# Generalisations of the 15-Puzzle
# (Sliding Tokens on Graphs)

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BIRS, Banff, 26 January 2017

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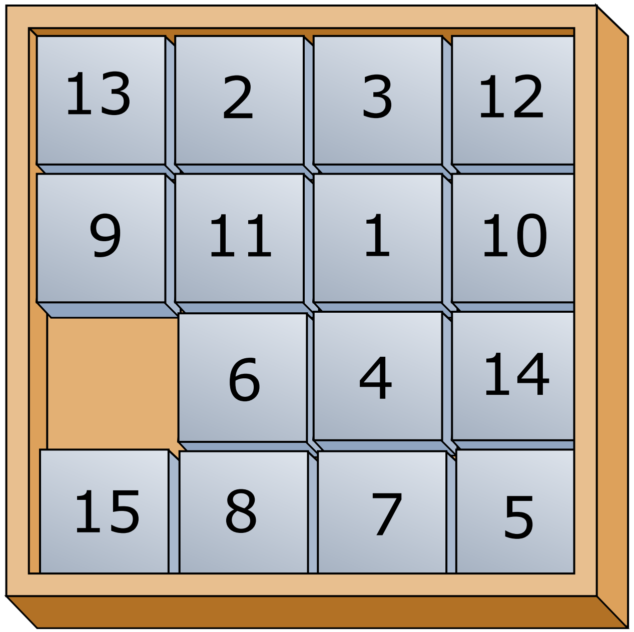
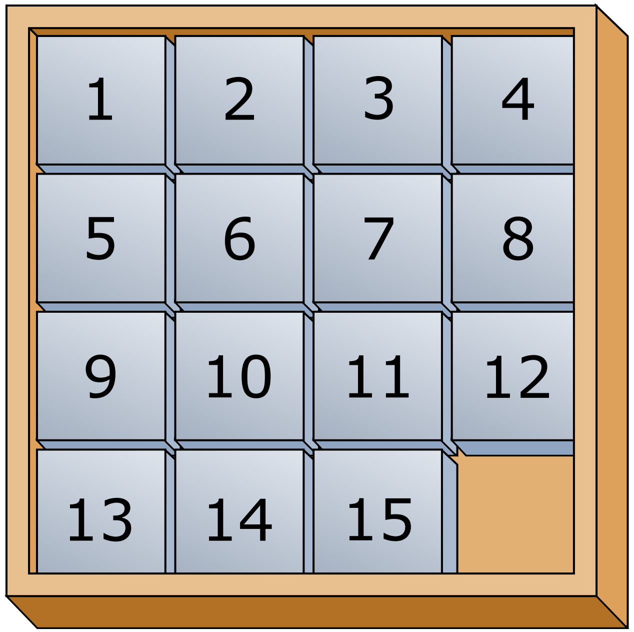
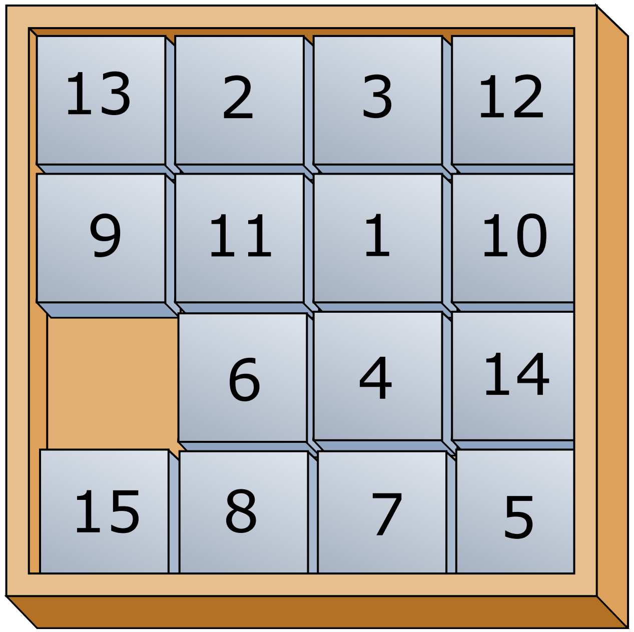
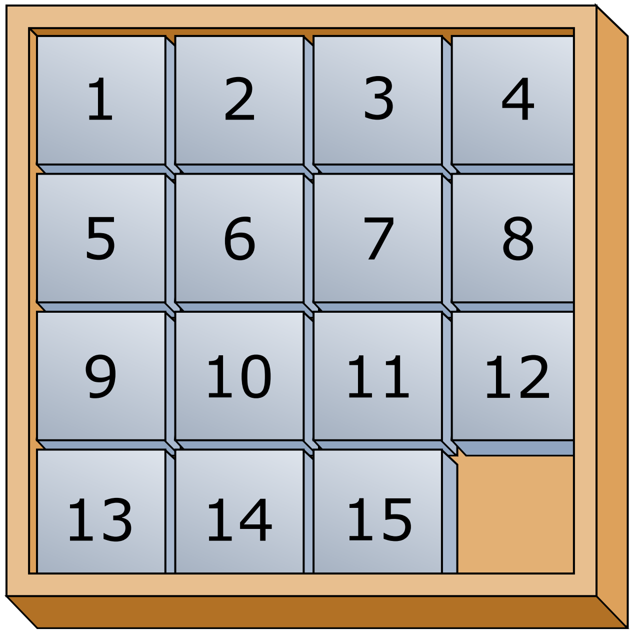
# *A classical puzzle: the 15-Puzzle*



