Deformations and Stability of Dufour Foliations

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Why Dufour foliations?

Morse foliations = singular folitations of dimension 1 which are locally given by the level sets of Morse functions.

Generalization to Morse-Bott foliations: locally given by Morse-Bott functions (still of codimension 1).

What about higher codimension analogs of Morse foliations? They do exist, and their local study has been initiated by Jean-Paul Dufour some 20 years ago. (Dufour & Z: Linearization of Nambu structures, Compositio, 1999; and Chapter 6 of book "Poisson structures and their normal forms", Birkhäuser, 2005)

This talk: deformations and stability of singular foliations in general, and of Dufour foliations in particular.

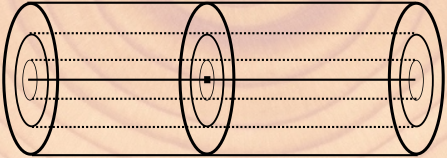
Outline

1. Singular foliations à la Stefan–Sussmann

Definition of singular foliations (Stefan-Sussmann)

A partition $M = \bigsqcup \mathcal{F}_i$ ($\mathcal{F}_i$ are leaves) satisfies :
$\forall x \in M$, $\mathcal{F}_x$: the leaf contains x, $\exists$ local coordinates $x_1, \ldots, x_n$ such that

- $\mathcal{F}_x = \{x_{d+1} = \ldots = x_n = 0\}$
- each disk $\{x_{d+1} = c_{d+1}, \ldots x_n = c_n\}$ is contained in some leaf



Regular case: all the leaves have the same dimension (each disk $\{x_{d+1} = c_{d+1}, \ldots x_n = c_n\}$ in the above definition is an open subset of some other leaf)

The local splitting condition

The Stefan–Sussmann definition may be reparaphrased as the following inductive local splitting condition:

Local splitting condition

Near each point x of rank d, the foliation is locally isomorphic (via a local diffeomorphism) to a direct product of $\mathbb{R}^d$ with a foliation on $\mathbb{R}^{n-d}$ such that the origin of $\mathbb{R}^{n-d}$ has rank 0.

Here, by definition, the rank of a point is the dimension of the leaf through it.

Compare with splitting/slice theorems for Poisson structures (Weinstein), Lie algebroids (Fernandes – Dufour), Dirac manifolds (Dufour – Wade), slice theorems for proper group actions (Palais), proper Lie groupoids (Weinstein – Zung)

Where do singular foliations come from?

- *Nature:* Earth layers; graphite; liquid crystals, composite materials, Saturn rings, trees, parallel worlds (?), etc. *Generators* and *cloning*.
- *Actions:* of Lie groups and algebras, Lie groupoids and algebroids, vector fields, etc.
- *Constraints:* level sets of functions and maps, singular fibrations, restrictions to subanifolds, (non)holonomic constraints, etc.

Proposition

Restriction (pull-back) of a singular foliation to a submanifold is again a singular foliation

- *Geometry and dynamical systems*: Riemannian foliations, stable and unstable manifolds, representatives of noncommutative spaces, etc.

Frobenius–Clebsch–Deahna theorem

Definition: Smooth singular distributions

$\mathcal{D}$ is a field of vector subspaces of tangent spaces, which is generated by a family $\{X_\alpha, \alpha \in I\}$ of smooth v.f.: $\mathcal{D}_x = Vect(X_\alpha(x), \alpha \in I)\ \forall x \in M$. $\mathcal{D}$ is called **involutive** if for any two vector fields X, Y tangent to $\mathcal{D}$, their Lie bracket $[X, Y]$ is also tangent to $\mathcal{D}$. $\mathcal{D}$ is **integrable** if it is the *tangent distribution* to a smooth singular foliation $\mathcal{F}$:
$\mathcal{D}_x = T_x S(x)\ \forall x \in M$, where $S(x)$ denotes the leaf through x.

If $\dim \mathcal{D}_x$ is constant then $\mathcal{D}$ is a **regular distribution** (subbundle of TM). It is clear that if $\mathcal{D} = \mathcal{D}_{\mathcal{F}}$ then $\mathcal{D}$ is involutive. (Being tangent to $\mathcal{D}$ means tangent to each leaf of $\mathcal{F}$ in this case, and the Lie bracket can be taken leaf by leaf). The converse is also true in the regular case:

Theorem (Frobenius 1877 – Clebsch 1860 – Deahna 1840)

If $\mathcal{D}$ is regular then it's integrable if and only if it's involutive.

Stefan–Sussmann theorem

Attention: The above theorem is FALSE in the smooth singular case.
Example: $\mathcal{D}$ on $\mathbb{R}^2$ given by $\mathcal{D}_{(x,y)} = Vect(\partial x)$ if $x \leq 0$ and $\mathcal{D}_{(x,y)} = Vect(\partial x, \partial y)$ if $x > 0$. Then $\mathcal{D}$ is involutive but not integrable.
Solution: Impose some additional conditions to avoid pathologies such as above, e.g. $\mathcal{D}$ is generated by locally finitely-generated involutive modules of vector fields (Hermann's theorem, 1963), or the following invariance condition (stronger than involutivity):

Theorem (Stefan-Sussmann 1973-74)

The following conditions are equivalent:

- $\mathcal{D} = D_{\mathcal{F}}$ *for some singular foliation* $\mathcal{F}$
- $\mathcal{D}$ *is generated by a family* C *of vector fields and is invariant with respect to the elements of* C

Example: Stefan–Sussmann condition is obviously satisfied for the family of Hamiltonian vector fields on Poisson manifolds, and for Lie algebroids.

