcal{N}(i)}$
- $\phi_j - \phi_i \leq u_j - u_i$

$\Rightarrow F^h(\phi_j - \phi_i) \geq F^h(u_j - u_i)$

Barles and Souganidis Framework

Theorem (Barles and Souganidis, 1990)

Let $F(x, u, \nabla u, D^2 u) = 0$ be a well-posed elliptic equation satisfying a comparison principle. Let $F^\epsilon[u^\epsilon]$ be a consistent, monotone approximation of the PDE with solutions bounded independent of ϵ. Then u^ϵ converges uniformly to the unique viscosity solution of the PDE as $\epsilon \to 0$.

- $F^\epsilon(x, u^\epsilon(x), u^\epsilon(x) - u^\epsilon(\cdot)) = 0, \; x \in \bar{\Omega}$
- Define envelopes

$$\bar{u} = \limsup_{\epsilon \to 0^+, y \to x} u^\epsilon(y), \quad \underline{u} = \liminf_{\epsilon \to 0^+, y \to x} u^\epsilon(y)$$

- Consistency and monotonicity $\Rightarrow$
  $\bar{u}$ ($\underline{u}$) is sub(super)solution
- Comparison principle $\Rightarrow \bar{u} \leq \underline{u}$

Outline

Globally Elliptic Extension

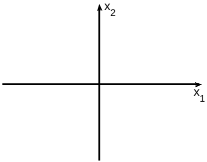
- Define "convexified" determinant

$$\det{}^+(M) = \begin{cases} \det(M), & M \geq 0 \\ < 0, & \text{o.w.} \end{cases}$$

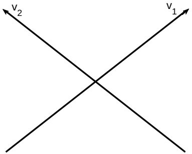
- Hadamard's inequality:

$$\det{}^+(D^2u) = \min\left\{\prod_{j=1}^{d} \max\{\nu_j^T(D^2u)\nu_j, 0\} + \min\{\nu_j^T(D^2u)\nu_j, 0\}\right\}$$

over orthogonal $\nu_1, \ldots, \nu_d$. [F and Oberman, *SINUM*, 2011]



$\det(D^2u) = u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}^2.$



$\det(D^2u) = u_{v_1v_1}u_{v_2v_2}.$

Minimal Stencils

Theorem (Motzkin and Wasow, *J. Math. Phys.*, 1952)

Given any stencil, there exists a linear elliptic operator that cannot be approximated in a consistent, monotone way on this stencil.

Theorem (Kocan, *Numer. Math.*, 1995)

Consider the degenerate linear elliptic operator $-u_{\nu\nu}$. On a Cartesian grid, the minimal width of a stencil on which this can be approximated in a consistent, monotone way is

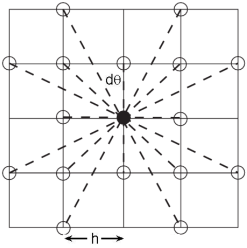
$$\begin{cases} \max\{n,m\} & \nu_1/\nu_2 = n/m,\ n,m \in \mathbb{Z},\ gcd(m,n) = 1 \\ \infty & \text{otherwise} \end{cases}$$

Wide Stencils

For grid directions,

$$u_{\nu\nu} \approx \frac{1}{|\nu h|^2} \left(u(\mathbf{x} + \nu h) + u(\mathbf{x} - \nu h) - 2u(\mathbf{x}) \right).$$

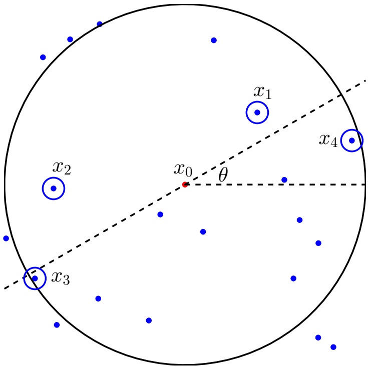
[F and Oberman, *SINUM*, 2011]



- Discretisation defined on uniform Cartesian grids.
- Difficult to handle different geometries.
- Challenging to implement near boundaries.

