Dirichlet property and dynamical system

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• Let K be a number field, that is, a finite extension field over $\mathbb{Q}$. Let O_K be the ring of integers in K. Let $K(\mathbb{C})$ be the set of all embeddings K into $\mathbb{C}$. For $\sigma \in K(\mathbb{C})$, $\bar{\sigma}$ is defined to be $\bar{\sigma}(x) = \overline{\sigma(x)}$ $(x \in K)$, where $^{-}$ is the complex conjugation. Let us consider the following equivalence relation $\sim$ on $K(\mathbb{C})$:

$$\sigma \sim \tau \quad \Longleftrightarrow \quad \sigma = \tau \text{ or } \bar{\tau}.$$

We set $s = \#(K(\mathbb{C})/\sim) - 1$.

Theorem (Dirichlet unit theorem)

The group $O_K^\times$ consisting of the units in O_K is a finitely generated abelian group of rank s.

Proof. Let us consider a map $L : O_K^\times \to \mathbb{R}^{K(\mathbb{C})}$ given by $L(x)_\sigma = \log|\sigma(x)|$. For a compact subset B in $\mathbb{R}^{K(\mathbb{C})}$, the set $\{x \in O_K^\times \mid L(x) \in B\}$ is finite ($\because$ every coefficients of $\prod_{\sigma \in K(\mathbb{C})}(T - \sigma(x))$ is bounded and belongs to $\mathbb{Z}$). Thus we can easily check that $O_K^\times$ is finitely generated.
Obviously the image of L is contained in the subspace

$$\Xi_K^0 = \left\{ (\xi_\sigma) \in \mathbb{R}^{K(\mathbb{C})} \;\middle|\; \sum_\sigma \xi_\sigma = 0,\ \xi_\sigma = \xi_{\bar{\sigma}}\ (\forall \sigma) \right\}$$

of dimension s. Thus the crucial point of the proof of the Dirichlet unit theorem is to show that, for any $\xi \in \Xi_K^0$, there are $a_1, \ldots, a_r \in \mathbb{R}$ and $x_1, \ldots, x_r \in O_K^\times$ such that $a_1 L(x_1) + \cdots + a_r L(x_r) = \xi$.

We set

$$\begin{cases} \Xi_K := \left\{ (\xi_\sigma) \in \mathbb{R}^{K(\mathbb{C})} \mid \xi_\sigma = \xi_{\bar{\sigma}} \ (\forall \sigma) \right\}, \\ \widehat{\mathrm{Div}}(O_K) := \mathrm{Div}(O_K) \times \Xi_K, \\ \widehat{\mathrm{Div}}(O_K)_{\mathbb{R}} := (\mathrm{Div}(O_K) \otimes_{\mathbb{Z}} \mathbb{R}) \times \Xi_K, \\ K_{\mathbb{R}}^{\times} := K^{\times} \otimes_{\mathbb{Z}} \mathbb{R}, \\ M_K^f := \text{the set of all maximal ideals of } O_K \text{ (finite places)}, \\ M_K^{\infty} = K(\mathbb{C}) \text{ (infinite places)}, \\ M_K = M_K^f \cup M_K^{\infty}. \end{cases}$$

For $\overline{D} = \left(\sum_{\mathfrak{p}\in M_K^f} a_{\mathfrak{p}}[\mathfrak{p}], (\xi_\sigma)_{\sigma\in M_K^\infty}\right) \in \widehat{\mathrm{Div}}(O_K)_{\mathbb{R}}$, we define $\overline{D} \geq 0$ and $\widehat{\deg}(\overline{D})$ to be

$$\overline{D} \geq 0 \quad \overset{\text{def}}{\Longleftrightarrow} \quad a_{\mathfrak{p}} \geq 0 \ (\forall \mathfrak{p}),\ \ \xi_\sigma \geq 0 \ (\forall \sigma)$$

and

$$\widehat{\deg}(\overline{D}) := \sum_{\mathfrak{p}\in M_K^f} a_{\mathfrak{p}} \log \#(O_K/\mathfrak{p}) + \frac{1}{2} \sum_{\sigma\in M_K^\infty} \xi_\sigma .$$

For $x \in K^\times$, we set

$$\widehat{(x)} := \left((x), -\log|x|^2\right),$$

where $\left(-\log|x|^2\right)_\sigma := -\log|\sigma(x)|^2$. Note that $\widehat{\deg}(\widehat{(x)}) = 0$ by the product formula. Moreover, this gives a homomorphism $\widehat{(\cdot)} : K^\times \to \widehat{\mathrm{Div}}(X)$, which naturally extends to the homomorphism

$$\widehat{(\cdot)} : K^\times_{\mathbb{R}} \to \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$$

given by $\widehat{(x_1^{a_1} \cdots x_r^{a_r})} = a_1\widehat{(x_1)} + \cdots + a_r\widehat{(x_r)}$ $(a_1, \ldots, a_r \in \mathbb{R})$.

Theorem (Arakelov geometric version of the Dirichlet unit theorem)

If $\widehat{\deg}(\overline{D}) \geq 0$ for $\overline{D} \in \widehat{\mathrm{Div}}(O_K)_{\mathbb{R}}$ (i.e. $\overline{D}$ is pseudo-effective), then there is $\phi \in K_{\mathbb{R}}^{\times}$ such that $\overline{D} + \widehat{(\phi)} \geq 0$.

Indeed, the above theorem implies the Dirichlet unit theorem. For $\xi \in \Xi_K^0$, we set $\overline{D}_\xi = (0, \xi)$. By the above theorem, there is $\phi \in K_{\mathbb{R}}^{\times}$ such that $\widehat{(\phi)} + \overline{D}_\xi \geq 0$. As $\widehat{(\phi)} + \overline{D}_\xi \geq 0$ and $\widehat{\deg}(\widehat{(\phi)} + \overline{D}_\xi) = 0$, we have $\widehat{(\phi)} + \overline{D}_\xi = (0, 0)$. Moreover, we can find $a_1, \ldots, a_r \in \mathbb{R}$ and $x_1, \ldots, x_r \in K^{\times}$ such that $\phi = x_1^{a_1} \cdots x_r^{a_r}$ and $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$.

