Algebraic Combinatorixx 2

Julie Beier (Earlham College),
Patricia Muldoon Brown (Armstrong State),
Rosa Orellana (Dartmouth),
Stephanie van Willigenburg (University of British Columbia)

May 14 - 19, 2017

1 Overview

Algebraic combinatorics is a large branch of mathematics with strong ties to many areas including representation theory, computing, knot theory, mathematical physics, symmetric functions and invariant theory. This workshop focused on the three areas below, inspired by the previous Algebraic Combinatorixx workshop and current trends.

- Algebraic combinatorics and representation theory: The representation theory of Lie algebras, quantum groups, Hecke algebras / double affine Hecke algebras, diagram algebras, symmetric groups and more utilize algebraic combinatorics and often take advantage of combinatorial objects such as crystals, Littlemann paths, tableaux, quivers, lattice paths, knots, and alternating sign matrices to discover new results.
- Algebraic combinatorics and geometry: This area includes the understanding of posets, lattices, simplicial complexes, CW-complexes, polytopes, and groups or arrangements also associated with these such as reflection groups, braid groups, hyperplane arrangements and descent algebras. This subarea of algebraic combinatorics connects to tropical geometry, Schubert calculus, Grassmannians and techniques in geometric representation theory.
- Algebraic combinatorics and combinatorial functions: In this category we are referring to symmetric functions such as Hall-Littlewood polynomials, Schur functions / k-Schur functions, and Weyl group characters as well as quasisymmetric functions such as those arising from random walks, P-partitions, 0-Hecke algebras and Hopf algebras. In addition, the Kazhdan-Lusztig polynomials fit into this category.

These three areas are clearly interconnected and all are of current interest in research. For example, the search for an algorithmic description of the coefficients that arise in the inner product of two Schur functions, known as Kronecker coefficients, has formed a very active research area over the past twenty years. Most recently this is due to deep connections with quantum information theory and the central role it plays within geometric complexity theory, which is an approach that seeks to settle the celebrated P versus NP problem – one of the several 1,000,000 Millennium Prize Problems set by the Clay Mathematics Institute. Meanwhile, the search for a combinatorial formula for the product of two Schubert polynomials is another longstanding open problem that has fuelled much research and development in the area due to its connections to algebraic geometry and the resolution of Hilbert's 15th Problem.

In addition to these mathematical goals, the workshop also sought to foster the community of women in algebraic combinatorics, cutting across the false research versus teaching institution divide. Thus, in addition to presentations, work was done during the week in teams; each team focusing on a different problem.

2 Presentation Highlights

The first day contained eight talks by the team leaders. In these, they presented background and problems that the teams would work on during the week. They are discussed in the following section. On the remaining days we heard a total of 15 excellent talks on current research. It is impossible to do justice to all of these talks but some highlights are mentioned below.

Hélène Barcelo spoke about discrete homotopy theory for graphs, known work of Babson, Barcelo, Kramer, de Longueville, Laubenbacher, Severs, Weaver and White, and argued that this theory is a good analogy to classical homotopy theory (cf. [3]). She went on to present a new idea developed by Barcelo, Capraro and White for a discrete homology on graphs and argued that it gives expected results such as Hurewicz Theorem and a discrete version of the Eilenbrerg-Steenrod axioms [4]. Moreover, she presents a conjecture that the path homologies of Grigor'yan, Muranov, and Yau are isomorphic to these discrete homologies for undirected graphs (cf. [10]).

Yui Cai presented work that was joint with Margaret Readdy and published this month on q-stirling numbers [8]. First, she argues that the q-Stirling numbers of the second kind can be understood using weights on allowable restricted growth words. Next we are presented with a new poset called a Stirling poset of the second kind. Cai was then able to give a poset explanation and homological interpretation to this work. Finally, we saw that using rook placements similar techniques could be used to produce the q-stirling numbers of the first kind.

The relationship between claw-contractible-free graphs and the e-positivity of a graph's chromatic symmetric function has been discussed since Stanley's 1995 paper [16]. Samantha Dahlberg presented results by Dahlberg, Foley and van Willigenburg, just recently posted on the archive, that settle this matter [9]. In this talk, we finally find the answer by learning of infinite families of graphs that are not claw-contractible and do not have chromatic symmetric functions that are e-positive. Moreover, we also saw one such family that is claw free but not e-positive.