Second Directional Derivatives

- Want monotone approximation of $\frac{\partial^2 u}{\partial e_\theta^2}$ at $x_0 \in \mathcal{G}$
- Consider some neighbourhood $B(x_0, \sqrt{h})$
- Find four points in $B(x_0, \sqrt{h}) \cap \mathcal{G}$ that best align with the line $x_0 + te_\theta,\ t \in \mathbb{R}$



Monotone Approximation

Look for approximation of the form

$$u_{xx} = \sum_{j=1}^{4} a_j(u(x_j) - u(x_0))$$

$$= \sum_{j=1}^{4} a_j \left[r_j \cos\theta_j u_x(x_0) + r_j \sin\theta_j u_y(x_0) + \frac{1}{2} r_j^2 \cos^2\theta_j u_{xx}(x_0) \right.$$

$$\left. + \mathcal{O}(r_j^3 + r_j^2 \sin\theta_j) \right]$$

Require

$$\begin{cases} \sum_{j=1}^{4} a_j r_j \cos\theta_j = 0 \\ \sum_{j=1}^{4} a_j r_j \sin\theta_j = 0 \\ \sum_{j=1}^{4} \frac{1}{2} a_j r_j^2 \cos^2\theta_j = 1 \\ a_j \geq 0 \end{cases}$$

Monotone Approximation of u_{xx}

- Relate neighbouring points using polar coordinates,

$$x_i - x_0 = (h_i, \theta_i)$$

- Define

$$C_i = h_i \cos\theta_i, \quad S_i = h_i \sin\theta_i$$

- A monotone scheme is

$$-u_{xx} \approx 2\frac{(C_3S_2 - C_2S_3)(S_1u_4 - S_4u_1) + (C_1S_4 - C_4S_1)(S_2u_3 - S_3u_2)}{(C_3S_2 - C_2S_3)(C_1^2S_4 - C_4^2S_2) - (C_1S_4 - C_4S_1)(C_3^2S_2 - C_2^2S_3)}$$

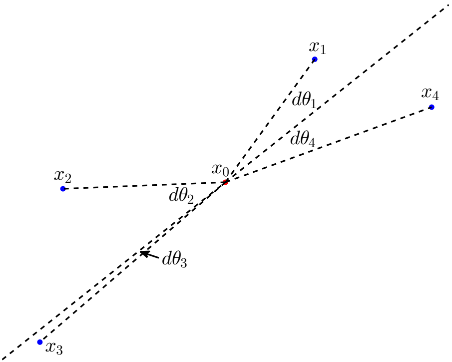
Monotone Approximation

- Can construct approximation of the form

$$\frac{\partial^2 u(x_0)}{\partial e_\theta^2} \approx \sum_{j=1}^{4} a_j (u(x_j) - u(x_0))$$

- All $a_j \geq 0$ as long as all $d\theta_j < \pi/2$

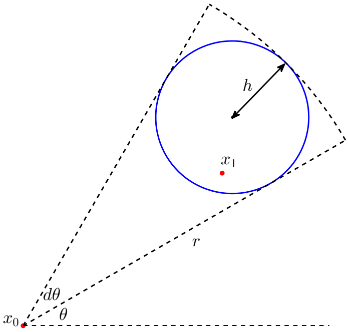
- Discretisation error is $\mathcal{O}(r + d\theta)$



Existence of Consistent, Monotone Scheme

Construction of monotone stencil requires existence of a discretisation point in the wedge

$$\{x_0 + te_\phi \mid \phi \in [\theta, \theta + d\theta], t \in (0, r]\}$$

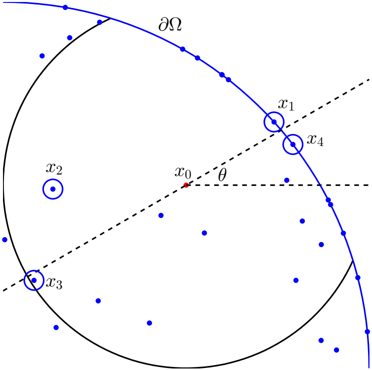


$$r = \sqrt{h}$$

$$\Rightarrow$$

$$d\theta = \mathcal{O}(\sqrt{h})$$

Admissible Point Clouds (Boundary)