We set $(x_j) = \sum_{k=1}^{l} \alpha_{jk}\mathfrak{p}_k$, where $\alpha_{jk} \in \mathbb{Z}$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ are distinct maximal ideals in O_K. Then

$$0 = a_1(x_1) + \cdots + a_r(x_r) = \left(\sum_{j=1}^{r} a_j\alpha_{j1}\right)\mathfrak{p}_1 + \cdots + \left(\sum_{j=1}^{r} a_j\alpha_{jl}\right)\mathfrak{p}_l.$$

Thus $\alpha_{jk} = 0$ for all j, k, which means that $x_1, \ldots, x_r \in O_K^\times$. Further, $\xi_\sigma + \sum_{i=1}^{r} a_i(-\log|\sigma(x_i)|^2) = 0$ for all σ, which implies that $\xi = 2a_1 L(x_1) + \cdots + 2a_r L(x_r)$. □

Remark

The analogue of the above theorem on a smooth projective curve does not hold in general. Indeed, let C be a smooth projective curve of genus $g \geq 1$ over $\mathbb{C}$ and D a divisor of degree 0 on C such that the order of $\mathscr{O}_C(D)$ in $\mathrm{Pic}^0(C)$ is infinite. Then there is no $\phi \in \mathrm{Rat}(C)^\times \otimes \mathbb{R}$ with $D + (\phi) \geq 0$.

• Let X be a d-dimensional, projective, smooth and geometrically integral variety over K. Let D be an $\mathbb{R}$-Cartier divisor on X, that is,

$$D \in \mathrm{Div}(X) \otimes \mathbb{R}.$$

• For $\sigma \in M_K^\infty = K(\mathbb{C})$, we set $K_\sigma := K \otimes_K^\sigma \mathbb{C}$ with respect to σ. Note that K_σ is naturally isomorphic to $\mathbb{C}$ via $a \otimes^\sigma z \mapsto \sigma(a)z$. Moreover, we set $X_\sigma := X \times_K K_\sigma$. Note that $X_\sigma = X \times_K^\sigma \mathrm{Spec}(\mathbb{C})$ with respect to $\sigma : K \hookrightarrow \mathbb{C}$. We set $X_\sigma^{an} := X_\sigma(\mathbb{C})$.

• Let $g : X_\sigma^{an} \setminus \mathrm{Supp}(D)_\sigma^{an} \to \mathbb{R}$ be a continuous function. We say g is a D-Green function of C^0-type on X_σ^{an} if there are an affine open covering $X = \bigcup U_i$ of X and a local equation f_i of D on U_i such that $g + \log|f_i|_\sigma^2$ extends to a continuous function on $(U_i)_\sigma^{an}$ for all i.

- For $\mathfrak{p} \in M_K^f$, the valuation $v_\mathfrak{p}$ of K at $\mathfrak{p}$ is given by

$$v_\mathfrak{p}(f) = \#(O_K/\mathfrak{p})^{-\operatorname{ord}_\mathfrak{p}(f)}.$$

Let $K_\mathfrak{p}$ be the completion of K with respect to $v_\mathfrak{p}$. We set

$$X_\mathfrak{p} := X \times_{\operatorname{Spec}(K)} \operatorname{Spec}(K_\mathfrak{p}),$$

which is also a projective, smooth and geometrically integral variety over $K_\mathfrak{p}$.

- Let $X_\mathfrak{p}^{an}$ be the analytification of $X_\mathfrak{p}$ in the sense of Berkovich. Let $g : X_\mathfrak{p}^{an} \setminus \operatorname{Supp}(D)_\mathfrak{p}^{an} \to \mathbb{R}$ be a continuous function. We say g is a D-Green function of C^0-type on $X_\mathfrak{p}^{an}$ if there are an affine open covering $X = \bigcup U_i$ of X and a local equation f_i of D on U_i such that $g + \log|f_i|_\mathfrak{p}^2$ extends to a continuous function on $(U_i)_\mathfrak{p}^{an}$ for all i.

• Let $\hat{O}_{K,\mathfrak{p}}$ be the completion of O_K at $\mathfrak{p}$. Let $\mathscr{X}_\mathfrak{p}$ be a model of $X_\mathfrak{p}$ over $\mathrm{Spec}(\hat{O}_{K,\mathfrak{p}})$, that is, $\mathscr{X}_\mathfrak{p}$ is a projective and flat integral scheme over $\mathrm{Spec}(\hat{O}_{K,\mathfrak{p}})$ such that the generic fiber of $\mathscr{X}_\mathfrak{p} \to \mathrm{Spec}(\hat{O}_{K,\mathfrak{p}})$ is $X_\mathfrak{p}$. Let $(\mathscr{X}_\mathfrak{p})_\circ$ be the central fiber of $\mathscr{X}_\mathfrak{p} \to \mathrm{Spec}(\hat{O}_{K,\mathfrak{p}})$. By using the valuative criterion, we have the natural map

$$r : X_\mathfrak{p}^{an} \to (\mathscr{X}_\mathfrak{p})_\circ,$$

which is called the reduction map.