# *A classical puzzle: the 15-Puzzle*

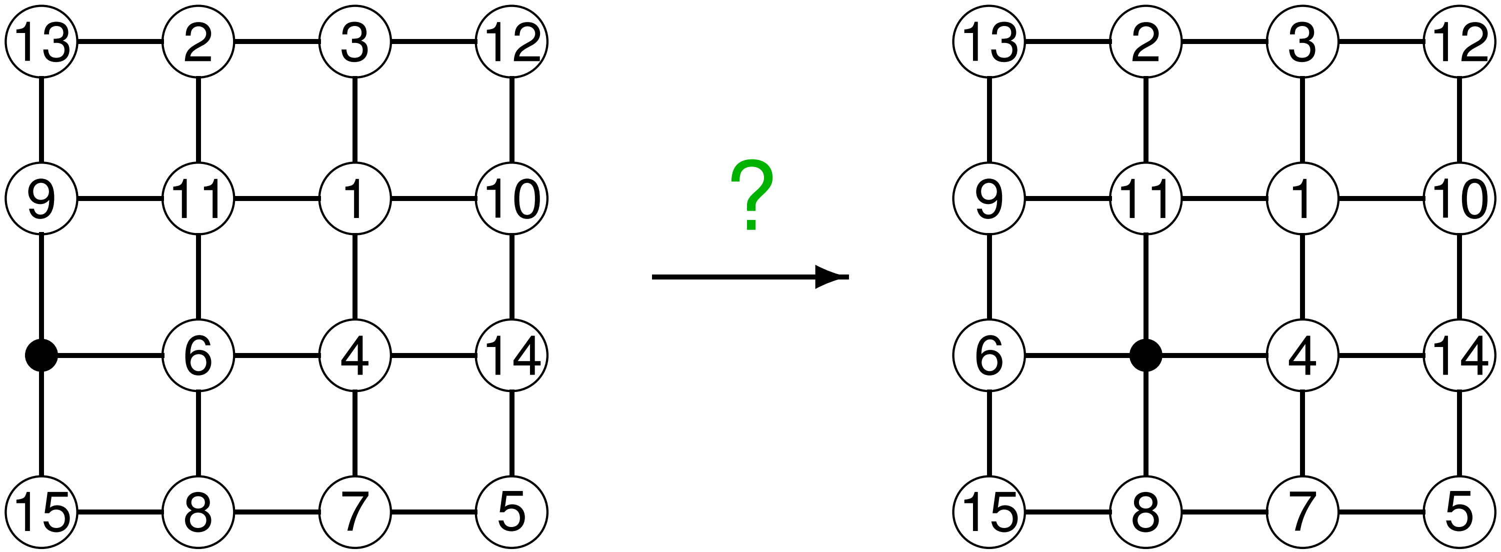


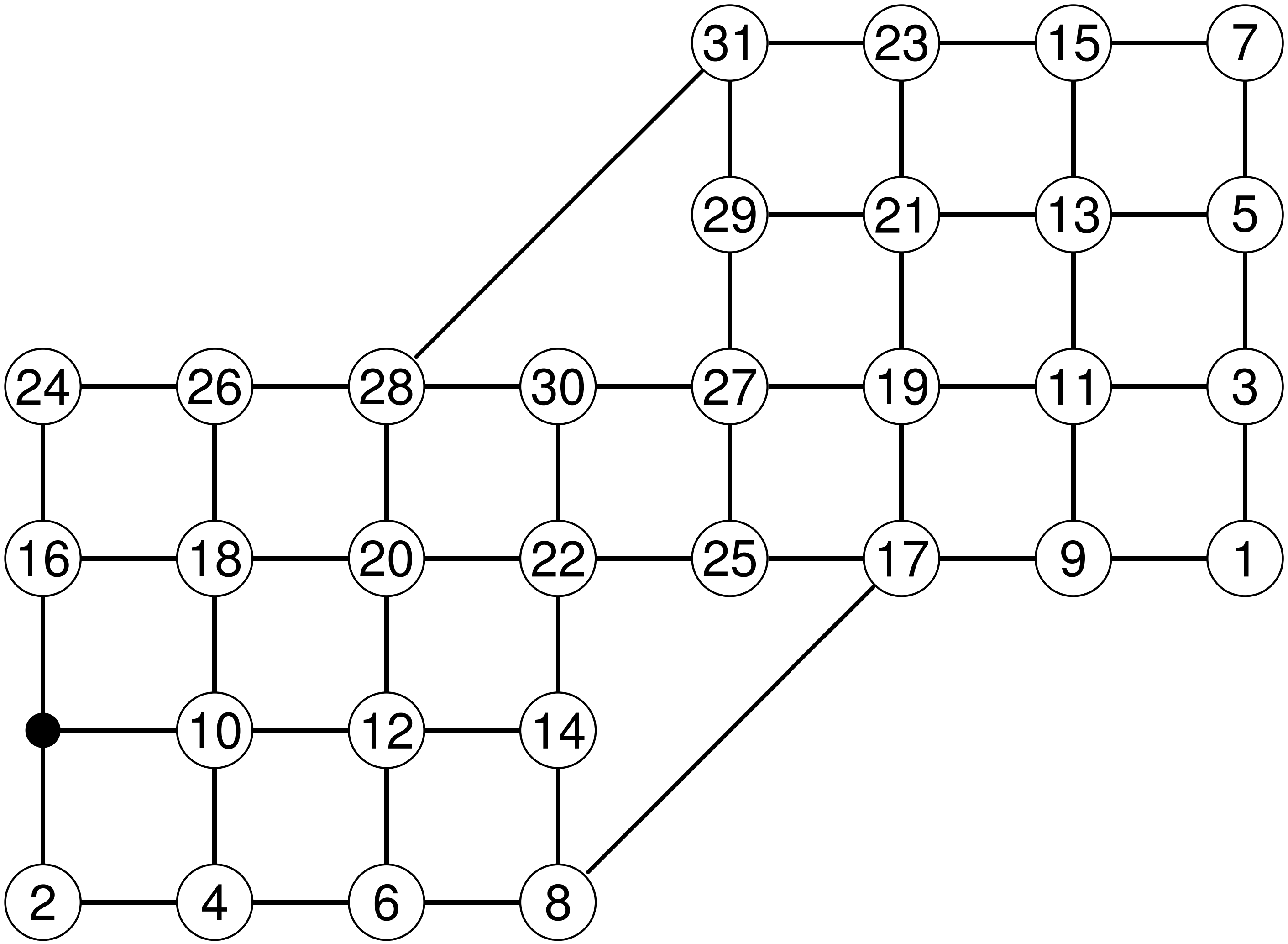

- can you always solve it?