- Near boundary: do not change approximation scheme
- Need boundary sufficiently well-resolved in order to preserve angular resolution
- Take $h_B = \mathcal{O}(h^{3/2})$

Convergence

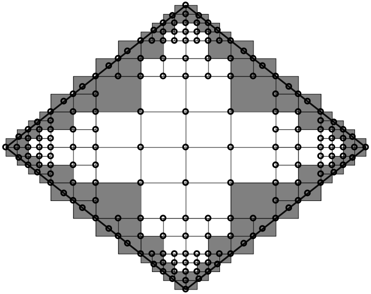
Theorem (F, 2015)

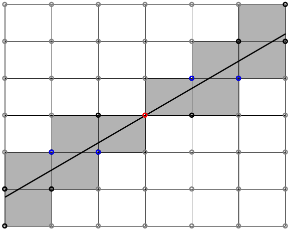
Let $F(x, u_{\nu\nu}) = 0$ be a well-posed elliptic equation satisfying a comparison principle. Let $\mathcal{G}_n \in \bar{\Omega}$ be a sequence of point clouds satisfying appropriate structure conditions, with $h_n \to 0$. Then it is possible to construct an approximation scheme $F_n[u_n] = 0$ such that u_n converges uniformly to the unique viscosity solution of the original PDE.

Quadtrees

Piecewise Cartesian grids augmented on boundary enable:

- Fast identification of stencils
- Easy construction of higher-order filtered schemes
- Simple strategies for mesh adaptation





[F and Salvador, in preparation]

Outline

1. Approximation Schemes

2. Uniqueness Results
   - Prescribed Gaussian Curvature
   - Quadratic Cost OT

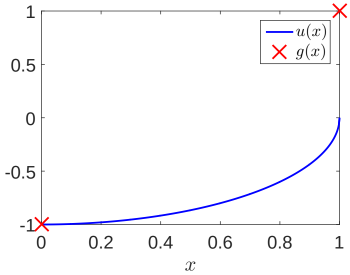
3. Computations

Outline

Prescribed Gaussian Curvature

$$F(x, \nabla u(x), D^2u(x)) \equiv -\det{}^+(D^2u(x)) + \kappa(x)(1+|\nabla u|^2)^{(n+2)/2} = 0$$

Eg: $\kappa = 1$,
$u(0) = -1$, $u(1) = 1$



Weak Dirichlet condition:

$$u(x) \leq g(x), \quad x \in \partial\Omega$$

and if $v(x)$ is any other solution of the PDE with

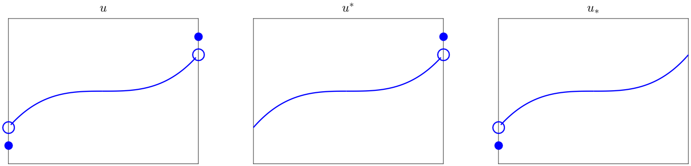
$$v(x) \leq g(x), \quad x \in \partial\Omega$$

then

$$v(x) \leq u(x), \quad x \in \Omega.$$

Viscosity Formulation

- Look at semi-continuous envelopes of solution.



- At boundary require

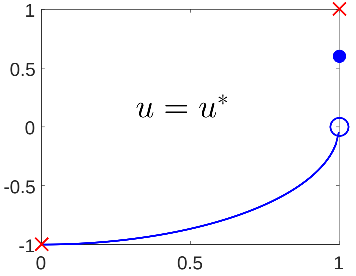
$$\min\{F(u^*), u^* - g\} \leq 0, \quad \max\{F(u_*), u_* - g\} \geq 0$$

in viscosity sense.