• We assume that there is an $\mathbb{R}$-Cartier divisors $\mathscr{D}_{\mathfrak{p}}$ on $\mathscr{X}_{\mathfrak{p}}$ such that

$$\mathscr{D}_{\mathfrak{p}} \cap X_{\mathfrak{p}} = D_{\mathfrak{p}} = (\text{the pullback of } D \text{ via } X_{\mathfrak{p}} \to X).$$

The pair $(\mathscr{X}_{\mathfrak{p}}, \mathscr{D}_{\mathfrak{p}})$ is called a model of $(X_{\mathfrak{p}}, D_{\mathfrak{p}})$ over $\mathrm{Spec}(\hat{O}_{K,\mathfrak{p}})$. For $x \in X_{\mathfrak{p}}^{an} \setminus \mathrm{Supp}(D)_{\mathfrak{p}}^{an}$, let f be a local equations of $\mathscr{D}_{\mathfrak{p}}$ at $\xi = r(x)$. We define $g_{(\mathscr{X}_{\mathfrak{p}}, \mathscr{D}_{\mathfrak{p}})}(x)$ to be

$$g_{(\mathscr{X}_{\mathfrak{p}}, \mathscr{D}_{\mathfrak{p}})}(x) := -\log |f(x)|^2.$$

It is easy to see that $g_{(\mathscr{X}_{\mathfrak{p}}, \mathscr{D}_{\mathfrak{p}})}$ is a D-Green function of C^0-type on $X_{\mathfrak{p}}^{an}$. We call it the Green function induced by the model $(\mathscr{X}_{\mathfrak{p}}, \mathscr{D}_{\mathfrak{p}})$.

• A pair $\overline{D} = (D, g)$ of an $\mathbb{R}$-Cartier divisor D on X and a collection of Green functions

$$g = \{g_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K} \cup \{g_\sigma\}_{\sigma \in M_K^\infty}$$

is called an adelic arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X if the following conditions are satisfied:

1. For each $\mathfrak{p} \in M_K$, $g_{\mathfrak{p}}$ is a D-Green function of C^0-type on $X_{\mathfrak{p}}^{an}$. In addition, there are a non-empty open set U of $\operatorname{Spec}(O_K)$, a model $\mathscr{X}_U$ of X over U and an $\mathbb{R}$-Cartier divisor $\mathscr{D}_U$ on $\mathscr{X}_U$ such that $\mathscr{D}_U \cap X = D$ and $g_{\mathfrak{p}}$ is a D-Green function induced by the model $(\mathscr{X}_U, \mathscr{D}_U)$ for all $\mathfrak{p} \in U \cap M_K$.
2. For each $\sigma \in M_K^\infty$, g_σ is a D-Green function of C^0-type on X_σ^{an}. Moreover, the function $\{g_\sigma\}_{\sigma \in M_K^\infty}$ is an F_∞-invariant, that is, for all $\sigma \in M_K^\infty$, $g_{\bar{\sigma}} \circ F_\infty = g_\sigma$, where $F_\infty : X_\sigma \to X_{\bar{\sigma}}$ is an anti-holomorphic map induced by the complex conjugation.

- For simplicity, a collection of Green functions

$$g = \{g_{\mathfrak{p}}\}_{\mathfrak{p} \in M_K^f} \cup \{g_\sigma\}_{\sigma \in M_K^\infty}$$

is often expressed by the following symbol:

$$g = \sum_{\mathfrak{p} \in M_K^f} g_{\mathfrak{p}}[\mathfrak{p}] + \sum_{\sigma \in M_K^\infty} g_\sigma[\sigma].$$

We denote the space of all adelic arithmetic $\mathbb{R}$-Cartier divisors of C^0-type on X by $\widehat{\mathrm{Div}}^a_{C^0}(X)_{\mathbb{R}}$.

- Let $\mathrm{Rat}(X)^{\times}_{\mathbb{R}} := \mathrm{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$. For $\varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$, we set

$$\widehat{(\varphi)} := \left((\varphi), \sum_{\mathfrak{p} \in M_K} (-\log |\varphi|^2_{\mathfrak{p}})[\mathfrak{p}] + \sum_{\sigma \in M_K^{\infty}} (-\log |\varphi|^2_{\sigma})[\sigma] \right).$$

Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on X.

$$\overline{D} \geq 0 \overset{\text{def}}{\iff} D \geq 0 \text{ and } g_v \geq 0 \text{ for all } v \in M_K.$$

We set

$$\hat{H}^0(X, \overline{D}) := \{\phi \in \mathrm{Rat}(X)^{\times} \mid \overline{D} + \widehat{(\phi)} \geq 0\} \cup \{0\}$$

and

$$\widehat{\mathrm{vol}}(\overline{D}) := \limsup_{n \to \infty} \frac{\log \# \hat{H}^0(X, n\overline{D})}{n^{d+1}/(d+1)!}.$$

- $\overline{D}$ is big $\overset{\text{def}}{\iff} \widehat{\mathrm{vol}}(\overline{D}) > 0$.
- $\overline{D}$ is pseudo-effective $\overset{\text{def}}{\iff} \overline{D} + \overline{A}$ is big for all big arithmetic $\mathbb{R}$-divisors $\overline{A}$ of C^0-type.

In the case where $d = 0$, we have the following:

- $\overline{D}$ is big $\iff \widehat{\deg}(\overline{D}) > 0$.
- $\overline{D}$ is pseudo-effective $\iff \widehat{\deg}(\overline{D}) \geq 0$.

Definition

We say $\overline{D}$ has the Dirichlet property if $\overline{D} + \widehat{(\varphi)} \geq 0$ for some $\varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$.

Fundamental question

Are the following conditions (1) and (2) equivalent ?

1. $\overline{D}$ *is pseudo-effective.*
2. $\overline{D}$ *has the Dirichlet property.*

Obviously (2) implies (1).

- If $\overline{D} + \widehat{(\varphi)} \geq 0$, then, for $v \in M_K$, $x \mapsto (|\varphi|_v \exp(-g_v/2))(x)$ is continuous. We denote $|\varphi|_v \exp(-g_v/2)$ by $|\varphi|_{g_v}$. Moreover, $\|\varphi\|_{g_v} := \sup_{x \in X_v^{an}} \{|\varphi|_{g_v}(x)\}$

Theorem

In the following cases, $\overline{D}$ has the Dirichlet property.

