Cutoff for the random-to-random shuffle

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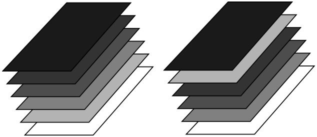
Women in Algebraic Combinatorixx 2

Random-to-random shuffle

Pick a card and position uniformly at random. Move the card there.



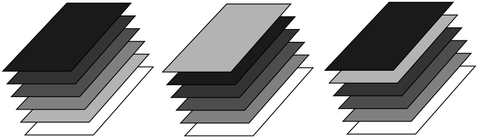
The walk on S_n is given by the matrix $K(g,h) = P(g^{-1}h)$ for:

$$P(g) = \begin{cases} \frac{1}{n} & g = e \\ \frac{2}{n^2} & g = (i, i+1) \text{ for some i} \\ \frac{1}{n^2} & g = (i, i+1, ..., i+j) \text{ for some } j > 1, i \\ 0 & \text{otherwise} \end{cases}$$

The distribution of the t^{th} step is: $K^t(e, \cdot) = P^{*t}(\cdot)$.

Related shuffles and walks

- Introduced by Diaconis and Saloff-Coste in 1993
- Symmetrization of random-to-top shuffle (Tsetlin library) with its inverse, top-to-random



- Random-to-random should mix faster, but this could not be shown...
- Random-to-top falls into a broader class: Bidigare-Hanlon-Rockmore hyperplane rearrangment random walks (easy to get eigenvalues of!)

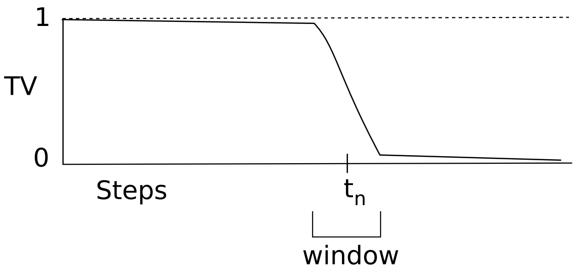
Cutoff

Total variation distance– how close to uniform after t steps:

$$||P_{\text{id}}^{*t} - \pi||_{TV} = \frac{1}{2} \sum_{g \in S_n} |P^{*t}(g) - \pi(g)|$$

P has TV cutoff if exists a seq. (t_n) s.t. for all $\epsilon > 0$:

$$\lim_{n\to\infty} ||P_{\text{id}}^{*t_n(1-\epsilon)} - \pi||_{TV} = 1 \qquad \lim_{n\to\infty} ||P_{\text{id}}^{*t_n(1+\epsilon)} - \pi||_{TV} = 0$$



Mixing bounds on random-to-random

- (Diaconis, Saloff-Coste 1993) t_{mix} is $O(n \log n)$
- (Uyemura-Reyes 2002) $\frac{1}{2} n \log n \leq t_{\text{mix}} \leq 4n \log n$
- (Diaconis 2005) Conjecture :

$$\left(\frac{3}{4} - o(1)\right) n \log n \leq t_{\text{mix}} \leq \left(\frac{3}{4} + o(1)\right) n \log n$$

- (Saloff-Coste and Zúñiga 2008) $t_{\text{mix}} \leq 2n \log n$
- (Subag 2013) $\left(\frac{3}{4} - o(1)\right) n \log n \leq t_{\text{mix}}$
- (Morris-Qin 2014) $t_{\text{mix}} \leq 1.5324 n \log n$

Theorem (B.-Nestoridi, 2017+)

$$t_{\text{mix}} \leq \frac{3}{4} n \log n + cn$$

Eigenvalues and the l^2 norm

Let K be a symmetric, transitive transition matrix of a random walk on Ω with eigenvalues $1 = \lambda_1 > \lambda_2 \geq ... \geq \lambda_{|\Omega|} \geq -1$, then:

$$4||K^t(x,\cdot) - \pi||_{TV}^2 \leq \left\|\frac{K^t(x,\cdot)}{\pi(\cdot)} - 1\right\|_2^2 = \sum_{j=2}^{|\Omega|} \lambda_j^{2t}$$

for every starting point $x \in \Omega$.

Spectral methods

If P constant on conjugacy class, by Shur's Lemma,

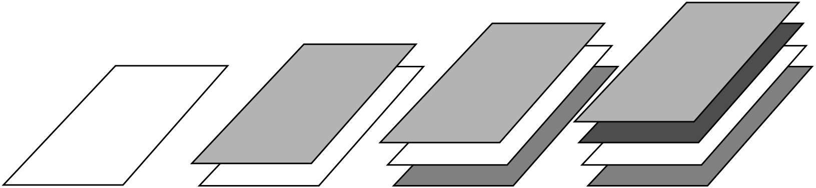
- Walk acts as a constant on each irreducible representation of S_n
- Eigenvalues are linear combinations of characters
- See e.g. (Diaconis-Shahshahani 1981) on transposition walk

But the random-to-random walk is not a conjugacy class walk and is not constant on irred. reps!