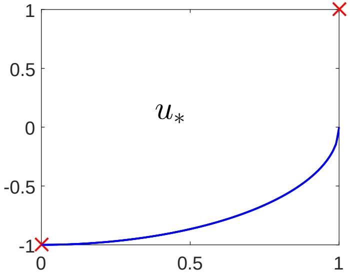
Example: 1D Ball

Example: $\kappa = 1$, $g(0) = -1$, $g(1) = 1$

$$u(x) = \begin{cases} -\sqrt{1-x^2}, & x \in [0,1) \\ a \in [0,1], & x = 1 \end{cases}$$



$u^*(1) \leq g(1)$



$u'_*(1) = \infty$

Generalised Solutions

Definition

A convex function u is a generalised solution of the prescribed Gaussian curvature equation if

$$\int_{\partial u(E)} (1+|p|^2)^{-(n+2)/2}\,dp = \int_E \kappa(x)\,dx$$

for every measurable $E \subset \Omega$.

Theorem (Bakelman, 1986)

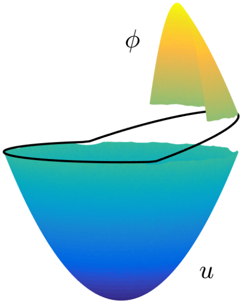
Under mild conditions on the data, the prescribed Gaussian curvature equation with Dirichlet data prescribed in the weak sense has a unique generalised solution.

Viscosity Subsolutions

Lemma

If u is a subsolution then $(u_*)^*(x) \leq g(x)$ *for* $x \in \partial\Omega$.

- Choose $x \in \partial\Omega$ and $\epsilon > 0$.
- u a sub-solution $\Rightarrow$ convex.
- Construct smooth, concave ϕ such that $u - \phi$ is maximised at $z \in \partial\Omega$ for some $|z - x| < \epsilon$.
- ϕ concave $\Rightarrow F[z, \phi] > 0$.
- $\min\{F[z, \phi], \phi(z) - g(z)\} \leq 0$

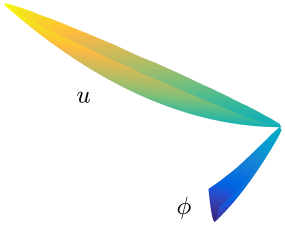


Viscosity Supersolutions

Lemma

Let u be a supersolution and $x \in \partial\Omega$. Then either $u(x) \geq g(x)$ or $\partial u(x)$ is empty.

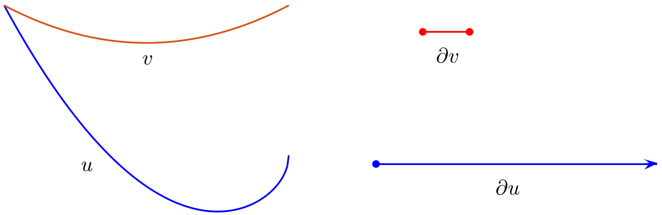
- Suppose both $u(x) < g(x)$ and $p \in \partial u(x)$.
- Construct smooth ϕ such that $u - \phi$ is minimised at x, $\nabla\phi(x) = p + n$, and $\det(D^2\phi(x))$ is arbitrarily large.
- $\max\{\phi(x) - g(x), F[x, \phi]\} < 0$, a contradiction.



Ordering of Subgradients

Lemma

Let $u \leq v$ be lower semi-continuous. Suppose that at each $x \in \partial E$ either $u(x) - v(x)$ or $\partial u(x)$ is empty. Then $\partial v(E) \subset \partial u(E)$.

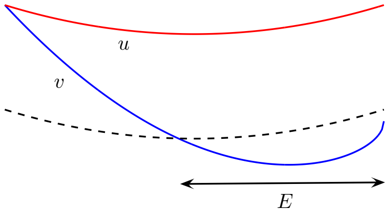


Uniqueness in Interior

Theorem

Let u be the maximal subsolution and v any other viscosity solution. Then $u = v$ on Ω.