1. *(the Dirichlet unit theorem)* $X = \mathrm{Spec}(K)$ *and* $\overline{D}$ *is pseudo-effective.*
2. *(Moriwaki)* $\overline{D}$ *is pseudo-effective and* D *is numerically trivial.*
3. *(Burgos, Moriwaki, Philippon and Sombra)* X *is a toric variety,* $\overline{D}$ *is pseudo-effective and* $\overline{D}$ *is of toric type (i.e.* D *is a toric divisor and* g *is invariant under the* $S^{\dim X}$*-action).*

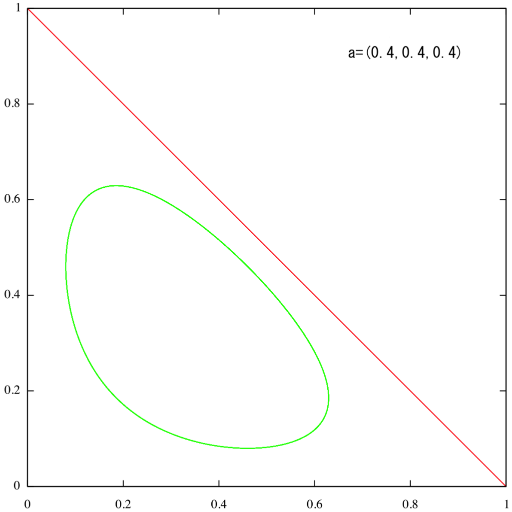
• Let $\mathbb{P}^2_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[T_0, T_1, T_2])$, $D = \{T_0 = 0\}$ and $z_i = T_i/T_0$ for $i = 1, 2$: Let us fix a sequence $\boldsymbol{a} = (a_0, a_1, a_2)$ of positive numbers. We define a D-Green function $g_{\boldsymbol{a}}$ on $\mathbb{P}^2(\mathbb{C})$ and an arithmetic divisor $\overline{D}_{\boldsymbol{a}}$ on $\mathbb{P}^2_{\mathbb{Z}}$ to be

$$g_{\boldsymbol{a}} := \log(a_0 + a_1|z_1|^2 + a_2|z_2|^2) \quad \text{and} \quad \overline{D}_{\boldsymbol{a}} := (D, g_{\boldsymbol{a}}).$$

Let $\vartheta_{\boldsymbol{a}} : \mathbb{R}^3_{\geq 0} \to \mathbb{R}$ be a function given by

$$\vartheta_{\boldsymbol{a}}(x_0, x_1, x_2) := \frac{1}{2}(-x_0 \log x_0 - x_1 \log x_1 - x_2 \log x_2 + x_0 \log a_0 + x_1 \log a_1 + x_2 \log a_2),$$

and let $\Theta_{\boldsymbol{a}} := \{(x_1, x_2) \in \Delta_2 \mid \vartheta_{\boldsymbol{a}}(1 - x_1 - x_2, x_1, x_2) \geq 0\}$, where $\Delta_2 := \left\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \mid x_1 + x_2 \leq 1\right\}$ (Newton-Okounkov body of $\mathscr{O}(D)$ at $(1 : 0 : 0)$).



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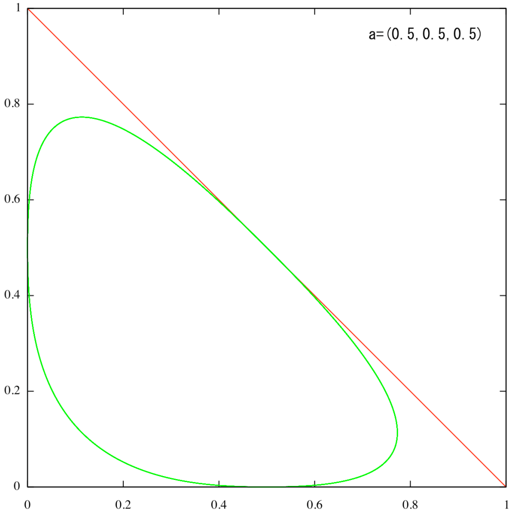
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1



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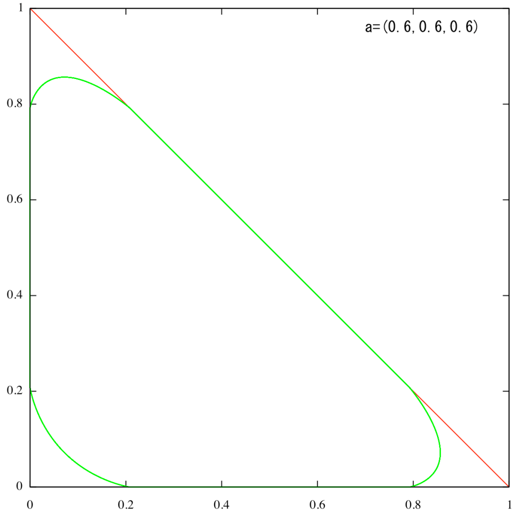
0.2
0.4
0.6
0.8
1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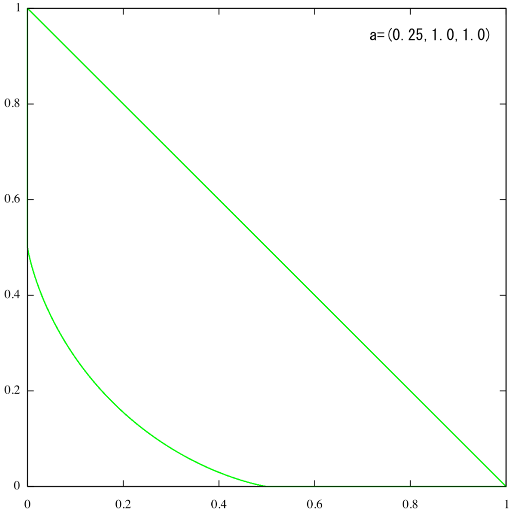
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1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0.2
0.4
0.6
0.8
1

• We set $H_i = \{T_i = 0\}$ for $i = 0, 1, 2$. The we have the following (1) – (4):

(1) For $(x_1, x_2) \in \Delta_2$,

$$\begin{cases} D + (z_1^{x_1} z_2^{x_2}) = (1 - x_1 - x_2)H_0 + x_1 H_1 + x_2 H_2, \\ g_{\boldsymbol{a}} + (-\log |z_1^{x_1} z_2^{x_2}|^2) \geq 2\vartheta_{\boldsymbol{a}}(1 - x_1 - x_2, x_1, x_2). \end{cases}$$

(2) $\widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) = 3! \int_{\Theta_{\boldsymbol{a}}} \vartheta_{\boldsymbol{a}}(1 - x_1 - x_2, x_1, x_2) dx_1 dx_2.$

(3) $\overline{D}_{\boldsymbol{a}}$ is big $\iff$ $a_0 + a_1 + a_2 > 1$.

