A new expression of the q-Stirling numbers

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Joint work with Margaret Readdy.

Classical q-binomial coefficient

Definition

The Gaussian polynomial or q-binomial is the familiar q-analogue of the binomial coefficient given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!},$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q \cdot [2]_q \cdots [n]_q$.

Theorem (MacMahon)

The q-binomial coefficient has the following combinatorial interpretation.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \mathfrak{S}(0^{n-k},1^k)} q^{\mathrm{inv}(w)},$$

where $\mathfrak{S}(0^{n-k}, 1^k)$ denotes the set of all 0-1 permutations consisting of $n-k$ zeros and k ones, and for $w = w_1 w_2 \cdots w_n \in \mathfrak{S}(0^{n-k}, 1^k)$ the number of inversions is $\mathrm{inv}(w) = |\{(i,j) : i < j \text{ and } w_i > w_j\}|$.

Theorem (Fu–Reiner–Stanton–Thiem)

$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \Omega(n,k)} q^{a(w)}(1+q)^{p(w)}$, *where* $\Omega(n,k)$ *is a subset of* $\mathfrak{S}(0^{n-k}, 1^k)$ *and* $(a(w), p(w))$ *is a bistatistic defined on* $\Omega(n,k)$.

Goal 1

Find compact q-$(1+q)$-encodings of classical q-analogues.

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Goal 2

Understand compact encodings of classical q-analogues via enumerative, poset theoretic and topological viewpoints.

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Understand compact encodings of classical q-analogues via enumerative, poset theoretic and topological viewpoints.

We do this for the q-Stirling numbers of the first and second kinds.

Set partitions

Definition

A *set partition* of the n elements $\{1, 2, \ldots, n\}$ is a decomposition of this set into mutually disjoint nonempty sets called blocks.

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Example

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Theorem

The Stirling number of the second kind $S(n, k)$ counts the number of set partitions of n elements into k blocks.

Restricted growth words

Definition

- Let $\pi = B_1/B_2/\cdots/B_k$ be a set partition of $\{1,\ldots,n\}$ in standard form, where the blocks are arranged such that $\min(B_1) < \min(B_2) < \cdots < \min(B_n)$. We denote the set of all partitions of $\{1,\ldots,n\}$ by Π_n.
- Given a partition $\pi \in \Pi_n$, we encode it using a *restricted growth word* $w(\pi) = w_1 \cdots w_n$, where $w_i = j$ if the element i occurs in the j-th block B_j of π.
- Let $\mathcal{R}(n,k)$ denote the set of all RG-words of length n with maximum letter k.

Example

The partition $\pi = 125/36/47$ has RG-word $w = 1123123$.

Restricted growth words are also known as restricted growth functions. They have been studied by Hutchinson, Milne and Rota.

q-Stirling numbers of the 2nd kind

Definition

The *q-analogue of the Stirling number of the second kind* is given by the recurrence formula

$$S_q[n,k] = S_q[n-1,k-1] + [k]_q S_q[n-1,k], \text{ for } 0 \leq k \leq n,$$

where $S_q[n,0] = \delta_{n,0}$. Setting $q = 1$ gives the familiar Stirling number of the second kind $S(n,k)$ which enumerates the number of $\pi \in \Pi_n$ with exactly k blocks.

Definition

For $w = w_1 \cdots w_n \in \mathcal{R}(n,k)$, define $\mathrm{wt}(w) = \prod_{i=1}^{n} \mathrm{wt}(w_i)$, where $\mathrm{wt}(w_1) = 1$ and for $2 \le i \le n$, we have

$$\mathrm{wt}(w_i) = \begin{cases} q^{w_i - 1} & \text{if } w_i \le \max\{w_1, \ldots, w_{i-1}\}, \\ 1 & \text{if } w_i > \max\{w_1, \ldots, w_{i-1}\}. \end{cases}$$

Example

Partition	RG-word w	$\mathrm{wt}(w)$
1/234	1222	$1 \cdot 1 \cdot q \cdot q = q^2$
12/34	1122	$1 \cdot 1 \cdot 1 \cdot q = q$
13/24	1212	$1 \cdot 1 \cdot 1 \cdot q = q$
14/23	1221	$1 \cdot 1 \cdot q \cdot 1 = q$
134/2	1211	$1 \cdot 1 \cdot 1 \cdot 1 = 1$
124/3	1121	$1 \cdot 1 \cdot 1 \cdot 1 = 1$
123/4	1112	$1 \cdot 1 \cdot 1 \cdot 1 = 1$

Using RG-words, we can compute that $S_q[4,2] = q^2 + 3q + 3$.