3 Scientific Progress Made

During the week, we were split into eight teams working on eight different projects. Approximately eight hours were specifically designated for work in teams, and a number of teams met outside of the other workshop activities to work farther. In this section we briefly describe each project and summarize the progress made so far.

3.1 Chromatic Symmetric Functions

Team Leader: Angèle Hamel
Team Members: Julie Beier, Samantha Dahlberg, Maria Gillespie, Stephanie van Willigenburg

Consider all of the proper colorings of a graph G. With each coloring, K, associate a monomial $x_{K(v_1)}x_{K(v_2)}\cdots$. The chromatic polynomial is defined to be the sum of these monomials, and is known to be symmetric. Stanley [and Gasharov] conjecture that all claw free graphs have chromatic polynomials that are Schur positive [17].

We know from the work of Schilling that if one can create a crystal structure on the proper colorings of a graph G then the chromatic polynomial will indeed be Schur positive. We started with a special simple case of claw free graphs that are already known to be Schur positive, paths on n vertices. We spent the week working on a definition of Kashiwara operators f_i, e_i and of ϕ_i, ε_i for path colorings. Subsequently, we started to show that these satisfy the Stembridge axioms, which would give us a crystal structure. At the end of our time, we are still working on refining our rules so that they will generalize to colorings of non-path graphs, and proving our rules for the path case.

3.2 Quasisymmetric Analogue of Macdonald Polynomials

Team Leader: Sarah Mason
Team Members: Cristina Ballantine, Zajj Daugherty, Angela Hicks, Elizabeth Niese

This project was initiated to work toward finding a quasisymmetric analogue to the Macdonald polynomials. It became quickly evident that the naive approach of combining suitable nonsymmetric Macdonald polynomials would not lead to the desired outcome. Since the initial definition of the Macdonald polynomials is based on a modification of the Hall inner product on the power sum symmetric functions, the team decided to change approaches and work toward an appropriate modification of the inner product on QSym and NSym. In order to do this, a quasisymmetric power sum basis is needed, so this is the task the group spent time on and made significant progess.

Calculation 1: An explicit calculation of the Cauchy kernel:

$$\sum_{\beta} H^N_\beta(X) M^Q_\beta(Y) = \prod_j \left(1 - \left(\sum_{i\geq 1} x_i\right) y_j\right)^{-1}$$

Result 1: A quasisymmetric power sum (type 1) P^Q_α such that $\langle P^Q_\alpha, P^N_\beta\rangle = \delta_{\alpha\beta} z_\alpha$ where $z_\alpha = z_{\widetilde{\alpha}}$.

Result 2: A monomial expansion for the quasisymmetric power sum (type 1):

$$P^Q_\alpha = \sum_{\alpha \leq \beta} c^\alpha_\beta M^Q_\beta$$

Conjecture 1: For $\lambda \vdash n$,

$$p_\lambda = \sum_{\widetilde{\alpha}=\lambda} P^Q_\alpha.$$

3.3 Algebraic Voting Theory & Representations of $S_m[S_n]$

Team Leader: Kathryn Nyman
Team Members: Hélène Barcelo, Megan Bernstein, Sarah Bockting-Conrad, Erin McNicholas, Shira Viel

We consider the problem of selecting a committee consisting of one member chosen from m candidates in each of n departments. Voters rank the possible committees and a positions scoring method (such as the Borda count) can then be used to select a winning committee. We can view the profile and results spaces of voting information as $\mathbb{Q}S_m[S_n]$ modules, where the wreath product acts on the committees by permuting candidates within a department and permuting the departments. Since the positional voting procedure is a $\mathbb{Q}S_m[S_n]$ module homomorphism between the spaces, Schur's Lemma provides information on the voting information lost in the election process as well as the voting information that determines the outcome.

Our first goal is to decompose the results space R into simple $\mathbb{Q}S_m[S_n]$-submodules. Let $p(m)$ denote the number of partitions of m, and enumerate the set of partitions as $(\gamma_1, \ldots, \gamma_{p(m)})$. These partitions index the irreducible representations of S_m. Note that m of these partitions are hooks; we refer to the others as non-hook. In addition, we define a flat partition of k to be the partition consisting of a single part of size k.