(4) $\overline{D}_{\boldsymbol{a}}$ is pseudo-effective $\iff$ $a_0 + a_1 + a_2 \geq 1$.

Thus, if $\overline{D}_{\boldsymbol{a}}$ is pseudo-effective, then the Dirichlet property holds. Indeed, if $a_0 + a_1 + a_2 = 1$, then $\overline{D}_{\boldsymbol{a}} + \widehat{(z_1^{a_1} z_2^{a_2})} \geq 0$ because $\vartheta_{\boldsymbol{a}}(a_0, a_1, a_2) = 0$.

• Let $x \in X(\overline{K})$ and $v \in M_K$. We denote the residue field of the image of $x : \mathrm{Spec}(\overline{K}) \to X$ by $K(x)$. Let $\{\phi_1, \ldots, \phi_n\}$ be the set of all K_v-algebra homomorphisms $K(x) \otimes_K K_v \to \overline{K}_v$. For each $i = 1, \ldots, n$, let w_i be the $\overline{K}_v$-valued point of X_v given by the composition of morphisms

$$\mathrm{Spec}(\overline{K}_v) \xrightarrow{\phi_i^a} \mathrm{Spec}(K(x) \otimes_K K_v) \xrightarrow{x \times \mathrm{id}_{K_v}} X_v.$$

We denote $\{w_1, \ldots, w_n\}$ by $O_v(x)$.

• For $w \in X_v(\overline{K}_v)$, we define $w^{an} \in X_v^{an}$ to be

$$w^{an} := \begin{cases} w & \text{if } v = \sigma \in K(\mathbb{C}), \\ \text{the unique extension of } v_{\mathfrak{p}} \text{ of } K_{\mathfrak{p}} & \text{if } v = \mathfrak{p} \in M_K^f. \end{cases}$$

- Let S be a subset of $X(\overline{K})$ and $v \in M_K$. We define the essential support $\mathrm{Supp}_{ees}(S)_v^{an}$ of S at v to be

$$\mathrm{Supp}_{ess}(S)_v^{an} := \bigcap_{Y \subsetneq X} \overline{\bigcup_{x \in S \setminus Y(\overline{K})} \{w^{an} \mid w \in O_v(x)\}},$$

where Y runs over all proper closed subscheme of X. It is not difficult to see that if we set $S_v = \bigcup_{x \in S}\{w^{an} \mid w \in O_v(x)\}$, then

$$\mathrm{Supp}_{ees}(S)_v^{an} = \bigcap_{Z \subsetneq X_v} \overline{\{w^{an} \mid w \in S_v \setminus Z(\overline{K}_v)\}}.$$

• For $x \in X(\overline{K})$, if $x \notin \mathrm{Supp}(D)$, we define the height of x with respect to $\overline{D}$ to be

$$h_{\overline{D}}(x) := \frac{1}{[K(x):K]} \sum_{v \in M_K} \sum_{w \in O_v(x)} \frac{1}{2} g_v(w^{an}).$$

In general, replacing $\overline{D}$ by $\overline{D} + \widehat{(\phi)}$ with $x \notin \mathrm{Supp}(D + (\phi))$, we can define it. Moreover, for $\lambda \in \mathbb{R}$,

$$X(\overline{K})^{\overline{D}}_{\leq \lambda} := \{x \in X(\overline{K}) \mid h_{\overline{D}}(x) \leq \lambda\}.$$

Theorem (Nondenseness of nonpositive points)

1. *If* $s \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ *with* $\overline{D} + \widehat{(s)} \geq 0$, *then*

$$\mathrm{Supp}_{ess}(X(\overline{K})^{\overline{D}}_{\leq 0})^{an}_{v} \cap \{x \in X^{an}_{v} \mid |s|_{g_v}(x) < 1\} = \emptyset$$

   for all $v \in M_K$.

2. *We assume that* D *is ample. If* $\overline{D}$ *has the Dirichlet property, then, for all* $v \in M_K$, *there is no closed algebraic curve* C_v *in* X_v *such that* $C^{an}_v \subseteq \mathrm{Supp}_{ess}(X(\overline{K})^{\overline{D}}_{\leq 0})^{an}_{v}$.

Proof. (1) We set $S = X(\overline{K})^{\overline{D}}_{\leq 0}$, $Y = \mathrm{Supp}(D + (s))$ and $g'_v = -\log |s|^2_{g_v}$. Then $g'_v \geq 0$ for all $v \in M_K$.