Lemma

The q-Stirling number of the second kind $S_q[n,k]$ is given by

$$S_q[n,k] = \sum_{w \in \mathcal{R}(n,k)} \mathrm{wt}(w).$$

Allowable RG-words

Definition

- An *allowable RG-word w* is of the form

$$w = u_1 \cdot 2 \cdot u_2 \cdot 4 \cdot u_3 \cdot 6 \cdot u_4 \cdots ,$$

  where u_{2i-1} is a word on the alphabet $\{1, 3, \ldots, 2i-1\}$. Denote the set of all allowable RG-words from $\mathcal{R}(n, k)$ by $\mathcal{A}(n, k)$.

- For an allowable word $w \in \mathcal{A}(n, k)$, we give it a new weight $\mathrm{wt}'(w) = \Pi_{i=1}^{n} \mathrm{wt}'(w_i)$, where $\mathrm{wt}'(w_1) = 1$ and for $2 \leq i \leq n$,

$$\mathrm{wt}'(w_i) = \begin{cases} q^{w_i - 1} \cdot (1+q) & \text{if } w_i < \max\{w_1, \ldots, w_{i-1}\}, \\ q^{w_i - 1} & \text{if } w_i = \max\{w_1, \ldots, w_{i-1}\}, \\ 1 & \text{if } w_i > \max\{w_1, \ldots, w_{i-1}\}. \end{cases}$$

The example $\mathcal{A}(4,2)$

Partition	RG-word w	$\mathrm{wt}(w)$	Allowed?	$\mathrm{wt}'(w)$
$1/234$	1222	$1 \cdot 1 \cdot q \cdot q = q^2$	No	N/A
$12/34$	1122	$1 \cdot 1 \cdot 1 \cdot q = q$	No	N/A
$13/24$	1212	$1 \cdot 1 \cdot 1 \cdot q = q$	No	N/A
$14/23$	1221	$1 \cdot 1 \cdot q \cdot 1 = q$	No	N/A
$134/2$	1211	$1 \cdot 1 \cdot 1 \cdot 1 = 1$	Yes	$(q+1)^2$
$124/3$	1121	$1 \cdot 1 \cdot 1 \cdot 1 = 1$	Yes	$(q+1)$
$123/4$	1112	$1 \cdot 1 \cdot 1 \cdot 1 = 1$	Yes	1

For the allowable words, we have

$$\begin{aligned}(q+1)^2 + (q+1) + 1 &= q^2 + 3q + 3 \\ &= S_q[4,2]\,.\end{aligned}$$

Theorem (Cai–Readdy)

$$S_q[n,k] = \sum_{w \in \mathcal{A}(n,k)} \mathrm{wt}'(w) = \sum_{w \in \mathcal{A}(n,k)} q^{A(w)} \cdot (1+q)^{B(w)}.$$

Stembridge's $q = -1$ phenomenon

If we set $q = -1$ in $S_q[n, k]$, then $(q + 1)$ will become 0, and we are left with words that are of the form

$$\pi = u_1 \cdot 2 \cdot u_3 \cdot 4 \cdot u_5 \cdot 6 \cdots ,$$

where u_{2i-1} is a word which only uses the letter $2i - 1$. We call such words *unmatched* words, and we have the following

Lemma (Cai–Readdy)

The number of all unmatched words in $\mathcal{A}(n, k)$ is $\binom{n-1-\lfloor k/2 \rfloor}{\lfloor (k-1)/2 \rfloor}$.