- Define $E = \{x \in \Omega \mid u(x) - u(x_0) + v(x_0) > v(x)\}$.
- $\partial u(E) \subset \partial v(E)$.
- $\int_{\partial v(E)} (1 + |p|^2)^{-(n+2)/2}\, dp = \int_{\partial u(E)} (1 + |p|^2)^{-(n+2)/2}\, dp$.
- $\partial u(x) = \partial v(x)$, $x \in \Omega$.



Interior Comparison and Convergence

Theorem (Comparison)

Let $u : \bar{\Omega} \to \mathbb{R}$ be a subsolution and $v : \bar{\Omega} \to \mathbb{R}$ a supersolution. Then $u \leq v$ on Ω.

Theorem (Convergence)

Let $\mathcal{G}_n \in \bar{\Omega}$ be a sequence of point clouds satisfying appropriate structure conditions, with $h_n \to 0$. Let $F_n[u_n] = 0$ be a consistent, monotone approximation scheme. Then the approximate solutions $u_n(x)$ exist and converge to the viscosity solution at all points $x \in \Omega$.

Consistency Condition

- The consistency condition is

$$\limsup_{h\to 0, y\in\mathcal{G}^h\to x, \xi\to 0} F^h(y, \phi(y)+\xi, \phi(y)-\phi(\cdot)) \leq \max\left\{F(y, \nabla\phi(y), D^2\phi(y)), \phi(y)-g(y)\right\},$$

$$\limsup_{h\to 0, y\in\mathcal{G}^h\to x, \xi\to 0} F^h(y, \phi(y)+\xi, \phi(y)-\phi(\cdot)) \geq \min\left\{F(y, \nabla\phi(y), D^2\phi(y)), \phi(y)-g(y)\right\}.$$

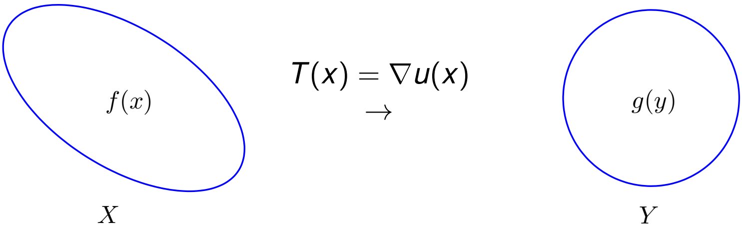
- Enforce Dirichlet BC in strong sense,

$$F^h(y, \phi(y), \phi(y)-\phi(\cdot)) = \phi(y)-g(y).$$

Outline

Quadratic Cost OT

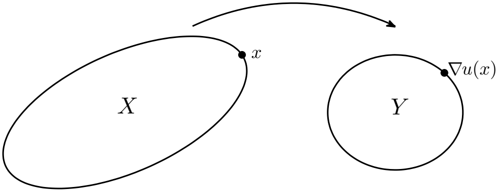
$$\begin{cases} \det(D^2 u(x)) = f(x)/g(\nabla u(x)) \\ u \text{ is convex} \\ \partial u(X) \subset \bar{Y} \end{cases}$$



Defining Function

- Introduce defining function

$$H(y) = \begin{cases} \text{dist}(y, \partial Y) & y \text{ outside } Y \\ -\text{dist}(y, \partial Y) & y \text{ inside } Y. \end{cases}$$



- Enforce constraint

$$H(\nabla u(x)) \le 0, \quad x \in X.$$

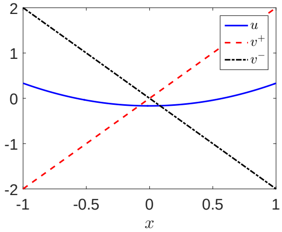
OT Boundary Conditions

- Option 1: Map boundary to boundary

$$H(\nabla u(x)) - \langle u \rangle = 0, \quad x \in \partial X$$

[Benamou, F, and Oberman, *JCP*, 2014]