Algebraization/tensorization of foliations

Idea: Represent foliations by elements of some vector space or algebra, in order to do deformation theory using analytico-algebraic machinary.

Posible approaches:

- Heafliger's *Gamma-structures*: in terms of cocycles with values in $\mathit{Diff}(\mathbb{K}^q)$ where q is the codimension. Gives holonomy and universal classifying space. But not every singular foliation can be given by a Haefliger structure ?!
- Skandalis (+ Androulidakis, Zambon): Via Lie algebroids and groupoids. Use *locally finitely-generated modules of tangent vector fields*. OK for holonomy. But what about normalization and deformation theory ?!
- Our approach: *Integrable differential forms* and *Nambu structures*. We claim that they are the right objects for studying general singular foliations.

2. Integrable differential forms and Nambu structures

Definition (Integrable 1-forms (codimension 1 foliations))

A differential 1-form α is called **integrable** if $\alpha \wedge d\alpha = 0$

At points where $\alpha \neq 0$, the involutivity of $\ker \alpha$ is equivalent to the above condition, hence we get a (singular) codimension 1 foliation. Cartan, Nemytskii (1940s), Kupka, Thom (1960s), etc.

Definition (Integrable q-forms / codimension q foliations)

A differential q-form ω is called **integrable** if:

$$\omega \wedge i_A\omega = 0 \;\; \& \;\; d\omega \wedge i_A\omega = 0 \quad \forall \; (q-1)\text{-vector} \;\; A$$

The kernel of an integrable q-form ω near a point x where $\omega(x) \neq 0$ is an involutive distribution of corank $q \to$ codimension q foliation.

Integrable q-forms are in use since 1970-80s only (?) Malgrange (wedge product of 1-forms), Camacho, Lins-Neto, Medeiros, etc.

Nambu structures via integrable differential forms

Fix a volume form Ω. Then for each q-form ω there is a unique p-vector field Λ (where $p + q = n$ is the dimension of the manifold) such that

$$\omega = \Lambda \lrcorner \Omega$$

Λ is called an **integrable p-vector field**, or a **Nambu structure** of order p, if its dual differential form $\omega = \Lambda \lrcorner \Omega$ is integrable. Equivalent definition:

Definition

A p-vector field Λ is called a Nambu structure iff near every point x such that $\Lambda(x) \neq 0$ there is a local coordinate system $(x_1, \ldots, x_n)$ such that

$$\Lambda = f \partial x_1 \wedge \ldots \wedge \partial x_p$$

(One can put $f = 1$). One may view a Nambu structure as a singular folitation + **leafwise contravariant volume element**.

A bit of history of Nambu structures

Nambu (1973): generalization of Hamiltonian formalisim from a binary bracket (the Poisson bracket) to a p-ary bracket (for $p = 3$). Takhtajan (1994) gave a definition of Nambu-Poisson structures in the general case using p-ary brackets in a way similar to Poisson structures.

When $p = 2$ a Nambu-Poisson structure is nothing but a Poisson structure, while a Nambu structure is our definition is a Poisson structure of rank 2. When $p \neq 2$ the definition of Takhtajan and our new definition coincide (theorem of Alekseevsky – Guha, Gautheron, Nakanishi).

Nambu p-ary bracket: $\{f_1, \ldots, f_p\} = \langle df_1 \wedge \cdots \wedge df_p, \Lambda \rangle$

Hamiltonian vector fields: $X_{f_1,\ldots,f_{p-1}} = (df_1 \wedge \cdots \wedge df_{p-1}) \lrcorner \Lambda$
Generalized Leibniz identity $\Leftrightarrow$ Hamiltonian vector fields preserve Λ

Some remarks on Nambu structures

- In general, associated Nambu structures do not lose singularities of foliations. Nor do they create new artificial singular points.
- The sheaf of local tangent Nambu strucures = the sheaf of local sections of a line bundle (the anti-canonical bundle of the foliations). If this line bundle is not globally trivial then a global Nambu structure doesn't exist, but it's no big deal: one can talk about twisted associated Nambu structures by taking tensor product with the dual (canonical) line bundle, so the theory still works.
- Nambu are good not only for foliations, but also for analysis (singularity theory). Example: manifold with boundary and corners can also be represented by Nambu. Near a corner, Λ is monomial:

$$\Lambda = x_1 \dots x_k \partial x_1 \wedge \dots \wedge \partial x_k \wedge \dots \wedge \partial x_n$$

One recovers the boundary strata as singular leaves of the associated foliation. (One can often views tratifications as singular foliations)

Associated Nambu stuctures to a foliation

Question: Given a singular foliation $\mathcal{F}$, how to associate to it a Nambu structure Λ, so that one can essentially recover $\mathcal{F}$ from Λ?