We conjecture R has the following decomposition into simple $\mathbb{Q}S_m[S_n]$-modules:

$$R = \oplus_{\left(\substack{a_1+a_2+\ldots+a_{p(m)}=n \\ a_i=0 \ \forall \text{ non-hook } \gamma_i}\right)} S^{(a_1,\ldots,a_{p_m})} \tag{1}$$

where $S^{(a_1,\ldots,a_{p(m)})}$ denotes the simple $\mathbb{Q}S_m[S_n]$-module indexed by the multi-partition $(\alpha_1, \ldots, \alpha_{p(m)})$ of n where each α_i is the flat partition of $|\alpha_i| = a_i$. A proof of Matt Davis shows that Equation (1) holds in the case when $m = 2$; we are working on generalizing his proof. Towards that end, our primary result is a computation of the character of V_n, the representation of $S_m[S_n]$ corresponding to R.

Let $g \in S_n[S_m]$, viewed as a permutation of $\{1_1, 1_2, \ldots, 1_m, \ldots, n_1, n_2, \ldots, n_m\}$. (Think of i_k as candidate k from department i). Writing g as a product of disjoint cycles, for each $i \in [n]$ define the *cycle family* $F_i(g)$ to be the set of cycles in g containing i_k for some $k \in [m]$. By definition of the action of the wreath product, π determines the permutation of the departments. Thus, if i_k and j_l appear together in some cycle of g, π sends department i to department j and for all $s \in [m]$, i_s is sent to j_t for some $t \in [m]$. We define $m(F_i(g)) = \#\{j \in [n] : F_j(g) = F_i(g)\}$, i.e. the number of departments appearing in cycles containing candidates from department i. If $F_i(g) = F_j(g)$ we will say j is a *member* of $F_i(g)$. Thus, $m(F_i(g))$ is the number of members in $F_i(g)$. Finally, we let $\mathcal{F}(g) = \{F_i(g) : i \in [n]\}$.

Theorem 1. *For each $g \in S_n[S_m]$,*

$$\chi_{V_n}(g) = \prod_{F \in \mathcal{F}(g)} \#\{\text{ cycles } c \text{ in } g : \text{the length of } c \text{ is } m(F)\}. \tag{2}$$

3.4 cd-Index of Eulearian Posets

Team Leader: Margaret Readdy
Team Members: Yue Cai, Anastasia Chavez, Gizem Karaali, Heather Russell

The Readdy group is currently looking at two questions: how to compute the cd-index of important families of Eulerian posets and to discover combinatorial interpretations of the coefficients of the cd-index invariant. A lot of time was spent getting the group to speed on the geometry and coalgebraic structure of flag enumeration for polytopes. We are in the midst of understanding the cd-index of the family of cyclic polytopes and finding a more streamlined way to compute the cd-index.

The cyclic polytopes are a family of polytopes central to the Upper Bound Theorem; that is, for fixed dimension n and vertices f_0 the cyclic polytope maximizes the face vector. Further demonstrating this family's importance, for the cd-index of polytopes, more generally, Eulerian posets, Billera and Ehrenborg showed the cyclic polytope maximizes each coefficient of the cd-index [5]. In essence, this is an Upper Bound Theorem for the cd-index. The Readdy group also began discussing a combinatorial interpretation for the cd-index of the permutahedron. This polytope will very likely have a more tractable answer as we can use its inherent group structure.

3.5 On the Kronecker quasi-polynomials

Team Leader: Mercedes Rosas
Team Members: Marni Mishna, Sheila Sundaram

A central result in the representation theory of the symmetric group says that the set of partitions λ of n label the irreducible representations S^λ of $\mathbb{S}_n$.

In this setting, a major open problem is to understand the decomposition into irreducibles for the tensor product of representations. The Kronecker coefficients $g^\lambda_{\mu,\nu}$ are the coefficients governing this decomposition,

$$S^\mu \otimes S^\nu \cong \bigoplus_{\mu,\nu} g^\lambda_{\mu,\nu} S^\lambda$$

The coefficients $g^\lambda_{\mu,\nu}$ are intriguing and poorly understood, and determining a satisfactory formula for them is one of the major open questions in algebraic combinatorics.