## *Sliding token puzzles*

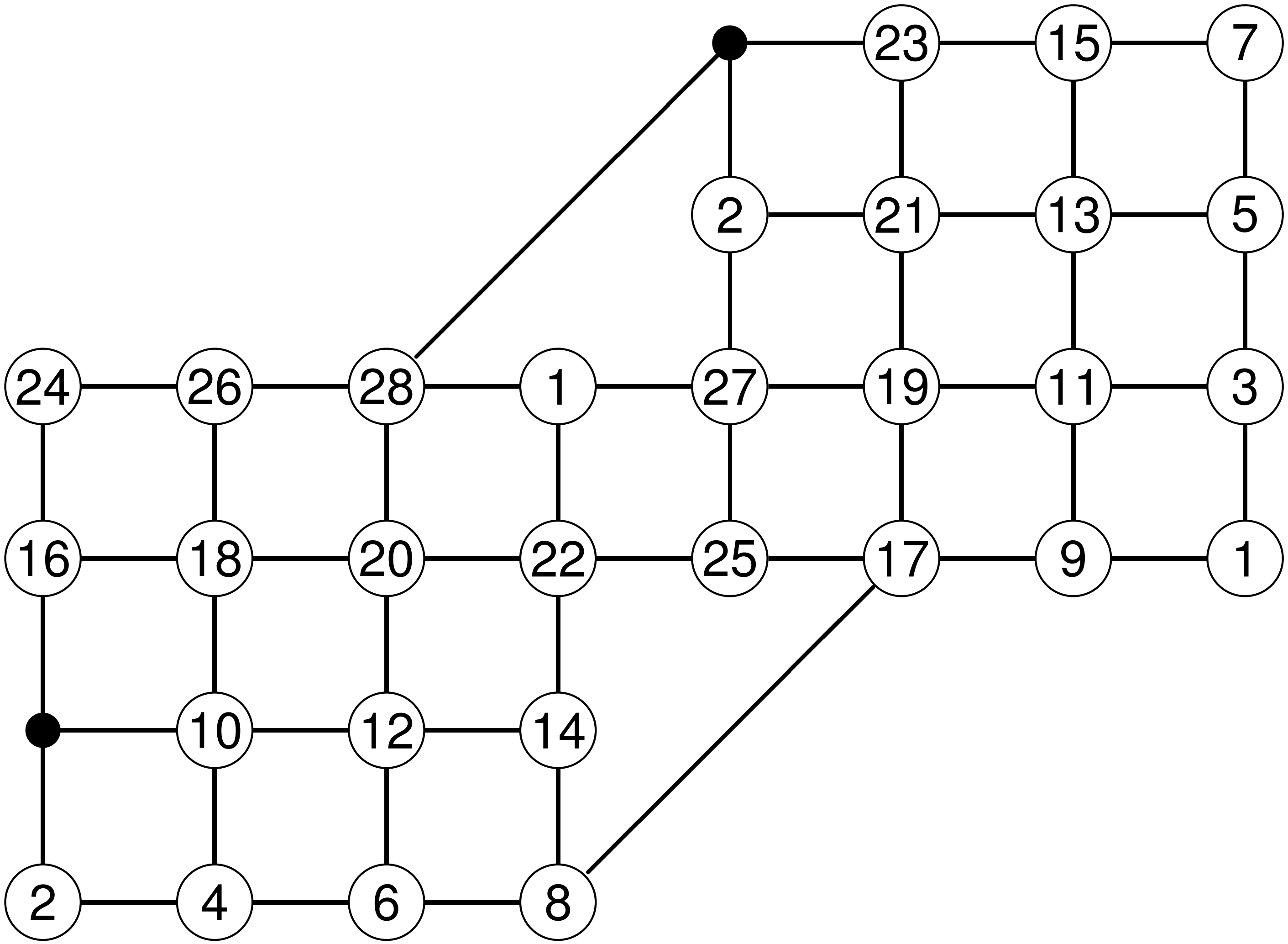
- we can interpret the 15-puzzle as a problem involving moving tokens on a given graph:



## *What if we would play on a different graph?*



## *And maybe more empty spaces and/or repeated tokens?*



## *Sliding token puzzles*

- for a given graph $G$ on $n$ vertices,

  define **puz($G$)** as the graph that has:

  - nodes: all possible placements of $n-1$ different tokens on $G$
  - adjacency: sliding one token along an edge of $G$ to an empty vertex

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- and our standard decision problems become:

  - are two token configurations in one component of puz($G$)?
  - is puz($G$) connected?

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**Theorem** (Wilson, 1974)

- if $G$ is a 2-connected graph, then puz($G$) is connected, except if:

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    (then puz($G$) has 2 components)

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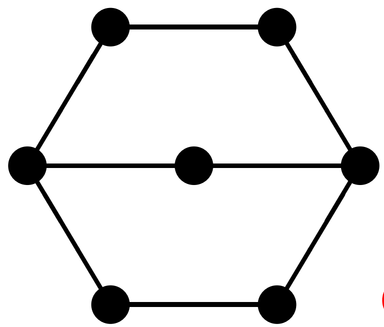
  - $G$ is the exceptional graph $\Theta_0$ (puz($\Theta_0$) has 6 components)



$\Theta_0$

## *Why does Wilson only consider 2-connected graphs?*

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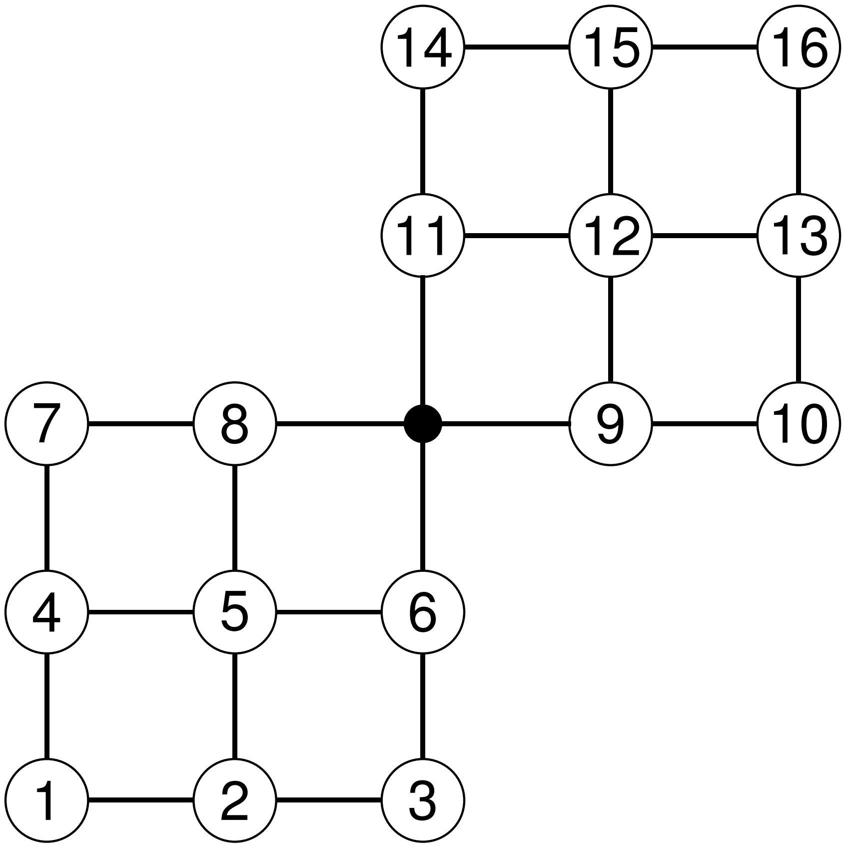
- since puz($G$) is never connected if $G$ has connectivity below 2:



## *Generalised sliding token puzzles*

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  - meaning: $k_1$ tokens with label 1, $k_2$ tokens with label 2, etc.
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  - we can assume that $k_1 \geq k_2 \geq \cdots \geq k_p$ and their sum is at most $n-1$
- the corresponding graph of all token configurations on $G$ is denoted by **puz($G$; $k_1, \ldots, k_p$)**

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**Theorem** (Brightwell, vdH & Trakultraipruk, 2013)

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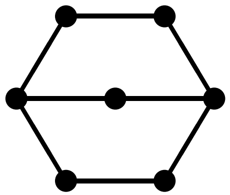

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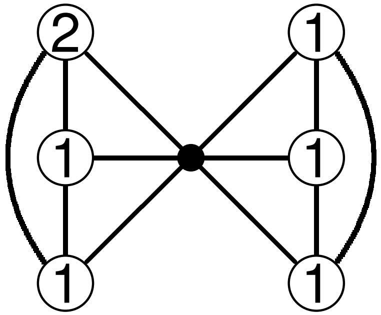
  - $G$ is not connected
  - $G$ is a path and $p \geq 2$
  - $G$ is a cycle, and $p \geq 3$, or $p = 2$ and $k_2 \geq 2$
  - $G$ is a 2-connected, bipartite graph with token set $(1^{(n-1)})$
  - $G$ is the exceptional graph $\Theta_0$ with some "bad" token sets
  - $G$ has connectivity 1, $p \geq 2$ and there is a "separating path preventing tokens from moving between blocks"

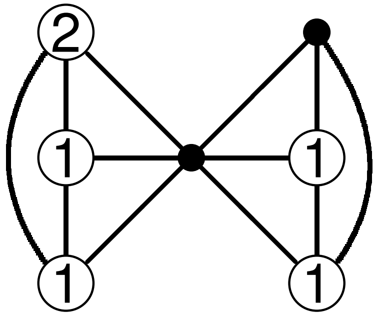
## *Generalised sliding token puzzles*

- "separating paths" in graphs of connectivity one:

bad:




good:



# *Generalised sliding token puzzles*

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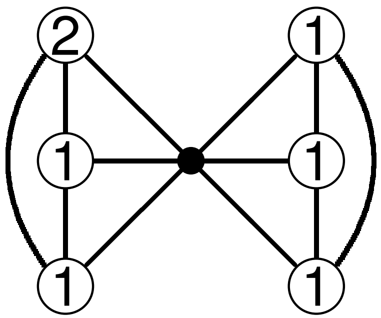

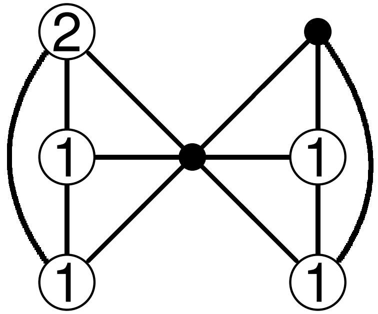

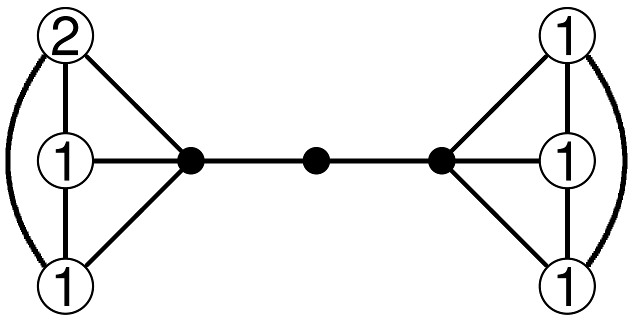

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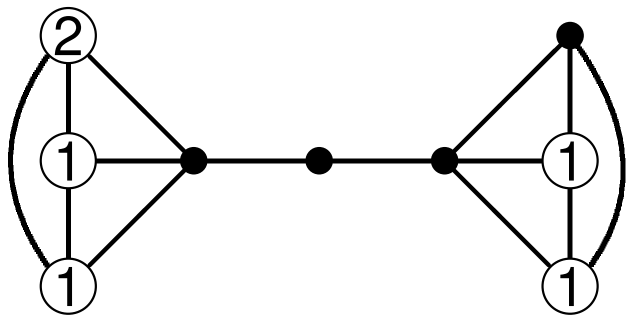


good

# The Structure of $T(\theta_0; (2, 1, 1, 1, 1))$

The following are the three groups of standard token configurations in the labelled token graph $T(\theta_0; (2, 1, 1, 1, 1))$.