- 1D example: $\begin{cases} -u'' + 1 = 0, & x \in (-1,1) \\ |u'| - 1 - \langle u \rangle = 0, & x = \pm 1 \end{cases}$
- Solution: $u(x) = \frac{x^2}{2} - \frac{1}{6}$, Supersolutions: $v(x) = \pm 2x$



PDE for Second BVP

Theorem

Under mild conditions on the data, a viscosity subsolution of

$$\max\left\{-g(\nabla u(x))\det^{+}(D^2 u(x)) + f(x), H(\nabla u(x))\right\} = 0$$

is a generalised solution of

$$\begin{cases} g(\nabla u(x))\det(D^2 u(x)) = f(x), \quad x \in X \\ \partial u(X) \subset \bar{Y} \\ u \text{ is convex.} \end{cases}$$

Moreover, viscosity subsolutions are uniquely defined on supp(f) up to additive constants.

Subsolutions of Second BVP

- Subsolutions generate “too much” mass

$$-g(\nabla u(x))\det{}^+(D^2u(x)) + f(x) \leq 0$$

$$\Rightarrow \int_{\partial u(E)} g(p)\,dp \geq \int_E f(x)\,dx, \quad E \subset X$$

- Subsolutions map into target set

$$H(\nabla u(x)) \leq 0 \Rightarrow p \in \bar{Y} \text{ for all } p \in \partial u(X)$$

- Subsolutions generate “too little” mass since $\partial u(X) \subset \bar{Y}$

$$\int_{\partial u(X)} g(p)\,dp \leq \int_Y g(p)\,dp = \int_X f(x)\,dx$$

Conclusion: Subsolutions are solutions!

Approximation of Second BVP

- Use consistent, elliptic, proper approximation $F^h[u^h] = 0$
- $\bar{u}(x) = \limsup_{h\to 0^+, y\to x} u^h(y)$ is a subsolution $\Rightarrow$ solution
- Use perturbed PDE to generate strict subsolutions

$$F^h[v^h] < 0$$

  that converge to generalised solution

$$\lim_{h\to 0} v^h(x) = u(x), \quad x \in \text{supp}(f)$$

- Discrete maximum principle: $v^h \leq u^h$
- Convergence: $u = \lim_{h\to 0} v^h \leq \underline{u} \leq \bar{u} = u$

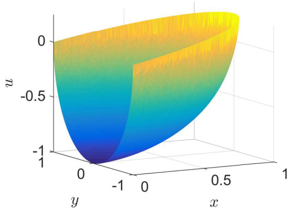
Outline

Prescribed Gaussian Curvature

$$\kappa(x, y) = 1$$

$$g(x, y) = -\sqrt{1 - x^2 - y^2} + \frac{1}{4}x$$

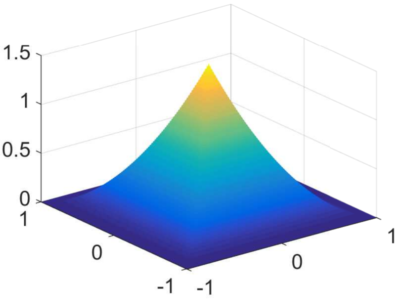
$$u(x, y) = -\sqrt{1 - x^2 - y^2}$$



h	$\|u - u^h\|_\infty$	$\|u - u^h\|_1$
2^{-3}	0.355	0.212
2^{-4}	0.333	0.183
2^{-5}	0.305	0.160
2^{-6}	0.290	0.133
2^{-7}	0.274	0.095

OT with Vanishing Densities

$$\max\left\{-g(\nabla u(x))\det{}^+(D^2u(x)) + f(x), H(\nabla u(x))\right\} = 0$$



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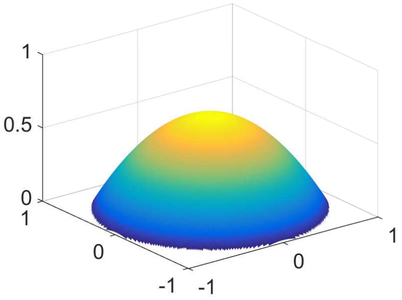