First let us see that $g'_v(y) = 0$ for all $y \in \bigcup_{x \in S \setminus Y(\overline{K})} \{w^{an} \mid w \in O_v(x)\}$. Indeed, we choose $x \in S \setminus Y(\overline{K})$ and $w \in O_v(x)$ with $y = w^{an}$. Then

$$0 \geq 2[K(x) : K] h_{\overline{D} + \widehat{(s)}}(x) = \sum_{v \in M_K} \sum_{w \in O_v(x)} g'_v(w^{an}),$$

and hence the assertion follows.
Here we assume the contrary, that is,

$$\mathrm{Supp}_{ess}(X(\overline{K})^{\overline{D}}_{\leq 0})^{an}_v \cap \{x \in X^{an}_v \mid |s|_{g_v}(x) < 1\} \neq \emptyset.$$

In particular, there is

$$y_\infty \in \overline{\bigcup_{x \in S \setminus Y(\overline{K})} \{w^{an} \mid w \in O_v(x)\}} \cap \{x \in X^{an}_v \mid |s|_{g_v}(x) < 1\}.$$

Thus we can find a sequence $\{y_m\}$ in X_v^{an} such that $y_m \in \bigcup_{x \in S \setminus Y(\overline{K})} \{w^{an} \mid w \in O_v(x)\}$ and $\lim_{m\to\infty} y_m = y_\infty$. By the previous assertion, $|s|_{g_v}(y_m) = 1$ for all m, so that $|s|_{g_v}(y_\infty) = \lim_{m\to\infty} |s|_{g_v}(y_m) = 1$, which is a contradiction.

(2) We assume that there is a closed algebraic curve C_v in X_v such that $C_v^{an} \subseteq \mathrm{Supp}_{ess}(X(\overline{K})^{\overline{D}}_{\leq 0})_v^{an}$, and hence $C_v^{an} \cap \{x \in X_v^{an} \mid |s|_{g_v}(x) < 1\} = \emptyset$ by (1). On the other hands, $\mathrm{Supp}(D + (s))_v^{an} \subseteq \{x \in X_v^{an} \mid |s|_{g_v}(x) < 1\}$, so that $C_v^{an} \cap \mathrm{Supp}(D + (s))_v^{an} = \emptyset$. As D is ample, $C_v \cap \mathrm{Supp}(D + (s))_v \neq \emptyset$. This is a contradiction. □

• Let $f : X \to X$ be an endomorphism of X. Let D be an $\mathbb{R}$-divisor on X such that $f^*(D) = dD + (\phi)$ for some $d \in \mathbb{R}_{>1}$ and $\phi \in \mathrm{Rat}(X)^\times_{\mathbb{R}}$.

Proposition

There is a unique family of D-Green functions $g = \{g_v\}_{v \in M_K}$ of C^0-type such that $f^*(D, g) = d(D, g) + \widehat{(\phi)}$.

• The pair $\overline{D} = (D, g)$ is called the canonical compactification of D. Note that if D is ample (i.e. there are ample Cartier divisors $D_1, \ldots, D_r$ and $a_1, \ldots, a_r \in \mathbb{R}_{>0}$ with $D = a_1 D_1 + \cdots + a_r D_r$), then $\overline{D}$ is pseudo-effective (more precisely $\overline{D}$ is nef).

- We assume that D is ample. For each $v \in M_K$, we set

$$\begin{cases} \mathrm{Prep}(f) := \left\{x \in X(\overline{K}) \mid f^n(x) = f^m(x) \text{ for some } n > m \geq 0\right\}, \\ \mathrm{Prep}(f_v) := \left\{x \in X_v(\overline{K}_v) \mid f_v^n(x) = f_v^m(x) \text{ for some } n > m \geq 0\right\}. \end{cases}$$

We have the following necessary condition of the Dirichlet property for $\overline{D}$:

Theorem

If $\overline{D}$ has the Dirichlet property, then, for all $v \in M_K$, there is no closed algebraic curve C_v in X_v such that $C_v^{an} \subseteq \mathrm{Supp}_{ess}(\mathrm{Prep}(f))_v^{an}$.

Proof. Note that, for $x \in \mathrm{Prep}(f)$, $h_{\overline{D}}(x) = 0$, so that $\mathrm{Prep}(f) \subseteq X(\overline{K})^{\overline{D}}_{\leq 0}$. Therefore,

$$\mathrm{Supp}_{ess}(\mathrm{Prep}(f))^{an}_{v} \subseteq \mathrm{Supp}_{ess}(X(\overline{K})^{\overline{D}}_{\leq 0})^{an}_{v}.$$

Therefore, the assertion follows from Nondenseness of nonpositive points.

Corollary

If $\overline{D}$ has the Dirichlet property, then $\mathrm{Prep}(f_v)^{an}$ is not dense in X_v^{an} for all $v \in M_K$.

Proof. We assume that $\mathrm{Prep}(f_v)^{an}$ is dense in X_v^{an}. Note that $\mathrm{Prep}(f_v) = \bigcup_{x \in \mathrm{Prep}(f)} O_v(x)$. Thus $\mathrm{Supp}_{ess}(\mathrm{Prep}(f))_v^{an} = X_v^{an}$. Therefore the assertion follows from the previous theorem. $\square$

- Let E be an elliptic curve over K. Let $X = E/[\pm 1]$ and $\pi : E \to X$ the canonical morphism. Note that $X \simeq \mathbb{P}^1_K$. Moreover, the homomorphism $[2] : E \to E$ $(x \mapsto 2x)$ descents to an endomorphism $X \to X$, that is, there is a morphism $f : X \to X$ such that the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{[2]} & E \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & X \end{array}$$

The endomorphism f is called a Lattés map.

• Let D be an ample Cartier divisor on X. Then $\pi^*(D)$ is symmetric because $\pi \circ [-1] = \pi$, so that $[2]^*(\pi^*(D)) = 4\pi^*(D) + (\phi')$ for some $\phi' \in \mathrm{Rat}(E)^\times$, that is, $\pi^*(f^*(D) - 4D) = (\phi')$. Therefore, if we set $\phi = \mathrm{Norm}(\phi')^{1/2} \in \mathrm{Rat}(X)^\times \otimes \mathbb{Q}$, then $f^*(D) = 4D + (\phi)$.

For $\sigma \in M_K^\infty$, $\mathrm{Prep}(f_\sigma)$ is dense in X_σ because $\pi(\mathrm{Prep}([2]_\sigma)) \subseteq \mathrm{Prep}(f_\sigma)$ and $\mathrm{Prep}([2]_\sigma)$ is dense in E_σ.

Therefore, the canonical compactification $\overline{D}$ does not have the Dirichlet property.