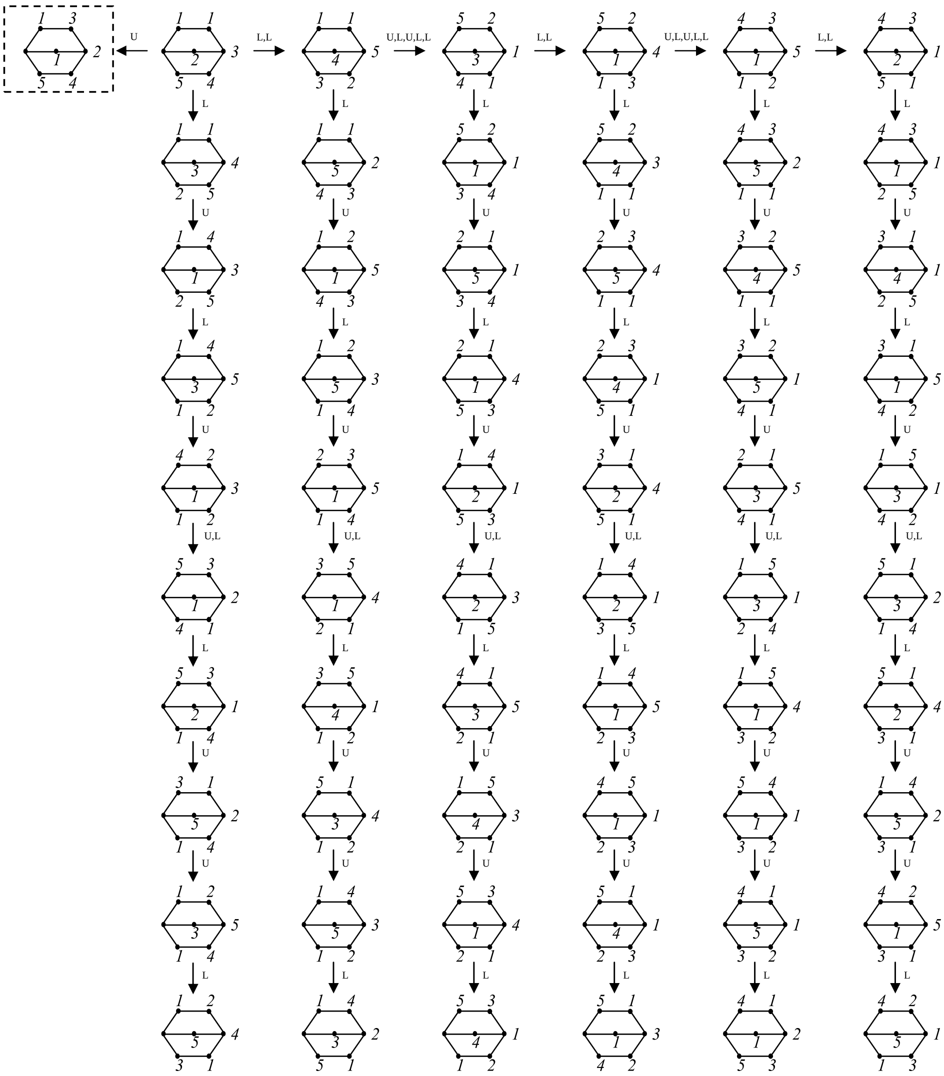


Figure A.8: Part 1 of Group $B_1$ in $T(\theta_0; (2, 1, 1, 1, 1))$

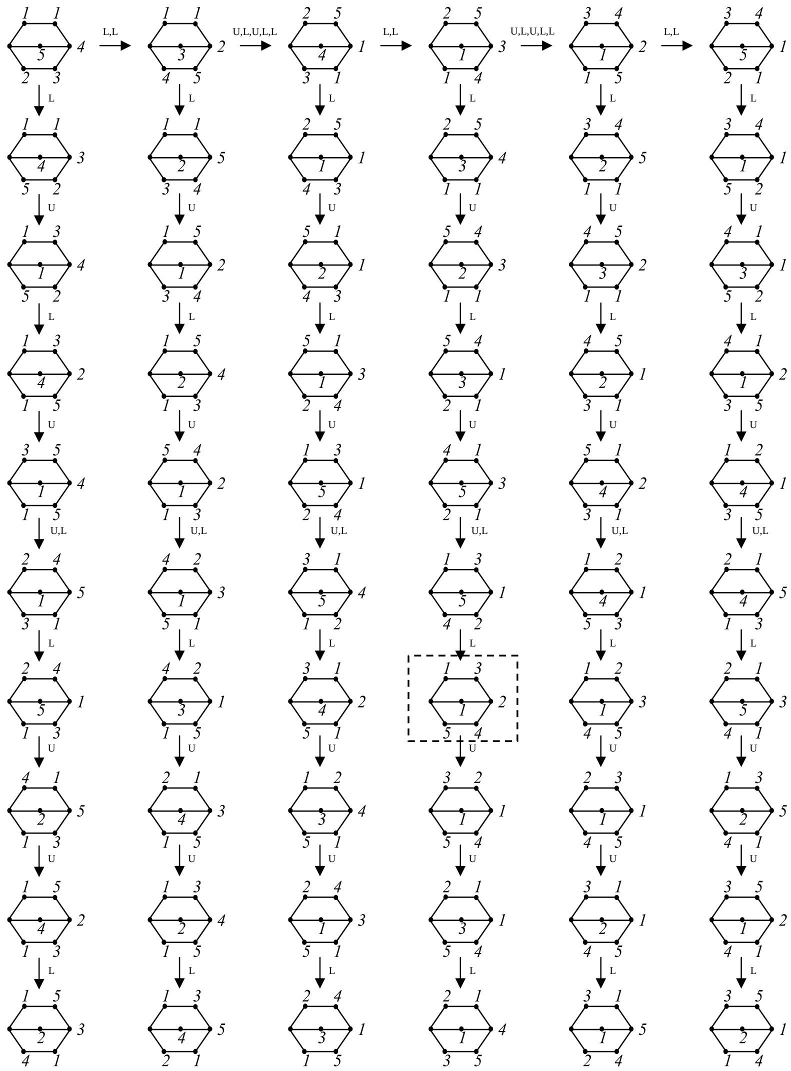


Figure A.9: Part 2 of Group $B_1$ in $T(\theta_0; (2,1,1,1,1))$

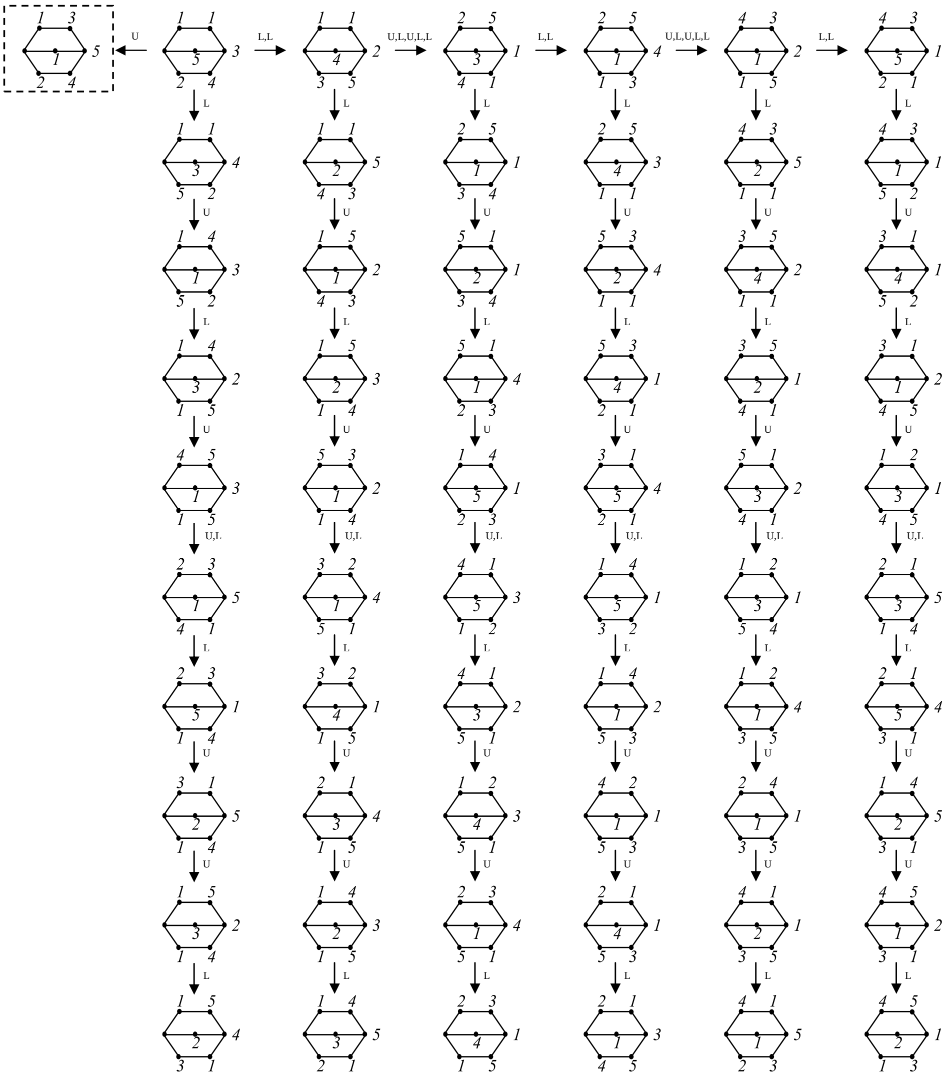


Figure A.10: Part 1 of Group $B_2$ in $T(\theta_0;(2,1,1,1,1))$

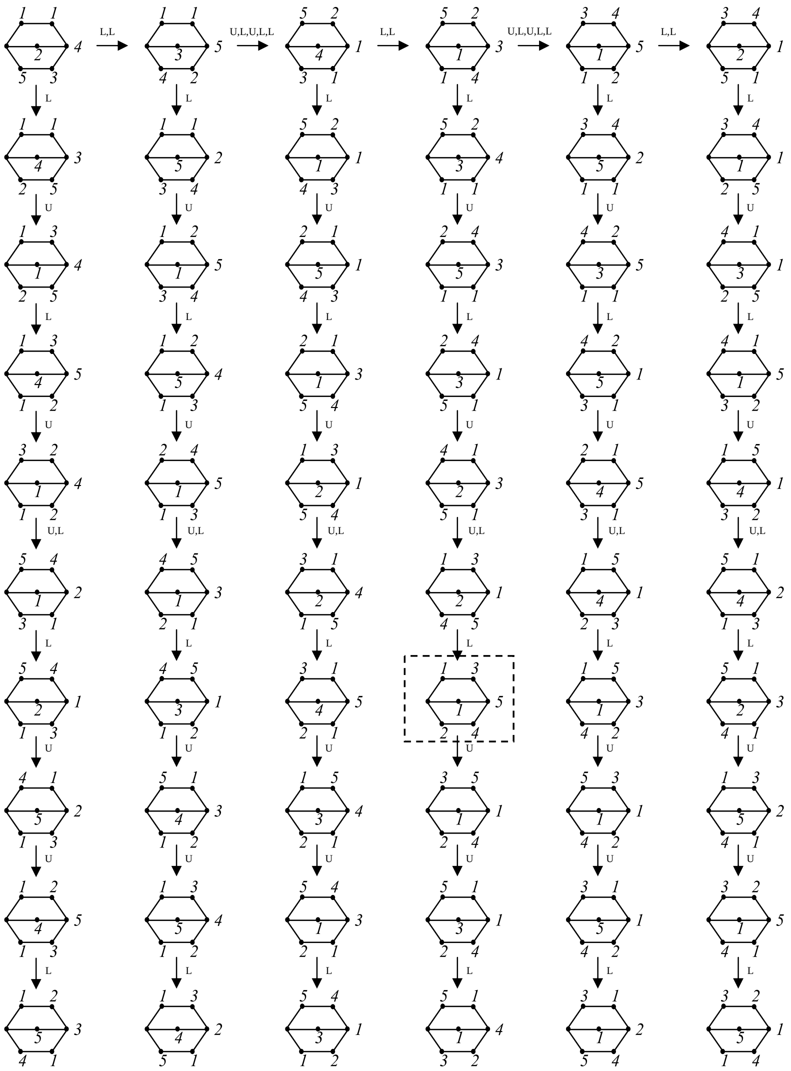


FIGURE A.11: Part 2 of Group $B_2$ in $T(\theta_0; (2,1,1,1,1))$

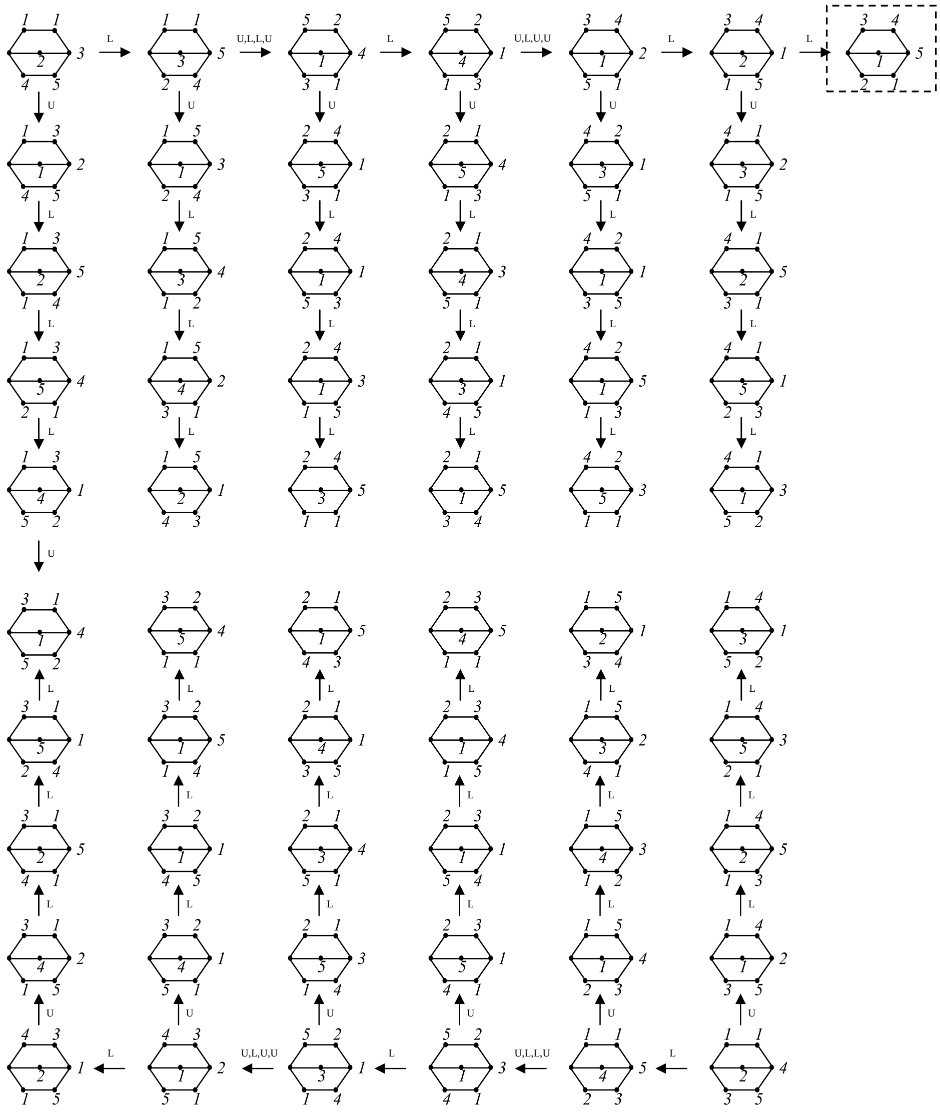


FIGURE A.12: Part 1 of Group $B_3$ in $T(\theta_0; (2,1,1,1,1))$