Intuitively, Λ would be tangent to $\mathcal{F}$ in the sense that at every regular point x of Λ (i.e. $\Lambda(x) \neq 0$) we can write $\Lambda = \partial x_1 \wedge \ldots \wedge \partial x_p$ in a local coordinate system such that $\partial x_1, \ldots, \partial x_p$ generate $\mathcal{F}$ near x. In particular, Λ vanishes at every singular point of $\mathcal{F}$

However, in some special situations the above intuitive tangency condition would imply that the singular set of Λ is too big compared to the singular set of $\mathcal{F}$.

Example: $\mathcal{F}$ in $\mathbb{C}^2$ with leaves $\{x = const \neq 0\}$, $\{x = 0, y \neq 0\}$ and $\{x = y = 0\}$. Then $\Lambda = f\partial y$. If Λ is analytic and vanishes at the singular point of $\mathcal{F}$ then the singular set of Λ (i.e. the level set $\{f = 0\}$ is of dimension (at least) 1 while the singular set of $\mathcal{F}$ is of dimension 0.

The tangency condition

Need a compromise between "being tangent everywhere" and "without unwanted singular points". The following definitions work well in the analytic case: Denote by $S(\Lambda)$ (resp. $S(\mathcal{F})$) the singular set of a Nambu structure Λ of order p (resp. of a p-dimensional foliation $\mathcal{F}$).

- Λ a **tangent Nambu structure** to $\mathcal{F}$ if $\mathrm{codim}(S(\mathcal{F}) \setminus S(\Lambda)) \geq 2$ and near each point $x \notin S(\Lambda) \cup S(\mathcal{F})$ there is a local coordinate system in which $\Lambda = \partial x_1 \wedge \ldots \wedge \partial x_p$ and $\mathcal{F}$ is generated by $\partial x_1, \ldots, \partial x_p$
- Moreover, if $\mathrm{codim}\big(S(\mathcal{F}) \setminus S(\Lambda)\big) \geq 2$, and is without multiplicity in the sense that Λ can't be written as $\Lambda = f^2\Lambda'$, where f is a function which vanishes somewhere, then we say that Λ is an **associated Nambu structure** to $\mathcal{F}$.

Existence and Uniqueness Theorem

In the holomorphic category, there always exists a local associated Nambu structure which is unique up to multiplication by an invertible function.

Construction of associated Nambu structures

- Take p local vector fields $X_1, \ldots, X_p$ tangent to $\mathcal{F}$ and linearly independent almost everywhere, and put $\Pi = X_1 \wedge \ldots \wedge X_q$.
- Decompose $\Pi = h\Lambda$, where $\operatorname{codim} S(\Lambda) \geq 2$.
- If $\operatorname{codim} S(\mathcal{F}) \geq 2$ then Λ is an associated Nambu structure of $\mathcal{F}$.
- If $\operatorname{codim} S(\mathcal{F}) = 1$, we find a reduced function s such that $S(\mathcal{F}) = \{s = 0\}$ then $s\Lambda$ is an associated Nambu structure of $\mathcal{F}$.

Example: $\mathcal{F}$ on $\mathbb{R}^3$ or $\mathbb{C}^3$ with leaves $\{x^2 + y^2 + z^2 = const\}$. Take two tangent vector fields $X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$, $Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$, and put

$$\Pi = X \wedge Y = z\left(x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\right)$$

then $\Lambda = \frac{\Pi}{z}$ is an associated Nambu structure of $\mathcal{F}$.

Global situation: Sheaf of tangent Nambu structures = sheaf of sections of the **anti-canonical line bundle** of the foliation. Associated Nambu structure = section which doesn't vanish anywhere.

Associated foliations

An "obvious" foliation generated by Λ consists of 2 kinds of leaves: regular leaves, and 0-dimensional leaves (singular points of Λ). But this foliation is "stupid". Need more sophisticated constructions.

Definition

We say that a vector field X is **tangent** to Λ if $X \wedge \Lambda = 0$

The set of tangent vector fields forms an integrable distribution. However, the foliation defined in by it may lose many singularities of Λ.

Example: $\Lambda = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$. Then $\mathrm{codim} S(\Lambda) = \mathrm{codim}\{x = 0\} = 1$, but the foliation $\mathcal{F}$ defined by the tangent vector fields of Λ consists of just one leaf, which is $\mathbb{C}^2$. Put an additional condition to avoid this situation:

Definition

A vector field X is called a **conformally invariant tangent** (CIT) vector field of a Nambu structure Λ if X is tangent to Λ and X conformally preserves Λ, i.e. $\mathcal{L}_X \Lambda = f\Lambda$ for some function f. The set of CIT vector fields of Λ will be denoted by $CIT(\Lambda)$.

Associated foliations

Theorem and Definition

The set of CIT vector fields of a Nambu structure Λ of order p generates an integrable singular distribution and hence defines a singular foliation $\mathcal{F}_\Lambda$ which will be called the **associated foliation** of Λ.