- Let E be an elliptic curve over $\mathbb{Q}$ and $\mathbb{P}^1_{\mathbb{Q}} := \mathrm{Proj}(\mathbb{Q}[x, y])$. Let D_1 be the Cartier divisor on E given by the 0-point of E, and $D_2 = \{x = 0\}$ on $\mathbb{P}^1_{\mathbb{Q}}$. Then $[2]^*(D_1) = 4D_1 + (\phi)$ for some $\phi \in \mathrm{Rat}(E)^\times$. Let $h : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ be the morphism given by $h(x : y) = (x^4 : y^4)$. Then $h^*(D_2) = 4D_2$. We set

$$X := E \times \mathbb{P}^1_{\mathbb{Q}}, \quad f : [2] \times h, \quad D := p_1^*(D_1) + p_2^*(D_2),$$

where $p_1 : X \to R$ and $p_2 : X \to \mathbb{P}^1_{\mathbb{Q}}$ are the projections. Note that $f^*(D) = 4D + (p_1^*(\phi))$. Then we have the following:

1. For all $v \in M_K$, $\mathrm{Prep}(f_v)^{an}$ is not dense in X_v^{an}.
2. For $\infty \in \mathbb{Q}(\mathbb{C})$ (the canonical embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$), $\mathrm{Supp}_{ess}(\mathrm{Prep}(f))^{an}_\infty = E(\mathbb{C}) \times \{(x : 1) \mid |x| = 1\}$.

By the above (2), $E(\mathbb{C}) \times \{(1 : 1)\} \subseteq \mathrm{Supp}_{ess}(\mathrm{Prep}(f))^{an}_\infty$. Thus, by the above theorem, the canonical compactification $\overline{D}$ does not have the Dirichlet property.

Problem

Here we do not assume the existence of the endmorphism $f : X \to X$. *We assume that* D *is ample and* $\overline{D}$ *is pseudo-effective. If, for all* $v \in M_K$, *there is no algebraic curve* C_v *in* X_v *with* $C_v^{an} \subseteq \mathrm{Supp}_{ess}(X(\overline{K})^{\overline{D}}_{\leq 0})_v^{an}$, *then does it follow that* $\overline{D}$ *has the Dirichlet property?*

From now on, we consider a functional approach.

Let V be a vector subspace of $\widehat{\mathrm{Div}}^{a}_{C^0}(X)_{\mathbb{R}}$ with $V \supseteq \{\widehat{(\varphi)} \mid \varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}\}$. Let V_+ denote the subset of all effective adelic arithmetic $\mathbb{R}$-Cartier divisors in V. Let $C_\circ$ be a subset of V verifying the following conditions :

1. for any $\overline{D} \in C_\circ$ and $\lambda > 0$, one has $\lambda\overline{D} \in C_\circ$;
2. for any $\overline{D}_0 \in C_\circ$ and $\overline{D} \in V_+$, there exists $\varepsilon_0 > 0$ such that $\overline{D}_0 + \varepsilon\overline{D} \in C_\circ$ for any $\varepsilon \in \mathbb{R}$ with $0 \leq \varepsilon \leq \varepsilon_0$;
3. for any $\overline{D} \in C_\circ$ and $\phi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$, one has $\overline{D} + \widehat{(\phi)} \in C_\circ$.

Assume given a map $\mu : C_\circ \to \mathbb{R}$ which verifies the following properties :

1. there exists a positive number a such that $\mu(t\overline{D}) = t^a \mu(\overline{D})$ for all adelic arithmetic $\mathbb{R}$-Cartier divisor $\overline{D} \in C_0$ and $t > 0$;
2. for any $\overline{D} \in C_\circ$ and $\phi \in \mathrm{Rat}(X)^\times_{\mathbb{R}}$, one has $\mu(\overline{D} + \widehat{(\phi)}) = \mu(\overline{D})$.

For $\overline{D} \in C_\circ$ and $\overline{E} \in V_+$, we define $\nabla^+_{\overline{E}} \mu(\overline{D})$ to be

$$\nabla^+_{\overline{E}} \mu(\overline{D}) = \limsup_{\epsilon \to 0+} \frac{\mu(\overline{D} + \epsilon \overline{E}) - \mu(\overline{D})}{\epsilon},$$

which might be $\pm\infty$.

In addition to (1) and (2), assume the following property:

3. there exists a map $\nabla_\mu : \widehat{\mathrm{Div}}^a_{C^0}(X)^+_{\mathbb{R}} \times C_\circ \to \mathbb{R} \cup \{\pm\infty\}$ such that

$$\nabla_\mu(\overline{E}, \overline{D}) = \nabla^+_{\overline{E}}\mu(\overline{D}) \quad \text{for } \overline{E} \in V_+ \text{ and } \overline{D} \in C_\circ,$$

where $\widehat{\mathrm{Div}}^a_{C^0}(X)^+_{\mathbb{R}}$ denotes the set of all effective adelic arithmetic $\mathbb{R}$-Cartier divisors.

We set

$$C_{\circ\circ} := \left\{ \overline{D} \in C_\circ \;\middle|\; \begin{array}{l} \nabla_\mu(\overline{E}_1, \overline{D}) \leq \nabla_\mu(\overline{E}_2, \overline{D}) \text{ for all} \\ \overline{E}_1, \overline{E}_2 \in \widehat{\mathrm{Div}}^{a}_{C^0}(X)^{+}_{\mathbb{R}} \text{ with } \overline{E}_1 \leq \overline{E}_2 \end{array} \right\}.$$

For any $v \in M_K$ and $f_v \in C^0(X_v^{\mathrm{an}})$, an adelic arithmetic $\mathbb{R}$-Cartier divisor $\overline{O}(f_v)$ is defined to be

$$\overline{O}(f_v) = \begin{cases} \left(0, f_v[v]\right) & \text{if } v \in M_K, \\ \left(0, \frac{1}{2} f_v[v] + \frac{1}{2} F_\infty^*(f_v)[\bar{v}]\right) & \text{if } v \in K(\mathbb{C}). \end{cases}$$

If $\overline{D}$ is an element in $C_{\circ\circ}$, then the map ∇_μ defines, for any $v \in M_K \cup K(\mathbb{C})$, a non-necessarily additive functional

$$\Psi^{\mu}_{\overline{D},v} : C^0(X_v^{an})_+ \longrightarrow [0, +\infty], \quad \Psi^{\mu}_{\overline{D},v}(f_v) := \nabla_\mu(\overline{O}(f_v), \overline{D}),$$

where $C^0(X_v^{an})^+$ denotes the cone of non-negative continuous functions on X_v^{an}.

