The Canonical Join Complex of the Tamari Lattice

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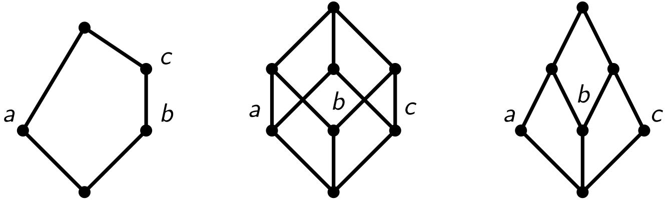
The canonical join representation (CJR)

- The CJR is a lattice-theoretic “minimal factorization” in terms of the join operation.
- There is an analogous factorization in terms of the meet called the **canonical meet representation**.
- Historically, these factorizations appeared as a tool for solving the *word problem* for free lattices.

The canonical join representation (CJR)

What is the canonical join representation for the top element?



Definition

The **canonical join representation** of an element w in L is the unique lowest irredundant expression $\bigvee A = w$.

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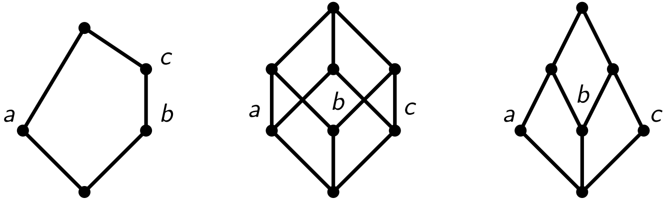


Observation

Each irredundant join of atoms is a canonical join representation.

Precise Definition

The **canonical join representation** of an element w in L is the unique lowest irredundant expression $\bigvee A = w$. More precisely:

- The expression $\bigvee A = w$ is a **join-representation** for w.
- The join $\bigvee A$ is **irredundant** if

$$\bigvee A' < \bigvee A \text{ for each proper subset } A' \subset A.$$

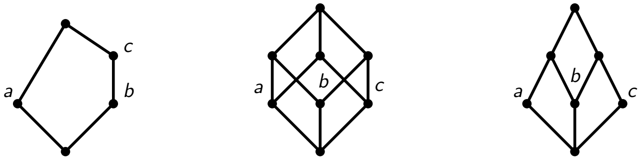
- If $\bigvee A$ is irredundant then A is an antichain.
- For $\bigvee A$ and $\bigvee B$ irredundant, we say A is "lower" than B if the order ideal generated by A is contained in the order ideal generated by B.

The canonical join complex

Definition

The **canonical join complex** $\Gamma(L)$ is the collection of subsets $A \subset L$ satisfying:

$$\bigvee A \text{ is a canonical join representation.}$$



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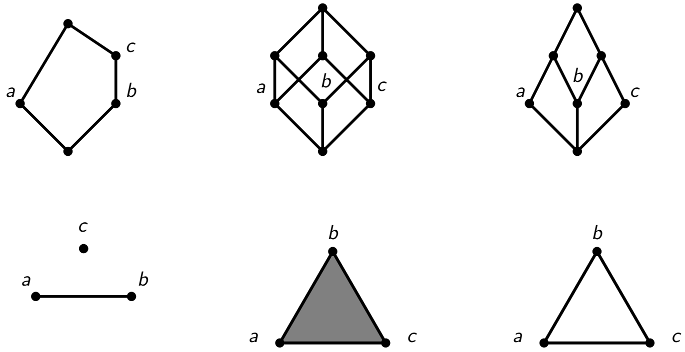
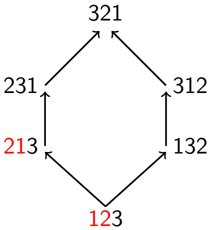


The weak order on S_n

Definition

- Write each permutation $w \in S_n$ as $w_1 \ldots w_n$ where $w_i = w(i)$.
- One moves up by a cover relation in the weak order by swapping consecutive entries in $w_1 \ldots w_n$ that are in order.