Our project focuses on a related function. Let $Q^\lambda_{\mu,\nu}$ be the stretching function defined by

$$Q^\lambda_{\mu,\nu}(n) = g^{n\lambda}_{n\mu,n\nu}$$

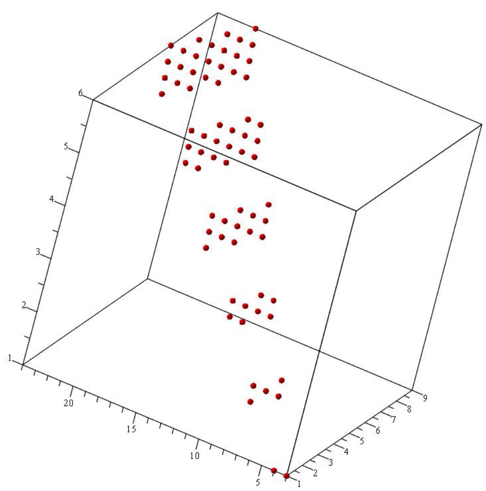
It is known that $Q^\lambda_{\mu,\nu}(n)$ is a quasi–polynomial in n, but little else is known about this function [11]. Recently, Balloni and Vergnes have developed algorithms to compute these polynomials when the length of the partitions is small [1, 2].

On the other hand, quasi–polynomials appear naturally in the study of dilations of rational polytopes. In this situation, Ehrhart quasi-polynomials count the number of integer points inside dilations of polytopes. Of course, not all quasi–polynomials correspond to such a counting function.

It is known that $Q^{\lambda}_{\mu,\nu}(n)$ is not, in general, the counting function for the number of integral points inside the dilations of any polytope, [11, 6].

In our research team we explored the *reduced* Kronecker coefficients, or equivalently, the $Q^{\lambda}_{\mu,\nu}(n)$ when μ, ν and λ are stable. Here, we refer to a stabilization phenomenon first observed by Murnaghan, [13]. These triples are important because they include all Littlewood–Richardson coefficients [12], they are themselves Kronecker coefficients, and the value of the usual Kronecker coefficients can be recovered from them, [7]. Finally, there is a wide consensus that the reduced Kronecker coefficients should be easier to understand than the general Kronecker coefficients.

In our work at Banff, we characterized the Kronecker polytope corresponding to those triples μ, ν and λ with $\ell(\mu), \ell(\nu) \leq 2$ and $\ell(\lambda) \leq 4$, i.e, $(2, 2, 4)$, and programmed our results into MAPLE. This is probably the situation where the Kronecker coefficients are best known; it belongs to the case of two-rowed shapes, one of a handful of cases for which explicit formulas exist [15]. In the following picture there is an example of the resulting polytope and some of its dilations.



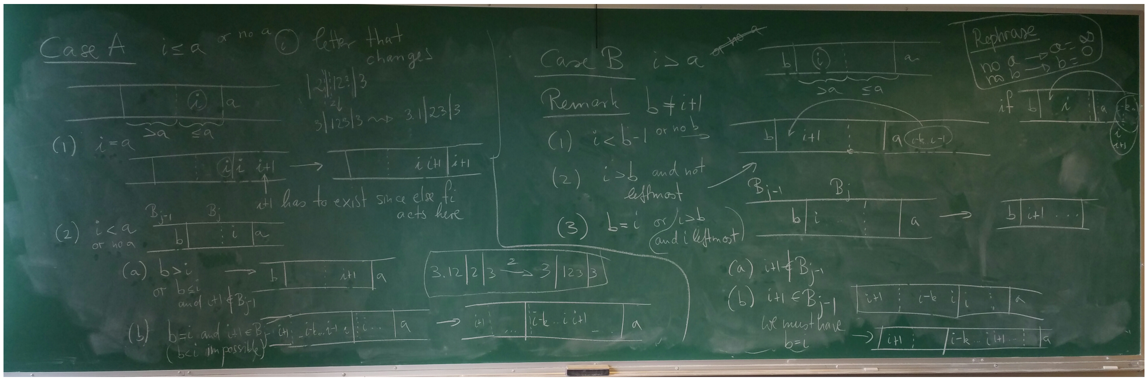
We plan to continue working on this project. Currently we are looking at the following case $(3, 3, 9)$. We are optimistic that getting a better understanding of this polynomial in this more unknown setting will lead to results in the general situation. In particular, we are trying to determine whether the resulting quasi-polynomial is the counting function for the dilations of a polytope, with the eventual goal of fully describing it.

