Metrics with Hessian Curvature of Type $\frac{1}{2}(1-\kappa^2)$

Example ((M^2, g) such that $\mathrm{Hess}_g\kappa = \frac{1}{2}(1-\kappa^2)g$: $G = \mathrm{SO}_2$)

- Structure equations:

$$\begin{cases} \mathrm{d}\theta^1 = -\theta^2 \wedge \eta \\ \mathrm{d}\theta^2 = \theta^1 \wedge \eta \\ \mathrm{d}\eta = -\kappa\theta^1 \wedge \theta^2 \\ \mathrm{d}\kappa = \kappa_1\theta^1 + \kappa_2\theta^2 \\ \mathrm{d}\kappa_1 = \frac{1}{2}(1-\kappa^2)\theta_1 - \kappa_2\eta \\ \mathrm{d}\kappa_2 = \frac{1}{2}(1-\kappa^2)\theta_2 + \kappa_1\eta \end{cases}$$

- η - Levi-Civita; $\theta = (\theta^1, \theta^2)$ - tautological form; $(\kappa, \kappa_1, \kappa_2) : F_{\mathrm{SO}_2}(M) \to \mathbb{R}^3$.

Example: (M^2, g) such that $\mathrm{Hess}_g\kappa = \frac{1}{2}(1-\kappa^2)g$

We saw that the Lie algebroid associated to the structure equations has:

- $X = \mathbb{R}^3$ with coordinates (k, k_1, k_2);
- $A = X \times (\mathbb{R}^2 \oplus \mathfrak{so}_2)$ with basis of sections $\alpha_1, \alpha_2, \beta$.
- The inner SO_2-action is generated by β.
- The bracket is given by

$$\begin{cases} [\alpha_2, \beta] = \alpha_1 \\ [\beta, \alpha_1] = \alpha_2 \\ [\alpha_1, \alpha_2] = \kappa\beta \end{cases}$$

- The anchor is given by

$$\begin{cases} \rho(\alpha_1) = \kappa_1\partial_\kappa + \frac{1}{2}(1-\kappa^2)\partial_{\kappa_1} \\ \rho(\alpha_2) = \kappa_2\partial_\kappa + \frac{1}{2}(1-\kappa^2)\partial_{\kappa_2} \\ \rho(\beta) = -\kappa_2\partial_{\kappa_1} + \kappa_1\partial_{\kappa_2} \end{cases}$$

Leaves

The function

$$F(k, k_1, k_2) = k_1^2 + k_2^2 + \frac{1}{3}k^3 - k = c.$$

is constant on the leaves. When $(k, k_1, k_2) \neq (\pm 1, 0, 0)$, the leaves are 2-dimensional.
The foliation has leaves of the following type:

- There are two fixed points: $(0, 0, 1)$, and $(0, 0, -1)$.
- Near to $(0, 0, 1)$ the leaves are spheres.
- There is a leaf near $(0, 0, -1)$ which is diffeomorphic to $\mathbb{R}^2 - \{0\}$.
- All other leaves are non-compact and contractible (diffeomorphic to $\mathbb{R}^2$).

Isotropy and Integrability

The isotropy Lie algebras are:

- $\mathfrak{so}_3$ at $(1,0,0)$
- $\mathfrak{sl}_2$ at $(-1,0,0)$
- Over all other points $\mathrm{Ker}\rho_{(k,k_1,k_2)} = \mathbb{R}$ and generated by

$$\xi = k_2\alpha_1 - k_1\alpha_2 + \frac{1}{2}(1-k^2)\beta.$$

A is integrable and but not weakly G-integrable:
The only leaves which can cause problems are the spheres.

Final Conclusions

A_L is G-integrable if and only if:

- L is not a sphere;
- L is a sphere such that

$$\frac{1-k_{\max}^2}{1-k_{\min}^2} \in \mathbb{Q}$$

It is possible to write down explicit formulas for the metrics we obtain..... but I will not bother you with this now!

Thank you!