Stirling poset of the second kind $\Pi(n,k)$

Definition

Let $\Pi(n,k)$ denote the poset with all the elements from $\mathcal{R}(n,k)$ and $u \prec w$ if $w = u_1 \cdots u_{i-1}(u_i + 1)u_{i+1} \cdots u_n$ for some index i. Also see that if $u \prec w$, then $\text{wt}(w) = q \cdot \text{wt}(u)$. We call this poset *the Stirling poset of the second kind*.

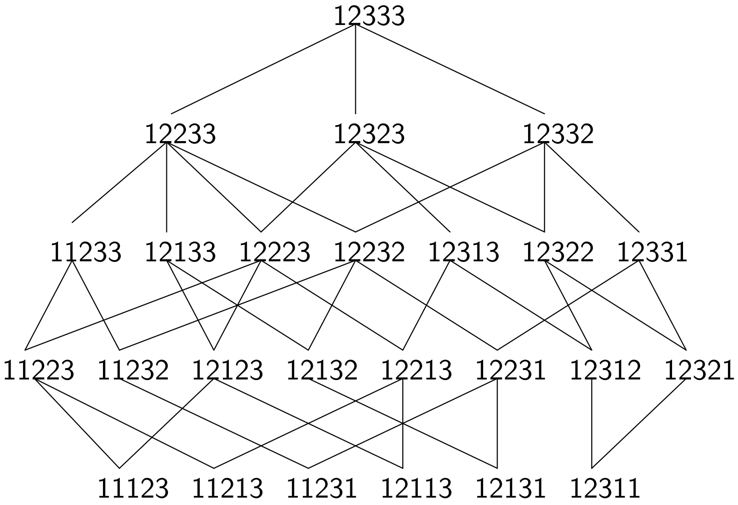

Figure: The Stirling poset of the second kind $\Pi(5,3)$.

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2^n is the number of subsets of an n-element set!

Theorem (Cai–Readdy)

The Stirling poset of the second kind $\Pi(n,k)$ can be decomposed into disjoint union of Boolean intervals

$$\Pi(n,k) = \bigcup_{w \in \mathcal{A}(n,k)}^{\cdot} [w, \alpha(w)].$$

Furthermore, if an allowable word $w \in \mathcal{A}(n,k)$ has weight $\mathrm{wt}'(w) = q^i \cdot (1+q)^j$, then the rank of the element w is i and the interval $[w, \alpha(w)]$ is isomorphic to the Boolean algebra on j elements.

Constructing a Boolean algebra

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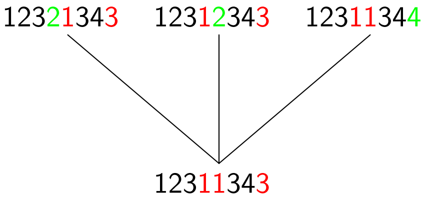
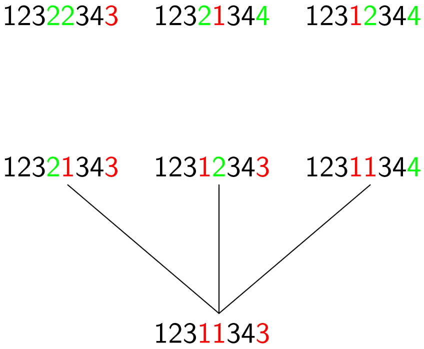
12321343 12312343 12311344

12311343

Constructing a Boolean algebra



Constructing a Boolean algebra



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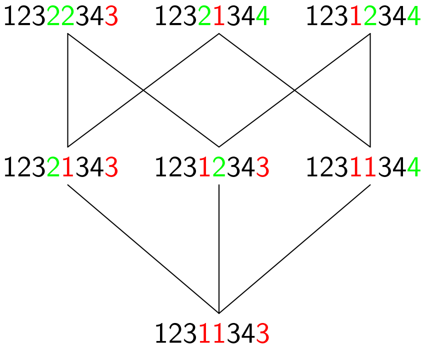
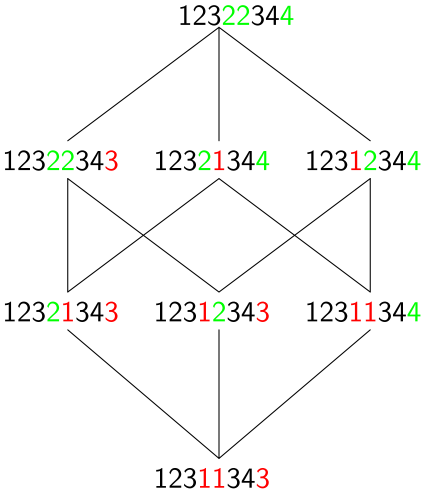