3.6 Minimaj Crystal

Team Leader: Anne Schilling
Team Members: Georgia Benkart, Laura Colmenarejo, Pamela Harris, Rosa Orellana, Greta Panova, Martha Yip, Meesue Yoo

Suppose f is a symmetric function defined on a combinatorial object C, and f has a positive integer expansion in terms of Schur functions. Our team explored the question of whether this implies C has a crystal structure. As a test case for this investigation, we focused on the particular case that C is the set of ordered multiset partitions of $\{1, 2, \ldots, n\}$ into $k + 1$ nonempty blocks. The motivation to consider this particular example was a recent paper by Rhoades, "Ordered Set Partition Statistics and the Delta Conjecture" [14].

We have defined operators on C that we conjecture to be crystal operators and that preserve the minimaj statistic (see the picture below). We have worked out many examples that indicate that the conjecture is correct. We intend to prove that they satisfy Stembridge's axioms, which will imply that C has a crystal structure. We have a representation theoretic interpretation of C that should serve as a guide.



4 Reduced Word Bounds

Team Leader: Bridget Tenner
Team Members: Susanna Fishel, Elizabeth Milićević, Rebecca Patrias

Let $R(w)$ denote the set of reduced words for a fixed permutation w in the symmetric group S_n. Identifying all reduced words which are related by a sequence of commutation relations, one obtains the well-studied set of *commutation classes* for w, defined by $C(w) = R(w)/[ij \sim ji]$ where $|i-j| > 1$. Similarly, one defines the set of *braid classes* for w to be $B(w) = R(w)/\left[i(i+1)i \sim (i+1)i(i+1)\right]$, which have received comparatively less attention in the literature. Our goal for the week at BIRS was to study the braid classes of a fixed permutation, and we decided to focus on an enumerative problem in particular.

Problem: What is the number $|B(w)|$ of braid classes in the set of all reduced words $R(w)$?

Our strategy for approaching this problem was to simultaneously study the number of commutation and braid classes. Since $R(w)$ is a finite set, we can organize the elements of $R(w)$ into a table which lists the elements in each distinct commutation class $C_1, C_2, \dots, C_k$ across in rows, and the elements in each distinct braid class $B_1, B_2, \dots, B_m$ in columns. If we only record nonempty classes, there must be at least one element of $R(w)$ in each row and column. It is straightforward to prove that each entry in this table contains at most one element of $R(w)$. Combining this observation with the fact that any two reduced words for w are related by a sequence of commutation relations and/or braid relations, we obtain sharp upper and lower bounds on the number of reduced words in terms of the number of braid and commutation classes.

Proposition: For any $w \in S_n$, we have $|C(w)| + |B(w)| - 1 \leq |R(w)| \leq |C(w)||B(w)|$.

Having proved this proposition early in the week, our goal for the remainder of the workshop was to determine precisely which permutations realize these upper and lower bounds. Since both $|R(w)|$ and $|C(w)|$ have been fairly well-studied, achieving this goal would then yield closed formulas for $|B(w)|$ for such families of permutations.

Theorem 2 (Fishel-Milićević-Patrias-Tenner)**.** *There is a classification of the elements $w \in S_n$ which achieve the above upper and lower bounds on $|R(w)|$:*

1. $|R(w)| = |C(w)||B(w)|$ *if and only if* $|C(w)| = 1$ *or* $|B(w)| = 1$.
2. $|R(w)| = |C(w)| + |B(w)| - 1$ *if and only if w lies in one of a few special families.*

Regarding those elements which achieve the upper bound, the permutations which have a single commutation class are called *fully commutative* and can be characterized in terms of pattern avoidance. Similarly, it is possible to completely describe those permutations which have a single braid class. For the lower bound, in fact we prove something stronger by listing all permutations such that $|R(w)| = |C(w)| + |B(w)| - e$, where $e \in \{0, 1\}$. For each such permutation, there exists an ordering on the rows and columns in the table discussed above such that the resulting pattern is "serpentine", and this visualization provided a useful platform upon which to build our analysis. We will continue by writing up these results and submitting them for publication.