Insight into representation theory

Random-to-random needs a more direct construction of its eigenspaces

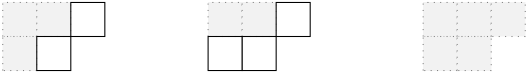
Using random insertion of new cards, can build a uniformly random deck



(Dieker-Saliola, 2015+) use this to recursively construct eigenvectors of random-to-random

Horizontal strips and diag

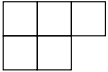
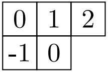
Horizontal strip: skew Young diagram λ/μ with at most one box per column. E.g.:



From now on, each λ/μ is a horizontal strip.

The diagonal (or content):

$$diag(\lambda) = \sum_{(i,j)\in\lambda} (j - i)$$

For $\lambda =$  $=$ 

, then $\mathrm{diag}(\lambda) = 0 + 1 + 2 - 1 + 0 = 2$.

Desarrangement tableaux

Desarrangement tableaux of μ (counted by d^μ)– tableaux with either:

- the 2nd entry of the first row is odd
- $|\mu|$ is even, $\mu_1 \leq 1$.

The tableau

1	3
2	

is a desarrangement tableaux while

1	2
3	

is not.

Theorem (Reiner-Saliola-Welker 2014)

$\sum_{\mu:\lambda/\mu \text{ is a horz strip}} d^\mu = d_\lambda$ *with bijection*

Theorem (Désarménien 1982)

$\sum_{\mu:|\mu|=n} d^\mu = d(n)$ *the number of derangements of* n

Eigenvector construction by Dieker-Saliola

- Get eigenvector for walk in $S^{\lambda+e_i}$ from one in S^λ from two functions.
- The first mirrors the random insertion of a new card:

$$\mathrm{sh}_i : S^\lambda \to M^{\lambda+e_i}$$

- And a replacement operator $\Theta_{i,j}$ used to project to $S^{\lambda+e_i}$
- Kernal of walk is dimension $d(n)$ (same as random-to-top) – basis indexed by desarrangements of μ $|\mu| = n$
- Get all eigenvectors indexed by λ/μ from adding blocks to μ basis, with $|\mu| \leq n$,
- Projection gives non-zero vector if λ/μ is a horizontal strip
- Old eigenvalue ϵ, new eigenvalue $\frac{1}{n^2}\left((n-1)^2\epsilon + n + \lambda_i - i\right)$

Eigenvalues of the walk

Theorem (Dieker-Saliola, 2015+)

The eigenvalue, $\mathrm{eig}(\lambda/\mu)$ *of the random to random walk corresponding to* (λ, μ)*, a horizontal strip, is*

$$\mathrm{eig}(\lambda/\mu) = \frac{1}{n^2}\left(\binom{n+1}{2} - \binom{|\mu|+1}{2} + \mathrm{diag}(\lambda) - \mathrm{diag}(\mu)\right)$$

and occurs with multiplicity $d_\lambda d^\mu$ *where* d_λ *is the number of standard Young tableaux of* λ *and* d^μ *is the number of desarrangement tableaux of* μ*.*

To use the l^2 bound on TV need bounds on eig, d_λ, and d^μ

Strategy for upper bound

We need to show for $t = \frac{3}{4} n \log(n) + cn$ that

$$\sum_{(\lambda,\mu)} d_\lambda d^\mu \left(\mathrm{eig}(\lambda/\mu)\right)^{2t} \leq Ce^{-2c}$$

- From spectral gap, $[n-1,1]/[2]$ eigenvalue $t = \frac{1}{2} n \log(n)$?
- With multiplicities of $[n-1,1]/\mu$, $t = n \log(n)$?
- $\mu = [k,1]$ for $k \leq \sqrt{n}$, $t = \frac{3}{4} n \log(n)$

- Cluster λ by λ_1, length of first row
- Get two bounds on eig in terms of λ_1: one for smallest μ, one if μ has $> n - \lambda_1$ more boxes
- Bound for d_λ (from Diaconis-Shahshahani)
- Get bound for d^μ utilizing bijection of Reiner-Saliola-Welker

$$\sum_{\mu : |\mu| = n - \lambda_1 + k} d^\mu \leq \binom{n - \lambda_1 + k}{n - \lambda_1 - 1} d_{\lambda/\lambda_1}$$

Open questions:

- Lower bound for random-to-random using the eigenvectors and/or with smaller window - currently $\frac{3}{4}n\log(n) + cn\log\log(n)$
- Symmetrizations of BHR hyperplane rearrangement walks. When does the symmetrization mix faster than the original?
- Ayyer-Schilling-Thery have a generalization of random-to-random to linear extensions of a finite poset. For posets that are not unions of chains, no longer rep. theory of S_n - can any of the eigenvector construction/mixing time bounds be mirrored there?
- Reiner-Saliola-Welker found two families of symmetrization of BHR walks $(\{[n-k, 1^k\}, \{[2^k, 1^{n-2k}]\})$ commute. Dieker-Saliola say their technique should find these eigenvectors as well. Mixing for arbitrary probability distributions on these families?

Thank you!