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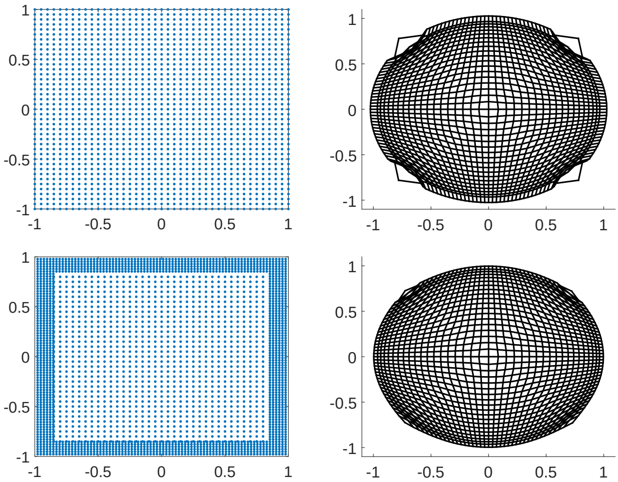
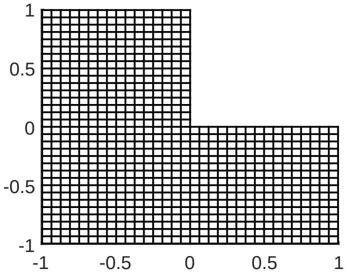
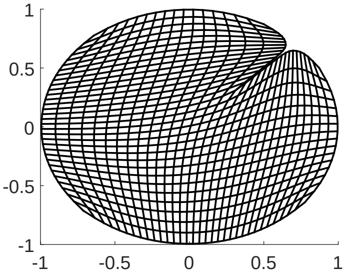
Vanishing Densities

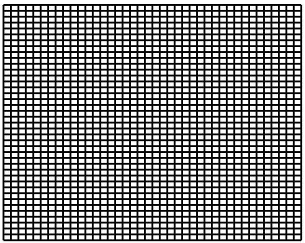


Non-Convex Sets

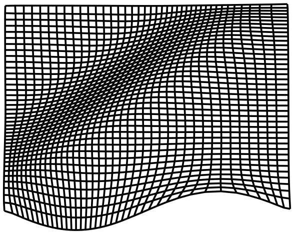


$\Rightarrow$





$\Rightarrow$

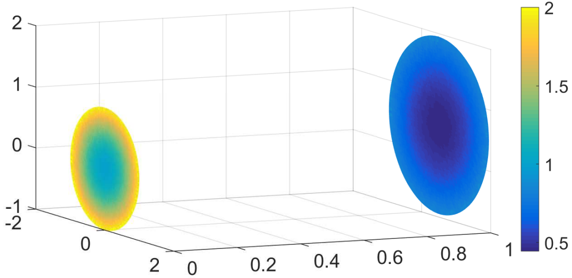


Optimal Transport (MTW Cost)

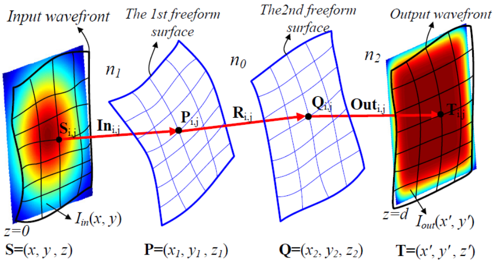
$$\begin{cases} g(T(x))\det\left(D^2c(x-T(x))+D^2u(x)\right) = \left|\det\left(D^2c(x-T(x))\right)\right| f(x) \\ D^2c(x-T(x))+D^2u(x)\geq 0 \\ T(x) = x+(\nabla c)^{-1}(\nabla u(x)) \end{cases}$$

Example:

$$\begin{cases} c(x,y) = \sqrt{|x-y|^2+L^2} \\ u(x) = \frac{(m-1)|x|^2+2x\cdot x_0}{\sqrt{L^2+|(m-1)x+x_0|^2}+\sqrt{L^2+|x_0|^2}} \\ T(x) = mx+x_0 \end{cases}$$