For previous example, $\Lambda = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$

- $\frac{\partial}{\partial x}$ is a tangent but not an associated vector field of Λ.
- $\mathcal{F}_\Lambda$ is generated by $\{x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ and consists two leaves $\{x = 0\}$ and $\{x \neq 0\}$.

Proposition

If Λ is a holomorphic Nambu structure and $\mathrm{codim}(\Lambda) \geq 2$, then every tangent vector of Λ is also CIT vector field of Λ.

$$\Lambda \to \mathcal{F}_\Lambda \to \Lambda_{\mathcal{F}_\Lambda}$$

We will say Λ **conformally preserves** a function f if there is Σ such that

$$[f, \Lambda] = f\Sigma,$$

where the bracket means the Schouten bracket. Denoted by $\mu(\Lambda)$ the set of functions which are conformally preserved by Λ.

Proposition

- If $\mathrm{codim} S(\Lambda) \geq 2$, then $\Lambda_{\mathcal{F}_\Lambda} = u\Lambda$ for some invertible function u.
- If $\Lambda = \prod f_i^{m_i} \prod g_j^{m_j} \Lambda_1$, where $\mathrm{codim} S(\Lambda_1) \geq 2$, f_i, g_j are irreducible, $f_i \in \mu(\Lambda_1)$, $g_j \notin \mu(\Lambda_1)$, then $\Lambda_{\mathcal{F}_\Lambda} = u \prod g_j \Lambda_1$ for some invertible function u.

From Nambu structures to singular foliations and back

Example: Let $f, g \in \mathcal{O}_2$ be irreducible and $(f, g) = 1$. Consider $\Lambda = fgX_f$ where $X_f = \frac{\partial f}{\partial y}\partial x - \frac{\partial f}{\partial x}\partial y$. Then $f \in \mu(X_f)$, $g \notin \mu(X_f)$ and $\Lambda_{\mathcal{F}_\Lambda} = gX_f$

From foliations to Nambu structures and back: $\mathcal{F} \to \Lambda_{\mathcal{F}} \to \mathcal{F}_{\Lambda_{\mathcal{F}}}$

Theorem

Let $\mathcal{F}$ be a holomorphic singular foliation and $\Lambda_{\mathcal{F}}$ be its associated Nambu structure. Suppose that $\mathcal{F}_{\Lambda_{\mathcal{F}}}$ is an associated foliation of $\Lambda_{\mathcal{F}}$ then $\mathcal{F}_{\Lambda_{\mathcal{F}}}$ is a saturation of $\mathcal{F}$. Moreover, if $\mathrm{codim} S(\mathcal{F}) \geq 2$ then $\mathrm{codim} S(\mathcal{F}_{\Lambda_{\mathcal{F}}}) \geq 2$.

Saturation means that each leaf of the latter foliation is saturated by the leaves of the former one.

Reference for Nambu$\leftrightarrow$foliation correspondence: Minh & Zung, "Commuting Foliations", Regular and Chaotic Dynamics 2013.

Morphisms, pull-backs, stratification, etc.

- The pul-back of a singular foliation by a map is again a singular foliation (via the pull-back of local associated integrable differential forms)
- Morphisms? Many kinds: isomorphisms (no problems); sending Nambu to Nambu (preserve the dimension), sending distribution to distribution, ...
- Foliation vs stratification of singular fibers of maps/fibrations: singular leaves of the associated singlar foliations are often strata of the (Whitney – Thom – Mather) stratification. It's true, for example, for nondegenerate singularities of integrable Hamiltonian systems.
- Counter-example (suggested by Mattei): Level sets of the function $f(x,y,z) = x(x-y)(x-2y)(x-zy)$ in $\mathbb{C}^3$. An 1-dimensional stratum of the singular level set is not a leaf, but the leaves in it are just points (due to changing biratios). Nothing wrong with singular foliations or Nambu structures, it's just that sometimes the stratification of a singular fiber can't be made foliated)

What are linear foliations?

There are two different non-equivalent notions of linear singular foliations:

- **Lie-linear**: foliations generated by linear vector fields = generatedby linear representations of Lie algebras = generated by linear actions of Lie groups. (Big theory of linear representations)
- **Nambu-linear**: associated to a linear Nambu structure, i.e. whose coefficients in a coordinate systems are linear.

Nambu-linear are also Lie-linear though the converse is not true (there are few Nambu-linear foliations): contraction of a linear Nambu p-vector field with constant $(p-1)$-forms give rise to generating linear vector fields.

Classification of Nambu-linear: Dufour–Z (Compositio Math. 1999). Some other people (Grabowski, ...) arrived at similar results.

Classification of linear Nambu structures: 2 types

Type I (piles of cabbages / parallel worlds):

$$\Lambda = \omega \lrcorner (\partial x_1 \wedge \cdots \wedge \partial x_n)$$

where

$$\omega = dx_1 \wedge \cdots \wedge dx_{p-1} \wedge dQ$$

where Q is a quadratic function
$x_1, \ldots, x_{p-1}$ are regular linear first integrals
Q is a quadratic first integral
In the nodegenerate case, Q depends only on $x_p, \ldots, x_n$
If Q is positive definite then the leaves are p-dimensional spheres
The foliation looks like **a pile of cabbages** or **parallel worlds**

Classification of linear Nambu structures: 2 types

Type II (open books):

$$\Lambda = \partial x_1 \wedge \cdots \wedge \partial x_{p-1} \wedge X$$

where

$$X = \sum_{i,j \geq p} a_{ij} x_i \partial x_j$$

is a linear vector field in the variables $x_p, \ldots, x_n$
This foliation can be splitted into direct product of a linear vector field with $\mathbb{R}^{p-1}$.
Looks like **open books** (especially if X is hyperbolic).