Constructing a Boolean algebra



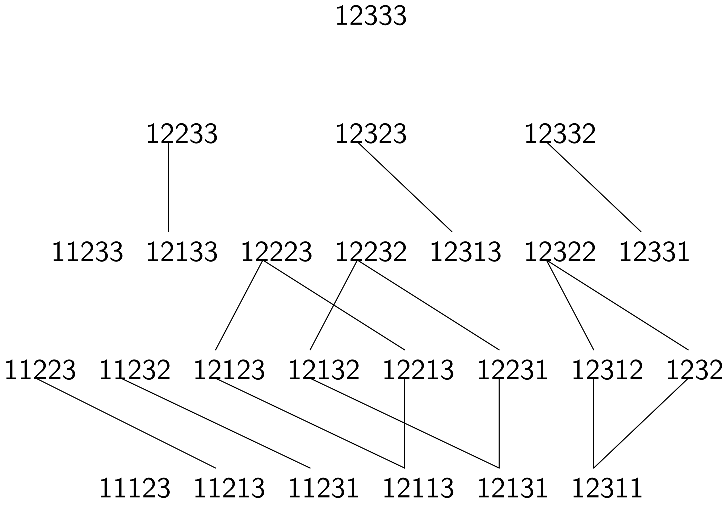


Figure: The decomposition of the poset $\Pi(5, 3)$ into Boolean algebras.

Definition

A *partial matching* on a poset P is a matching on the underlying graph of the Hasse diagram of P, that is, a subset $M \subseteq P \times P$ satisfying the following.

1. The ordered pair $(a, b) \in M$ implies $a \prec b$.
2. Each element $a \in P$ belongs to at most one element in M.

When $(a, b) \in M$, we write $u(a) = b$. A partial matching on P is *acyclic* if there does not exist a cycle

$$a_1 \prec u(a_1) \succ a_2 \prec u(a_2) \succ \cdots \succ a_n \prec u(a_n) \succ a_1$$

with $n \geq 2$, and the elements $a_1, a_2, \ldots, a_n$ are distinct.

An acyclic matching on a poset is also called a *discrete Morse matching*.

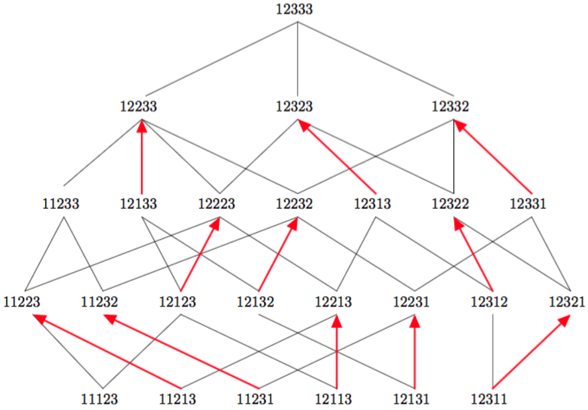


Figure: The matching on the poset $\Pi(5,3)$.

Theorem (Cai–Readdy)

This matching on $\Pi(n, k)$ is an acyclic matching.

Algebraic complex

Definition

Let P be a graded poset and W_i denote the rank i elements. We say the *poset P supports a chain complex* (C, ∂) *of* $\mathbb{F}$*-vector space* C_i if each C_i has basis indexed by the rank i elements W_i and $\partial : W_i \longrightarrow W_{i-1}$ is a boundary map. Furthermore, for $x \in W_i$ and $y \in W_{i-1}$ the coefficient $\partial_{x,y}$ of y in $\partial_i(x)$ is zero unless $y <_P x$ in the poset.