4.1 Ehrart Theory of Alcoved and Tropical Polytopes

Team Leader: Josephine Yu
Team Members: Emily Barnard, Carolina Benedetti, Patricia Brown

Our project focuses on characterizing the Ehrhart polynomials of alcoved and tropical polytopes. To this end we started our analysis focusing on 2-dimensional alcoved polytopes. It is known that the leading coefficient of the Ehrhart polynomial of a lattice polytope P is the (Euclidean) volume of the polytope and that the second leading coefficient is half of the boundary volume (normalized so that the standard simplex in a lattice has volume 1). For example, the standard 3-dimensional simplex has Ehrhart polynomial $x^3 + 3x^2 + 3x + 1$. We obtained the following lemma whose proof we omit.

Lemma: Let A be the normalized area (twice Euclidean area) and P be the (lattice) perimeter of an alcoved polygon. Then the pair (A, P) satisfies the following relations.

1. If the alcoved polygon has no interior lattice points, then $A + 2 = P$.
2. If the alcoved polygon contains an interior lattice point and is not three times a smallest simplex, then $A + 8 \geq 2P$ (Scott's inequality).
3. $6A \leq P^2$, and equality is achieved exactly for regular hexagons.

We know that not every pair (A, P) satisfying the conditions above can be achieved by an alcoved polygon. In view of this, we are making steps towards the following problems:

1. Characterize the pairs (A, P) that come from alcoved polygons. The next section may be useful.
2. Study higher dimensions.
3. Study h^* vector, which coincides with (in reverse order) the h-vector of the alcoved triangulation (or any unimodular triangulation) of the polytope.
4. Understand families of alcoved polytopes whose Ehrhart and/or h^* polynomial are palindromic and unimodal.
5. Some alcoved polytopes arise from matroids. Using the known Hopf algebras structures on matroids, can we define similar operations on (families of) alcoved polytopes?

Some of the results above can be extended in the setting of tropical polytopes as follows.

Lemma: For lattice tropical polygons, the area A and perimeter P satisfy $6A \leq P^2$, where the perimeter P of a pendant line segment is defined as twice its lattice length. The Ehrhart coefficients have the same interpretation.

We can decompose an alcoved polygon into ribbons by slicing it with parallel hyperplanes (lines). An alcoved polygon is thus determined by the numbers of triangles in the layers. A sequence of positive integers come from an alcoved polygon this way if and only if

1. it is unimodal,
2. only the peak can be even, and
3. only the peak can repeat.

Since there are three different hyperplane directions, different sequences may give the same alcoved polygon. We plan to characterize ”canonical” sequences giving rise to alcoved polytopes.

Each alcoved polytope has an “alcoved triangulation” obtained by slicing with all hyperplanes of the form $x_i - x_j = c$ where $c \in \mathbb{Z}$. The dual graph of an alcoved triangulation is a subgraph of the tiling of $\mathbb{R}^n/(1, 1, \ldots, 1)$ by regular permutohedra. Recall that all the two dimensional faces of a permutohedron (in any dimension) is either a square or a hexagon.

Conjecture: A subgraph of the permutohedral tiling is dual to the alcoved triangulation of an alcoved polytope if and only if

1. it is connected and induced,
2. if three vertices of a square are in the subgraph, then so is the fourth, and
3. if four vertices of a hexagon are in the subgraph, then so are the other two.

This gives us a way to study higher dimensional alcoved polytopes using graphs only.

Open problems.

1. Make connections to reduced words in affine Weyl groups.
2. Generalize to Weyl groups of other types.

5 Outcomes of the Meeting

As detailed above, all teams made significant progress on their problems. According to our team reports, they all have concrete plans to continue to work together to finish at least the problem they started at the meeting. Several teams are planning to submit papers within the next year and some have already scheduled times to meet in person again. Moreover, many people commented that they learned new mathematics or who to talk to about their work. This is fantastic and was an important goal.