In a sense, the two types are dual to each other: in Type 2 the Nambu structure is decomposabe, in Type 1 the integrable differential form is decomposable. Very often, singular foliations are locally of these two types, because of the linearization theorems (see Lecture 2).

(Quasi-)homogeneous foliations

What is a homogeneous singular foliation of degree k? Generated by homogeneous vector fields of degree k?

The problem is that, the Lie bracket of two vector fields of degree k is a vector field of degree $2k-1 \neq k$ unless $k=1$. So a family of vector fields of degree k can't be involutive in general, and if one generates an involutive family from some vector fields of degree k by taking Lie brackets, we will get vector fields of different degrees.

To avoid this problem: replace vecto fields by **(quasi-)homogeneous Nambu structures** or integrable differential forms.

Example: $\omega = dF1 \wedge \cdots \wedge dF_q$, where $F_1, \ldots, F_q$ are homogeneous polynomial functions.

Example: Any Lie-linear foliation is a (Nambu-)homogeneous foliation.

Example: Diract product of homogeneous foliations is again homogeneous.

H-degree

H-degree $=$ degree of the associated homogeneous Nambu structure. (A discrete invariant for linear representations)

Example: Let $\mathfrak{g}$ be a semisimple Lie algebra, and consider the associated coadjoint foliation on $\mathfrak{g}^*$. Leaves $=$ coadjoint orbits. The singular set of codimension 3. First integrals $=$ Casimir functions. The associated Nambu structure is

$$\Lambda = \wedge^m \Pi$$

where m is half the dimension of coadjoint orbits. The H-degree is also m, because Π is linear. Up to a multiplicative constant, the dual integrable differential form is

$$dF_1 \wedge \cdots \wedge dF_d$$

where $F_1, \ldots, F_d$ are generators of the algebra of Casimir functions. (d is the dimension of the Cartan subalgebra).

Example: $\mathfrak{g} = gl(n, \mathbb{K})$

The coordinates are $x_{ij}, 1 \le i, j \le n$. The tangent vector fields are (for $i \neq j$)

$$X_{ij} = x_{ij}(\partial x_{jj} - \partial x_{ii}) + (x_{ii} - x_{jj})\partial x_{ji} + \sum_{k \neq i,j} (x_{ik}\partial x_{jk} - x_{kj}\partial x_{ki})$$

The tangent Nambu structure $\wedge_{i \neq j} X_{ij}$ is divisible by $\prod_{i<j}(x_{ii} - x_{jj})$ and

$$\Lambda = \wedge_{i \neq j} X_{ij} / \prod_{i<j}(x_{ii} - x_{jj})$$

is the associated Nambu structure of order $n(n-1)$ (equal to the dimension of the generic orbits), which is homogeneous of degree $n(n-1)/2$ (half of the order). It is proportional to $\wedge^{n(n-1)/2}\Pi$ where Π is the associated linear Poisson structure on the dual of the Lie algebra.

3. Deformation cohomology of singular foliations

We want to develop a general deformation theory for singular foliations. Fundamental tool at the infinitesimal level: *deformation cohomology*.

General idea: Given a certain structure S

- Infinitesimal deformations: D such that the formal deformation $S + \epsilon D$ of S also satisfies structural equations modulo ϵ^2

- Trivial deformations: terms of the type $(1 + \epsilon X)_* S - S$ modulo ϵ^2

- **Deformation cohomology:**

$$H_{def}(S) = \frac{\{\text{infinitesimal deformations}\}}{\{\text{trivial deformations}\}}$$

may be interpreted as the formal tangent space to the moduli space of deformations.

- If $H_{def}(S) = 0$ then S is called **infinitesimally rigid**. In many situations, infinitesimal rigidity implies rigidity (Richardson–Nijenhuis, Mather, etc.).

Infinitesimal deformations of Nambu structures

Let Λ be a Nambu structure of order p.

- A multi-vector field Π of order p is called an **infinitesimal deformation** of Λ if $\Lambda + \epsilon\Pi$ is a Nambu structure modulo ϵ^2. The condition "$\Lambda + \epsilon\Pi$ is Nambu modulo ϵ^2" is a linear system of equations on Π (linear first order PDEs + linear algebraic equations), so the set of infinitesimal deformations is a vector space.
- If $\Pi = \mathcal{L}_X\Lambda$ for some vector field X, then Π is called a trivial deformation of Λ. The set of all trivial deformations is also a vector space.
- If $\Pi = \mathcal{L}_X\Lambda + f\Lambda$ for some vector field X and some function f, then Π is called a trivial deformation of the associated foliation $\mathcal{F}_\Lambda$. The set of all trivial deformations of the foliation is also a vector space.