The type-B Coxeter group

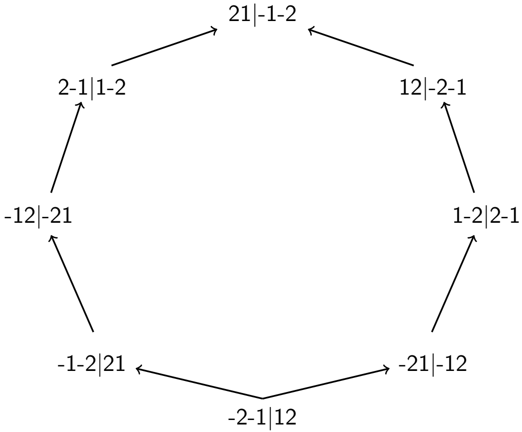


Figure: The weak order on the type-B Coxeter group B_2.

The Tamari lattice

Definition

- A permutation $w = w_1 \dots w_n$ avoids the 312-pattern if it has no subsequence of entries

$$w_i > w_l > w_k \text{ with } 1 \leqslant i < k < l \leqslant n.$$

- The **Tamari lattice** T_n is the subposet of the weak order on S_n induced by the permutations that avoid the 312-pattern.

- The **type-B Tamari lattice** T_n^s is the subposet of the weak order on B_n induced by the signed permutations that avoid:

  1. the 312-pattern where the "2" is positive and
  2. the 231-pattern where the "2" is negative.

Main results

Theorem [B.]

The canonical join complex of the Tamari lattice T_n is shellable.

- It is contractible when n is even.
- It is homotopy equivalent to a wedge of $\frac{1}{r+1}\binom{2r}{r}$ many spheres, all of dimension $r-1$, when $n = 2r+1$.

Theorem [B.]

The canonical join complex of the type-B Tamari lattice T_n^s is shellable.

The canonical join complex is homotopy equivalent to:

1. a wedge of $\mathrm{Cat}(B_r) = \binom{2r}{r}$ many spheres all of dimension $r-1$, when $n = 2r$;
2. a wedge of $\mathrm{Cat}^+(B_r) - \mathrm{Cat}(A_{r-2}) = 2\binom{2r-2}{r-2}$ many spheres, equally distributed in dimensions $r-1$ and $r-2$, when $n = 2r-1$ for $r > 1$.

Main results

Definition

- For each finite Coxeter group W, each c-Cambrian lattice is a lattice quotient of the weak order on W that is parametrized by an orientation c of the associated Coxeter diagram.
- When W is S_n and c is a linear orientation, we recover the Tamari lattice T_n.

Theorem [B.]

For each orientation c of the type-A Coxeter diagram, the canonical join complex of the corresponding c-Cambrian lattice is vertex decomposable.

Modeling CJR's in S_n

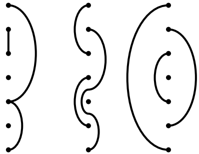
Noncrossing Arc Diagrams

- Draw n nodes in a vertical column and label them in increasing order from bottom to top.
- Each diagram consists of a (possibly empty) collection of curves called **arcs**.
- Each pair of arcs α and α' satisfies:

(C1) α and α' do not share the same top endpoint or the same bottom endpoint;

(C2) α and α' do not intersect in their interiors.

Modeling CJR's in S_n



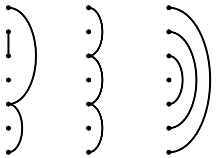
Definition

A collection of arcs are **compatible** if there is a noncrossing arc diagram containing them.

Theorem [Reading]

The canonical join complex of the weak order on S_n is isomorphic to the complex of compatible arcs on n nodes.

The canonical join complex of T_n



- A **right arc** is an arc that does not pass to the left of any node.
- A **simple arc** is an arc that connects consecutive nodes.

Corollary [Reading]

The canonical join complex of the Tamari lattice T_n is isomorphic to the complex of compatible right arcs on n nodes.

Notation

Write $\Delta(n)$ for the complex of compatible right arcs on n nodes.