Beam Shaping

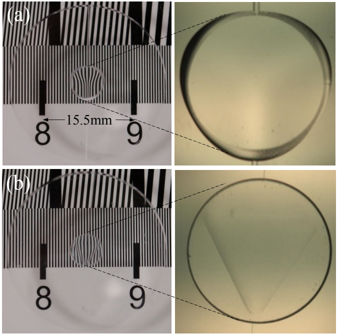


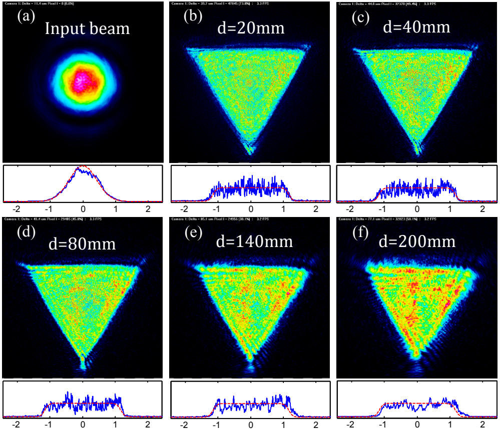
- Step 1: Determine ray mapping $(x, 0) \to (T(x), d)$ that conserves energy

$$I_{in}(x) = I_{out}(T(x)) \det(\nabla T(x))$$

- Step 2: Design lens(es) that produces this ray mapping

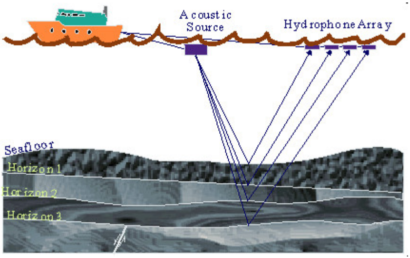
Beam Shaping

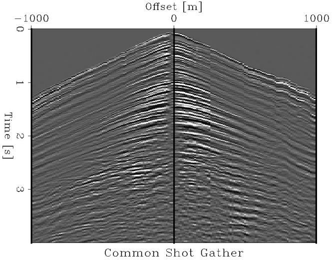




[Feng, F, Huang, Ma, and Liang, *Appl. Optics*, 2015]

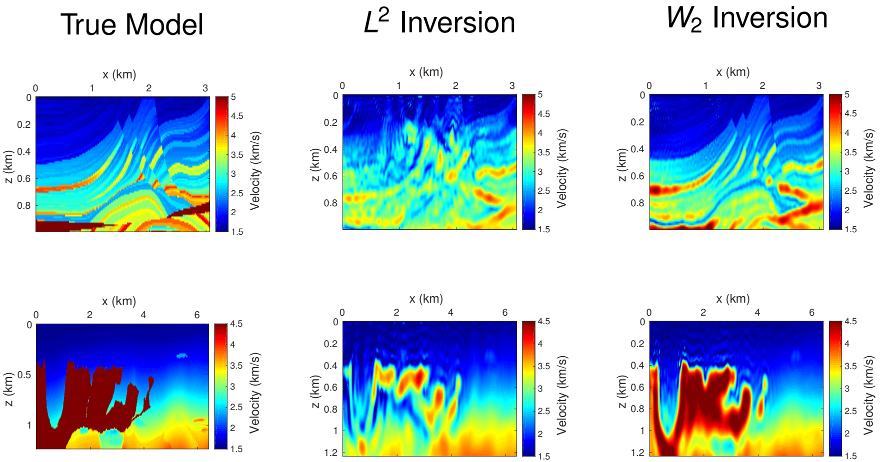
Seismic Full Waveform Inversion





Goal: Find v to minimise the misfit $\mathcal{M}(d_{obs}, d_{calc}(v))$.

Seismic Full Waveform Inversion



[Yang, Engquist, Sun, and F, 2016]

Thanks!