Deformation cohomology of Λ and $\mathcal{F}_\Lambda$

Definition: Deformation cohomology of Λ and $\mathcal{F}_\Lambda$

$$H_{def}(\Lambda) = \frac{\{\text{Infinitesimal deformations of } \Lambda\}}{\{\mathcal{L}_X \Lambda\}},$$

$$H_{def}(\mathcal{F}_\Lambda) = \frac{\{\text{Infinitesimal deformations of } \Lambda\}}{\{\mathcal{L}_X \Lambda + f\Lambda\}}.$$

Remark: If a local associated Nambu structure doesn't exist globally (because the anti-canonical line bundle is non-trivial, use a "twisted associated Nambu structure" (twisted by the canonical bundle).

Problems: Computations of deformation cohomologies, relations to problems of rigidity and (true) deformations, comparison with other cohomology theories, characteristic classes and indices, etc.

If the deformation cohomology is finite-dimensional, one expects that the moduli space of deformations/normal forms of perturbations is also locally finite-dimensional.

Examples of deformation cohomology

(Computations worked out with **Ph. Monnier** and **Truong Hong Minh**)

• The regular case: J. L. **Heitsch** (1973-75) defined a differential complex whose first cohomology class is the deformation cohomology space of a *regular* foliation $\mathcal{F}$: It is nothing else but the *algebroid cohomology* of the natural linear action of the tangent Lie algebroid $T\mathcal{F}$ on the normal bundle $N_{\mathcal{F}} = TM/T\mathcal{F}$ of $\mathcal{F}$.
Assume that there is a global Nambu structure Λ associated to $\mathcal{F}$ (i.e. regular leafwise volume contravariant form). Then $\mathcal{F} = \mathcal{F}_{\Lambda}$ and our deformation cohomology for the foliation coincides with Heitsch's:

Theorem

$H_{def}(\mathcal{F}) = H^1(T\mathcal{F}, N_{\mathcal{F}})$

The differential complex here is $\Omega^*(T\mathcal{F}, N_{\mathcal{F}})$ of leafwise differential forms with values in the normal bundle: it's very similar to the usual De Rham complex of differential forms with values in $\mathbb{R}$

Examples of deformation cohomology

The above theorem is still valid when a global assocated Nambu structure doesn't exist: replace Nambu by a twisted Nambu in this case. $H_{def}(\Lambda)$ can be much larger than $H_{def}(\mathcal{F}_\Lambda)$, also in the regular case.

For example, let $M = P \times Q$ compact, where Q is simply-connected, and the foliation $\mathcal{F}$ is given by the projection to P, i.e. the leaves are $\{pt\} \times Q$. Then the deformation cohomology of the foliation is trivial (it follows from the above theorem, and agrees with Reeb stability theorem). On the other hand, $H_{def}(\Lambda)$ becomes the deformation cohomology of a function f on P (the value of f at a point x in P equals the volume of $\{x\} \times Q$ with respect to the contravariant volume form Λ), and we have:

Theorem

Let $f : P \to \mathbb{R}$ be a smooth simple Morse function on a compact manifold P. Then $\dim H_{def}(f)$ is the number of singular points of f.

Examples of deformation cohomology

- The regular case of top order:

If Λ is regular of top order, then only 1 leaf (the manifold itself), the foliation is trival (no deformation), but Λ can be deformed (by changing the total volume)

Theorem

If Λ is a regular Nambu structure of top degree on a compact manifold then $H_{def}(\Lambda) = \mathbb{R}$ and $H_{def}(\mathcal{F}_\Lambda) = 0$

Consistent with Moser [1964]: Two volume forms ω_1 and ω_2 on a compact manifold M are diffeomorphic $\Leftrightarrow$ they have the same global volume:

$$\int_M \omega_1 = \int_M \omega_2$$

Examples of deformation cohomology

• The case of top order with nondegenerate singularities:
Locally $\Lambda = x_1 \partial x_1 \wedge \partial x_2 \wedge \cdots \wedge \partial x_n$ near singular points (Type II). Two kinds of leaves: regular n-dimensional domains, and singular $(n-1)$-submanifolds.

Classification of these structures is done by Olga Radko (2002, for $n = 2$: Poisson surfaces) and David Martinez Torres (2004, for n arbitrary). Numerical invariants of the classifiation (besides topological ones):

- Regularized Liouville volume of the manifold
- $(n-1)$-dimensional volume of each singular leaf (induced from Λ)

Theorem

Let Λ be a Nambu structure of top order with nondegenerate singularities on a compact manifold M. Then $\dim H_{def}(\mathcal{F}_\Lambda) = 0$ and $\dim H_{def}(\Lambda) = k + 1$, where k is the number of $(n-1)$-dimensional leaves.

Examples of deformation cohomology

- The case of top order, local coholmology.

Assume that

$$\Lambda = f\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$$

where $f(0) = 0$ and moreover 0 is a singular point of f, i.e. $df(0) = 0$. We will work here in the local holomorphic category, with germs of functions. In this case we have:

- $$H_{def}(\mathcal{F}_\Lambda) \cong \frac{\mathcal{O}_n}{\left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle}$$

  and $\dim H_{def}(\mathcal{F}_\Lambda) = \tau(f)$ is the **Tjurina number** of f at 0.