Decreasing dimension

Observation

- For each $n > 2$, the complex $\Delta(n)$ is not pure.
- Any shelling of $\Delta(n)$ must begin with the diagram consisting of all simple arcs (i.e. the largest facet).

Theorem [B.]

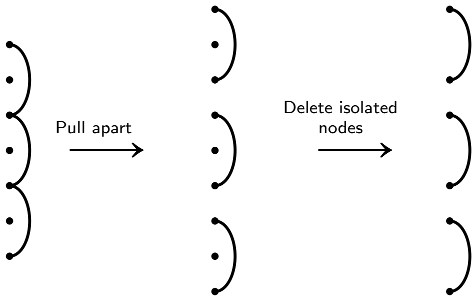
Suppose that $\mathcal{L} = F_1, \ldots, F_m$ is linear ordering of the facets of $\Delta(n)$ satisfying:

$$\text{If } |F_i| > |F_k| \text{ then } i < k.$$

Then $\mathcal{L}$ is a shelling of $\Delta(n)$, and F_i is a homology facet if and only if F_i contains no simple arcs.

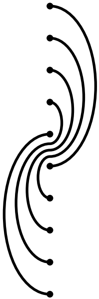
Counting homology facets

- When $n = 2r$, we prove by induction that each facet of $\Delta(n)$ contains a simple arc.
- When $n = 2r + 1$, we define a bijection from the set of homology facets to the set of noncrossing perfect matchings on $\{1, \ldots, 2r\}$.



Proof sketch for the type-B Tamari lattice

- Argue that the canonical join complex of T_n^s is shellable.
- Argue that a facet F is a homology facet if and only if F does not contain a simple symmetric arc.
- Define a bijection from the set of homology facets onto certain symmetric noncrossing perfect matchings.



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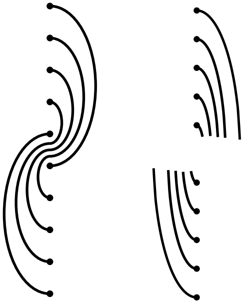

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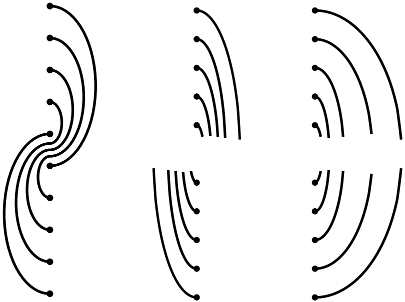

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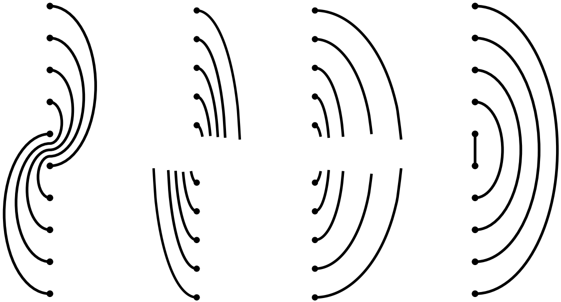

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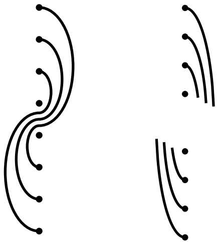

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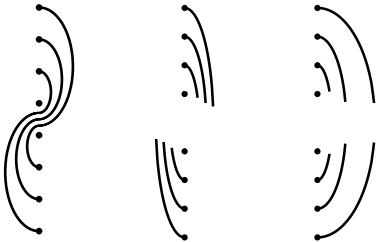

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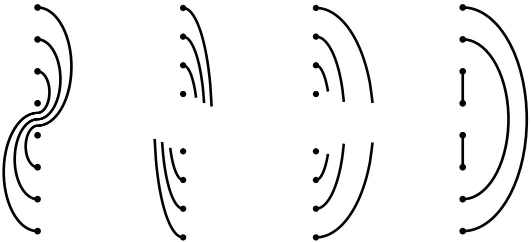

Thank you!