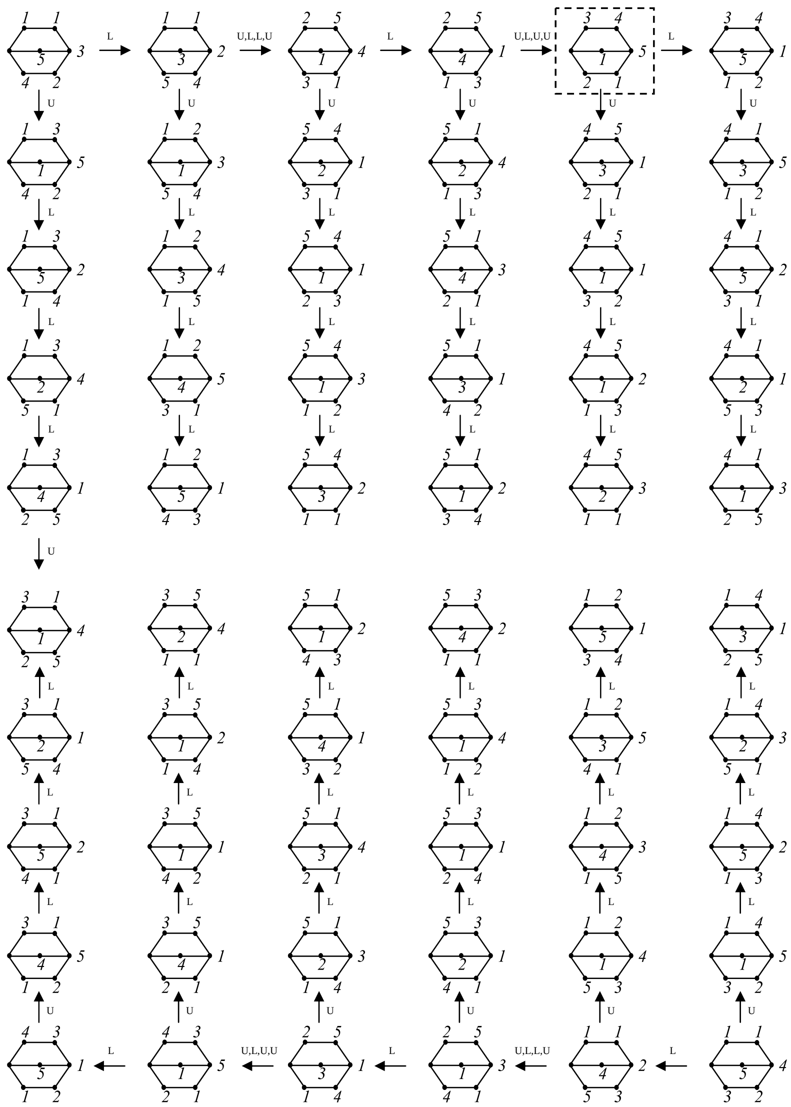


Figure A.13: Part 2 of Group $B_3$ in $T(\theta_0; (2,1,1,1,1))$

## *Generalised sliding token puzzles*

- we can also characterise:
  - given a graph $G$, token set $(k_1, \ldots, k_p)$, and two token configurations on $G$,
  - are the two configurations in the same component of $\mathrm{puz}(G; k_1, \ldots, k_p)$?

configuration $\alpha$, let $\alpha_i$ be a token configuration obtained from $\alpha$ by moving some tokens (if necessary) to make all the vertices on $P_i$ unoccupied.

Let $G$ be a connected graph with connectivity 1, $n(G) - (k_1 + k_2 + \cdots + k_p) = 1$, and $B$ a block in $G$. Then $B$ contains at least one cut-vertex of $G$. Let $v_B$ be one of these cut-vertices. Given a token configuration $\alpha$, let $\alpha_{v_B}$ be a token configuration obtained from $\alpha$ by moving some tokens (if necessary) to make $v_B$ unoccupied.

We denote the multiset of all the tokens used in a token configuration $\alpha$ by $\tau(\alpha)$. For example, if $\alpha$ is any of the token configurations in Figure 2.4, then $\tau(\alpha) = \{1, 1, 2, 2, 3, 3\} = (2, 2, 2)$.