- $$H_{def}(\Lambda) \cong \frac{\mathcal{O}_n}{\{X(f) - (divX)f | X \in \mathfrak{X}\}}$$

  and $\dim H_{def}(\Lambda) =$ **some number** of f (?! Don't know the name)

Examples of deformation cohomology

• The case of order 0 (i.e, functions). $\Lambda = f$ is a 0-vector field. Locally:

$$H_{def}(f) = \frac{\mathcal{O}_n}{\{X(f) | X \in \mathfrak{X}\}} = \frac{\mathcal{O}_n}{\left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle},$$

$$H_{def}(\mathcal{F}_f) = \frac{\mathcal{O}_n}{\{X(f) + cf | X \in \mathfrak{X}, c \in \mathcal{O}_n\}} = \frac{\mathcal{O}_n}{\left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle}.$$

$\dim H_{def}(f) = \mu(f)$, $\dim H_{def}(\mathcal{F}_f) = \tau(f)$ (**Milnor** and **Tjurina** number).

• The case of vector fields (order 1) If $\Lambda = X$ is a vector field, the leaves of the foliation are integral curves of X. Normalization of the foliation = Orbital normalization of X.
Local deformation cohomology (for germs of vector fields) is given by resonant terms, and can be infinite-dimensional. If the vector field has a non-resonant linear part, then the local deformation cohomology is trivial. Global deformation = complicated dynamical problem.

Decomposable Nambu structures with small singularities

- Λ is a Nambu structure and $\omega = i_\Lambda \Omega$, Ω is a volume form.
- If ω is decomposable (i.e. $\omega = \omega_1 \wedge \ldots \wedge \omega_q$) and $\mathrm{codim}(\omega) \geq 3$ then by Malgrange (1977): $\omega = u df_1 \wedge \ldots \wedge df_q$

Proposition

Let $\omega = u df_1 \wedge \ldots \wedge df_q$ be an integrable q-form and η is an infinitesimal deformation ω. If $\mathrm{codim} S(\omega) \geq q + 2$ then

$$\eta = a_0 df_1 \wedge \ldots \wedge df_q + u \sum_{i=1}^{q} df_1 \wedge \ldots \wedge df_{i-1} \wedge da_i \wedge df_{i+1} \wedge \ldots \wedge df_q.$$

It means that $\omega + \epsilon\eta$ is also decomposable and admits first integrals modulo ϵ^2.

Decomposable Nambu structures with small singularities

Corollary

If $\omega = df$ (Nambu structure of order $n-1$) and $\mathrm{codim} S(df) \geq 3$ then

$$H_{def}(\mathcal{F}_{df}) = \frac{\mathcal{O}_n}{\left\{a_i \frac{\partial f}{\partial_{x_1}} + \ldots + a_n \frac{\partial f}{\partial_{x_n}} + h \circ f | a_i \in \mathcal{O}_n, h \in \mathcal{O}\right\}}$$

In particular, $\mu(f) \geq \dim H_{def}(\mathcal{F}_{df}) \geq \tau(f) - 1$.

Corollary

If O is an isolated singular point of $\omega = u df_1 \wedge \ldots \wedge df_q$ then

$$\dim H_{def}(\mathcal{F}_\omega) < \infty$$

Some open problems concerning deformation

- Interpret deformation cohomology as part of a bigger cohomology theory (with a differential complex) in the singular case ?
- Computation of deformation cohomology for linear actions of Lie algebras, and for other situations?
- Rigidity of singular foliations given by semisimple compact group actions: The group actions are rgid, but if we forget the group, is the foliation still rigid ?
- Singular Reeb stability (for singular points and leaves) ? Linearized models along a leaf ?
- Local and global (Infinitesimal) rigidity of orbit-like foliations with simply-connected leaves ?
- Etc.

4. Dufour foliations

A **Dufour foliation** of dimension p is a foliation which can be locally given near every singular point by a Nambu structure of order p which has a nondegenerate linear part. If every singular point of a Dufour foliation is of type 1 (resp., type 2), then we say that it is a Dufour foliation of **type 1** (resp., **type 2**). If there are singular points of both types then we say that it is a Dufour foliation of **mixed type**.

Examples:
1) Morse fucntions and Morse foliations are Dufour foliations of Type 1 and codimension 1. On can also define Dufour-Bott foliations. Topology and stability of Morse and Morse-Bott foliations are studied by many people, also quite recently (**Reeb** stability, Thurston, Wagneur, Camacho, Scardua, Seade, Mafra, Fukui, Rosati, ...)
2) Contravariant fields of top order and nondegenerate singularities (studied by O. Radko, D. Martinez, ...) are Dufour foliations of Type 2 and codimension 0.