Topological $q = -1$ phenomenon

Theorem (Hersh–Shareshian–Stanton)

Let P be a graded poset supporting an algebraic complex (C, ∂). Assume the poset P has a discrete Morse matching M such that for all matched pairs (y, x) with $y < x$ one has $\partial_{y,x} \in \mathbb{F}^*$. If all unmatched poset elements occur in ranks of the same parity, then $\dim(H_i(C, \partial)) = |P^{un\ M}|$, *that is, the number of unmatched elements of rank i.*

Integer homology of $\Pi(n, k)$

Theorem (Cai–Readdy)

For the algebra complex (C, ∂) supported by the Stirling poset of the second kind $\Pi(n, k)$, a basis for the integer homology is given by the increasing allowable RG-words in $\mathcal{A}(n, k)$. Furthermore, we have

$$\sum_{i \geq 0} (\dim H_i(C, \partial; \mathbb{Z})) q^i = \begin{bmatrix} n-1-\lfloor \frac{k}{2} \rfloor \\ \lfloor \frac{k-1}{2} \rfloor \end{bmatrix}_{q^2} .$$

q-Stirling numbers of the first kind

We have a similar analysis for the q-Stirling numbers of the first kind via rook placements.

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Theorem (de Médicis–Leroux)

The q-Stirling number of the first kind $c_q[n,k]$ is given by

$$c_q[n,k] = \sum_{T \in P(n,n-k)} q^{s(T)},$$

where the sum is over all rook placements of $n-k$ rooks on a staircase board of length n.

Theorem (Cai–Readdy)

The q-Stirling number of the first kind is given by

$$c_q[n,k] = \sum_{T \in Q(n,n-k)} q^{s(T)} \cdot (1+q)^{r(T)},$$

where the sum is over all rook placements of $n-k$ rooks on an alternating shaded staircase board of length n.

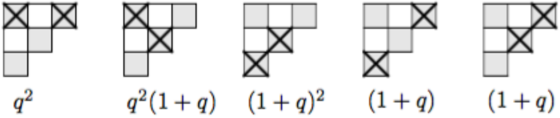

Figure: Computing the q-Stirling number of the first kind $c_q[4, 2]$ using allowable rook placements.

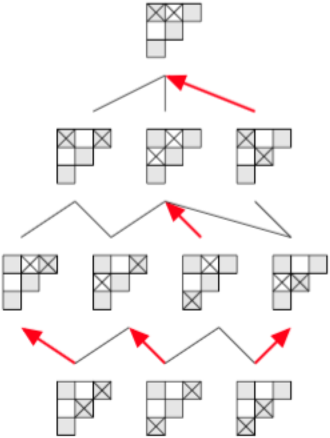

Figure: The Stirling poset of the first kind $\Gamma(4,2)$

Integer homology of $\Gamma(m, n)$

Theorem (Cai–Readdy)

For the algebraic complex $(\mathcal{C}, \partial)$ supported by the Stirling poset of the first kind $\Gamma(m, n)$, a basis for the integer homology is given by the rook placements in $P(m, n)$ having all of the rooks occur in shaded squares in the first row. Furthermore,

$$\sum_{i \geq 0} \dim(H_i(\mathcal{C}, \partial; \mathbb{Z})) \cdot q^i = q^{n(n-1)} \cdot \begin{bmatrix} \lfloor m/2 \rfloor \\ n \end{bmatrix}_{q^2}.$$

References

- Y. Cai and M. Readdy, q-Stirling numbers: A new view, *Advances in Applied Mathematics* **86** (2017), 50–80.
- S. Fu, V. Reiner, D. Stanton and N. Thiem, The negative q-binomial, *Electronic Journal of Combinatorics* **19** (2012), P36.
- S. Milne, Restricted growth functions and incidence relations of the lattice of partitions of an n-set, *Advances in Math.* **26** (1977), 290–305.
- R. Forman, A User's Guide to Discrete Morse Theory, *Séminaire Lotharingien de Combinatoire* **B48c** (2002), 35 pp.

Thank you!