**Theorem 2.3**

*Let $G$ be a connected graph with $n(G) \geq 3$, $k_1 \geq k_2 \geq \cdots \geq k_p$ positive integers for some integer $p \geq 2$, and $k_1 + k_2 + \cdots + k_p \leq n(G) - 1$. Then two token configurations $\alpha$ and $\beta$ are in the same component of $T(G; (k_1, k_2, \ldots, k_p))$ if and only if at least one of the following conditions holds:*

1. *$T(G; (k_1, k_2, \ldots, k_p))$ is connected;*

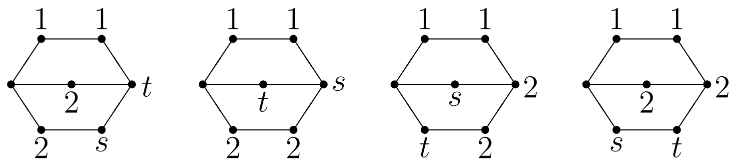
2. *$G$ is a path, and the orders of tokens on $G$ of $\alpha$ and $\beta$ are the same;*

3. *$G$ is a cycle, and the cyclic orders of tokens on $G$ of $\alpha$ and $\beta$ are the same;*

4. *$G$ is the graph $\theta_0$, and*

    *(a) $(k_1, k_2, \ldots, k_p) = (2, 2, 2)$ or $(2, 2, 1, 1)$, and for any (1,1)-standard token configurations $\alpha'$ and $\beta'$ which can be reached from $\alpha$ and $\beta$, respectively, we have that $\alpha'$ and $\beta'$ are in the same group from the following two groups:*

***Group*** $a_1$*: (1,1)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is* $(2,2,s,t)$*, where* $s,t \in \{3,4\}$*. I.e., token configurations which have the following forms:*




***Group*** $a_2$*: (1,1)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is* $(2,s,2,t)$*, where* $s,t \in \{3,4\}$*,*

*(b)* $(k_1,k_2,\ldots,k_p) = (2,1,1,1,1)$*, and for any (1,1)-standard token configurations* $\alpha'$ *and* $\beta'$ *which can be reached from* $\alpha$ *and* $\beta$*, respectively, we have* $\alpha'$ *and* $\beta'$ *are in the same group from the following three groups:*

***Group*** $b_1$*: (1,1)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is* $(2,3,4,5)$ *or* $(2,5,4,3)$*;*

***Group*** $b_2$*: (1,1)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is* $(2,4,3,5)$ *or* $(2,5,3,4)$*;*

***Group*** $b_3$*: (1,1)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is* $(2,3,5,4)$ *or* $(2,4,5,3)$*;*

*(c)* $(k_1,k_2,\ldots,k_p) = (1,1,1,1,1,1)$*, and for any (1,6)-standard token configurations* $\alpha'$ *and* $\beta'$ *which can be reached from* $\alpha$ *and* $\beta$*, respectively, we have* $\alpha'$ *and* $\beta'$ *are in the same group from the following six groups:*

***Group*** $c_1$*: (1,6)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is* $(2,3,4,5)$*;*

***Group*** $c_2$*: (1,6)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is* $(2,5,4,3)$*;*

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***Group*** $c_4$*: (1,6)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is* $(2,5,3,4)$*;*

***Group*** $c_5$*: (1,6)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is* $(2,3,5,4)$*;*

***Group*** $c_6$*: (1,6)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is* $(2,4,5,3)$*.*

5. *$G$ is a 2-connected bipartite graph other than a cycle, there are $n(G)-1$ different tokens, and one of the following holds:*

   (a) *$\alpha$ and $\beta$ have their unoccupied vertices at even distance in $G$, and $\alpha\beta^{-1}$ is an even permutation;*

   (b) *$\alpha$ and $\beta$ have their unoccupied vertices at odd distance in $G$, and $\alpha\beta^{-1}$ is an odd permutation.*

6. *$G$ is a connected graph with connectivity 1 other than a path, $n(G)-(k_1+k_2+\cdots+k_p)=l\geq 2$, $P_1,P_2,\ldots,P_m$ are all the separating paths of size $l$ in $G$, and $\tau(\alpha_i|_{G_{i,1}})=\tau(\beta_i|_{G_{i,1}})$ and $\tau(\alpha_i|_{G_{i,2}})=\tau(\beta_i|_{G_{i,2}})$ for all $i=1,2,\ldots,m$.*

7. *$G$ is a connected graph with connectivity 1 other than a path, $n(G)-(k_1+k_2+\cdots+k_p)=1$, for each block $B$ in $G$, $\tau(\alpha_{v_B}|_B)=\tau(\beta_{v_B}|_B)$, and at least one of the following conditions holds:*

   (a) *$T(B;\tau(\alpha_{v_B}|_B))$ is connected;*

   (b) *$B$ is a cycle, and the cyclic orders of tokens of $\alpha_{v_B}|_B$ and $\beta_{v_B}|_B$ are the same;*

   (c) *$B$ is the graph $\theta_0$, and $\alpha_{v_B}|_B$ and $\beta_{v_B}|_B$ satisfy 4(a), 4(b), or 4(c) above;*

   (d) *$B$ is a 2-connected bipartite graph other than a cycle, there are $n(B)-1$ different tokens in $\alpha_{v_B}|_B$ and $\beta_{v_B}|_B$, and $\alpha_{v_B}|_B\cdot(\beta_{v_B}|_B)^{-1}$ is an even permutation.*

## *Generalised sliding token puzzles*

- we can also characterise:
  - given a graph $G$, token set $(k_1, \ldots, k_p)$, and two token configurations on $G$,
  - are the two configurations in the same component of $\mathrm{puz}(G; k_1, \ldots, k_p)$?
- so recognising connectivity properties of $\mathrm{puz}(G; k_1, \ldots, k_p)$ is easy
- can we say something about the number of steps we would need?

## *The length of sliding token paths*

- **SHORTEST-A-TO-B-TOKEN-MOVES**

*Input*: a graph $G$, a token set $(k_1, \ldots, k_p)$,
two token configurations $A$ and $B$ on $G$,
and a positive integer $N$

*Question*: can we go from $A$ to $B$ in at most $N$ steps?

## *The length of sliding token paths*

**Theorem** (Goldreich, 1984-2011)

- restricted to the case that there are $n-1$ different tokens, SHORTEST-A-TO-B-TOKEN-MOVES is **NP-complete**

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**Theorem** (vdH & Trakultraipruk, 2013; probably others earlier)

- restricted to the case that all tokens are the same, SHORTEST-A-TO-B-TOKEN-MOVES is in **P**

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**Theorem** (vdH & Trakultraipruk, 2013)

- restricted to the case that there is just one special token and all others are the same:

  SHORTEST-A-TO-B-TOKEN-MOVES is already **NP-complete**

## *Robot motion*

- the proof of that last result uses ideas of the proof of

**Theorem** (Papadimitriou, Raghavan, Sudan & Tamaki, 1994)

- SHORTEST-ROBOT-MOTION-WITH-ONE-ROBOT is **NP-complete**

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- **Robot Motion** problems on graphs are sliding token problems,
  - with some special tokens (the robots)
    - that have to end in specified positions
  - all other tokens are just obstacles
    - and it is not important where those are at the end