Some questions on Dufour foliations

- Local deformations/stability at singular points? Done (local linearization problem).
- Deformation cohomology and global structural stability of Dufour foliations? (Extention of results from the Morse case)
- Realization via Lie algebroids? Not clear. (When order $= 2$ then realized via Poisson structures and their cotangent Lie algebroids. In general how can one use surgery to *glue Lie algebroids* together? Use categorical approach to Lie algebroids?)
- Existence of Dufour foliation of a given codimension on arbitrary manifolds?
- Topology and Morse theory? Restriction to Dufour foliations of Type 1 with compact leaves already gives a large family of foliations to study topology.
- Extensions to Dufour-Bott foliations

Linearization of singular points of Type 2

Splitting proposition

If the linear part Λ_1 of a Nambu structure Λ which vanishes at O is of Type 2, $\Lambda_1 = \partial x_1 \wedge \cdots \wedge \partial x_{p-1} \wedge \sum_{i,j \geq p} a_{ij} x_i \partial x_j$, with non-vanishing trace $\sum_{i \geq p} a_{ii} \neq 0$, then then Λ is $(p-1)$-splittable, i.e. it can be written locally, in some coordinate system, as

$$\Lambda = \partial x_1 \wedge \cdots \wedge \partial x_{p-1} \wedge X$$

The above proposition is a particular case ofthe so-called **generalized Kupka's phenomenon**. After the splitting, the linearization problem for Λ becomes the linearization problem for a vector field X, so we get:

Theorem (Dufour–Z 1999)

If $\sum_{i,j \geq p} a_{ij} x_i \partial x_j$ is non-resonant then Λ is smoothly linearizable. If moreover it satisfies a Diophantine condition and Λ is analytic then analytically linearizable.

Linearization of singular points of Type 1

$\Lambda = \Lambda_1 + h.o.t$ is of Type 1, i.e.

$$\Lambda_1 \lrcorner (dx_1 \wedge \cdots \wedge dx_n) = dQ \wedge dx_{p+2} \wedge \cdots \wedge dx_n$$

where Q is a quadratic function. We will consider only the nondegenerate case, i.e. Q is nondegenerate quadratic in variables $x_1, \ldots, x_{p+1}$

Theorem (Dufour–Z 1999, Z 2013)

a) If Λ is formal then it's formally linearizable
b) If Λ is analytic then it's analytically linearizable
c) If Λ is smooth and the signature of Q is different from $(2, *)$ then Λ is smootly linearizable. If the signature is $(2, *)$ then there are counter-examples.

Of course, under the assumptions of the above theorems, the deformation cohomology is trivial. From that infinitesimal rigidity to linearizability, we need more work.

Tools used in the proof of local linearization

- Division theorems (De Rham, Saito)
- Decomposition of the dual integrable differential form
- Godbillon–Vey algorithm (to formally linearize the foliation)
- Malgrange's "Frobenius with singularity" theorem (for the existence of analytic first integrals)
- Blowing up (in the compact case, when Q is positive definite)
- Equivariant smooth linearization of vector fields (Sternberg–Belitskii–Kopanskii)
- **Levi decomposition** (for the existence of $SO(p+1)$ symmetry group, similar to the one used in the linearization of Lie algebroids.) Cerveau was first to use it for singular foliations.
- **Slicing method** (for dealing with the smooth noncompact case: turning non-compact leaves into compact leaves by slicing).

Stability of Dufour foliations: the elliptic case

Theorem

Consider a Dufour foliation of type 1 of dimension $p \geq 2$ with only elliptic singular points. Then:
i) All regular fibers are diffeomorphic to S^p
ii) The foliation is completely integrable (i.e., it admits a complete set of first integrals)
iii) The deformation cohomology is trivial
iv) The foliation is structurally stable (any nearby singular foliation will be diffeomorphic to it)
v) The foliation can be blown up to a regular S^p fibration over a base space = manifold with boundary: the boundary corresponds to singular points of the foliation.
iv) The classification of such filiations is equivalent to the classification of S^p fibrations over manifolds with boundary.

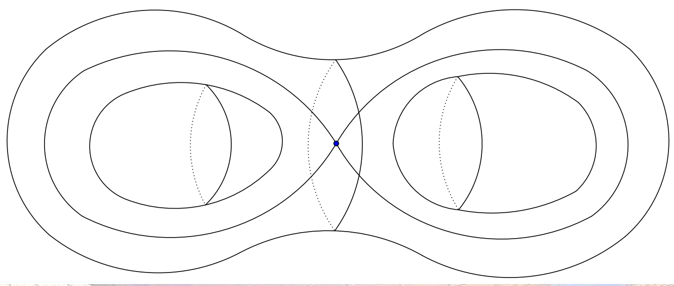
Remark: The dimension of the base space is the codimension of the foliation. The Reeb case is when this base space is a closed interval.

Deformations and stability of Dufour foliations: the general case

- Deformation cohomology $\rightarrow$ possible infinitesimal holonomy.
- If there is no room for infinitesimal holonomy (i.e. when some leaves are known to be closed simply-connected) then the deformation cohomology is trivial.
- Under some conditions (transversality of heteroclinic leavs + some simply-connected fibers), all the leaves will be automatically closed, the foliation will be completely integrable, infinitesimaly rigid, and structurally stable.

A simple example

There are Dufour foliations which are locally non-structurally-stable but globally structurelly stable



THANK YOU!