Definition

We define the support of $\Psi^{\mu}_{\overline{D},v}$ to be the set $\mathrm{Supp}(\Psi^{\mu}_{\overline{D},v})$ of all $x \in X_v^{an}$ such that $\Psi^{\mu}_{\overline{D},v}(f_v) > 0$ for any non-negative continuous function f_v on X_v^{an} verifying $f_v(x) > 0$.

Note that $\mathrm{Supp}(\Psi^{\mu}_{\overline{D},v})$ is closed in X_v^{an}.

Theorem

Let $\overline{D}$ be an element of $C_{\circ\circ}$ with $\mu(\overline{D}) = 0$. If s is an element of $\mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ with $\overline{D} + \widehat{(s)} \geq 0$, then

$$\mathrm{Supp}(\Psi^{\mu}_{\overline{D},v}) \cap \{x \in X_v^{an} \mid |s|_{g_v} < 1\} = \varnothing$$

for any $v \in M_K$.

Proof. We set $\overline{D}' = \overline{D} + \widehat{(s)} = (D', g')$ and $f_v = \min\{g'_v, 1\}$. Thus, as

$$0 \leq \overline{O}(f_v) \leq \overline{D}'$$

and $\overline{D} \in C_{\circ\circ}$, one has

$$0 = \nabla_{\mu}((0,0), \overline{D}) \leq \Psi^{\mu}_{\overline{D},v}(f_v) = \nabla_{\mu}(\overline{O}(f_v), \overline{D}) \leq \nabla_{\mu}(\overline{D}', \overline{D}) = \nabla^{+}_{\overline{D}'}\mu(\overline{D}).$$

On the other hand, by using the properties (1) and (2), one obtains

$$\mu(\overline{D} + \epsilon \overline{D}') - \mu(\overline{D}) = \mu(\overline{D} + \epsilon \overline{D}) - \mu(\overline{D}) = ((1+\epsilon)^a - 1)\mu(\overline{D}),$$

and hence $\nabla^{+}_{\overline{D}'}\mu(\overline{D}) = a\mu(\overline{D}) = 0$. Therefore, $\Psi_{\overline{D},v^{\mu}}(f_v) = 0$, so that

$$\mathrm{Supp}(\Psi^{\mu}_{\overline{D},v}) \cap \{x \in X_v^{an} \mid f_v(x) > 0\} = \varnothing.$$

Note that $g'_v = -\log |s|^2_{g_v}$. Thus, we can see that

$$\{x \in X_v^{an} \mid f_v(x) > 0\} = \{x \in X_v^{an} \mid |s|_{g_v} < 1\},$$

as required. □

We have the following examples of μ:

1. $V := \widehat{\mathrm{Div}}^a_{C^0}(X)_{\mathbb{R}}$ and $C_\circ := \{\overline{D} \in \widehat{\mathrm{Div}}^a_{C^0}(X)_{\mathbb{R}} \mid D \text{ is big}\}$. Let ζ be an $\mathbb{R}$-Cartier divisor on $\mathrm{Spec}(K)$ with $\widehat{\deg}(\zeta) = 1$. For $\overline{D} \in C_\circ$, we set

$$\mu^{\mathrm{asy}}_{\max}(\overline{D}) := \sup\{t \in \mathbb{R} \mid \overline{D} - t\pi^*(\zeta) \text{ has the Dirichlet property}\},$$

   where π is the canonical morphism $X \to \mathrm{Spec}(K)$. Note that the above definition does not depend on the choice of ζ. $\mu(\overline{D}) := \mu^{\mathrm{asy}}_{\max}(\overline{D})$ is an example.
2. $V := \widehat{\mathrm{Div}}^a_{C^0}(X)_{\mathbb{R}}$ and $C_\circ := \{\overline{D} \in \widehat{\mathrm{Div}}^a_{C^0}(X)_{\mathbb{R}} \mid D \text{ is big}\}$. $\mu(\overline{D}) := \widehat{\mathrm{vol}}(\overline{D})$ is an example.
3. $V = C_\circ := \{\overline{D} \in \widehat{\mathrm{Div}}^a_{C^0}(X)_{\mathbb{R}} \mid \overline{D} \text{ is integrable}\}$. $\mu(\overline{D}) := \widehat{\deg}(\overline{D}^{d+1})$ is an example.

Note the following facts:

Remark

If D is ample, $\overline{D}$ is nef and $X(\overline{K})^{\overline{D}}_{\leq 0}$ is Zariski dense, then

$$\mathrm{Supp}(\Psi^{\widehat{\mathrm{vol}}}_{\overline{D},v}) \subseteq \mathrm{Supp}(\Psi^{\mu^{\mathrm{asy}}_{\max}}_{\overline{D},v}) \subseteq \mathrm{Supp}_{ess}(X(\overline{K})^{\overline{D}}_{\leq 0})^{an}_{v}.$$

for all $v \in M_K$.

Problem

We assume that D is ample and $\mu^{\mathrm{asy}}_{\max}(\overline{D}) = 0$. If, for all $v \in M_K$, there is no algebraic curve C_v in X_v with $C_v^{an} \subseteq \mathrm{Supp}(\Psi^{\mu^{\mathrm{asy}}_{\max}}_{\overline{D},v})$, then does it follow that $\overline{D}$ has the Dirichlet property?

Thank you for your attention.