MSRI generously agreed to provide $25,000 of funding for participants from this workshop to meet as teams this summer for continued work. We began accepting applications on June 1st and have selected four teams that will be meeting at MSRI during July and August of 2017. We are appreciative of the MSRI's support and believe their support illustrates the significance of the research completed during this workshop.

We also would like to acknowledge that the Association for Women in Mathematics, AWM, provided funding for several of the US based participants to attend this workshop. This was absolutely vital to us creating a community with such diversity, and we appreciate their support. The financial support came from both their established travel funding program and from the new Advance grant program, NSF-HRD 1500481 - AWM ADVANCE grant.

In addition to the mathematical outcomes, it is clear from the evaluations that the community was enhanced as almost every respondent referred to the value of the networking, mentoring and conversations that occurred during the week. We had participants from numerous countries, people at many different points in their career, and women who work at a wide range of institutions. Respondents overwhelmingly valued the team structure and felt the conference was well organized. Every respondent said that they believed this was a valuable use of their time and several commented that they strongly hope there will be another Algebraic Combinatorixx in the future.

The comments from the week and in the evaluations were incredibly positive, and we share the participants' gratitude to BIRS for providing such a fantastic space to work and amazing staff to help. Below is a small sample of some of the comments about the significance of the week for them:

- “I feel like I learned more algebraic combinatorics in 4 days than in years before. I learned so much. I also made personal and professional connections that I will maintain going forward”
- “It has helped to re-energize me. ... I learned a lot of cool math, and my group made serious progress on our project and has plans to continue.”

- (Listed as a strength of the workshop) "Bringing together an extraordinary group of mathematicians - [the] talks were so impressive and informative"
- "I think my group will write a paper and continue to collaborate."
- "My team made great progress and I am confident that we will be able to write a nice paper."
- "The supportive community here was exceptional."
- "The discussions at meals, in groups, in the evening discussions were all amazingly friendly and helpful."
- "I most appreciated the way the organizers and more experienced attendees fostered a positive, supportive environment for collaboration and mentoring. I received so much valuable advice and mathematical insight."
- "I really appreciated that there were different kinds of mathematicians here - those from liberal arts colleges [and] those from R1 schools. That mix was helpful; I benefited from many perspectives and many different ways of doing math."
- "This networking is alone worth the trip."
- "I just really enjoyed the experience overall and felt energized professionally and personally."
- "I think this has been my favorite conference/workshop!"
- "I would do this every summer if I could."

References

[1] V. Baldoni and M Vergne, Multiplicity of compact group representations and applications to Kronecker coefficients, *ArXiv e-prints*, (2015).

[2] V. Baldoni, M. Vergne, and M. Walter, Computation of Dilated Kronecker Coefficients, *ArXiv e-prints*, (2016).

[3] E. Babson, H. Barcelo, M. de Longueville and R. Laubenbacher, Homotopy theory for graphs, *J. Algebraic Comb.* **24** (2006) 31-44.

[4] H. Barcelo, V. Capraro, and J. White, Discrete homology theory for metric spaces, *Bulletin of the London Math. Soc.* **46** (2014) 889-905.

[5] L. Billera, and R. Ehrenborg, Monotonicity of the cd-index for polytopes, *Mathematische Zeitschrift* **233** (2000), 421-441.

[6] , E. Briand, R. Orellana, and M. Rosas, Reduced Kronecker coefficients and counter–examples to Mulmuley's strong conjecture SH, *ArXiv:0810.3163* (2008)

[7] E. Briand, R. Orellana, and M. Rosas, The stability of the Kronecker product of Schur functions, *J. Algebra*, **331** (2011), 11–27.

[8] Y. Cai and M. Readdy, q-Stirling numbers: A new view, *Advances in Applied Mathematics* **86** (2017), 50 - 80.

[9] S. Dahlberg, A. Foley, and S. van Willigenburg, Resolving Stanley's e-positivity of claw-contractible-free graphs, *arXif:1703.05770 [math.CO]* (2017).

[10] A. Grigor'yan, Y. Muranov, and S.-T. Yau, Homologies of digraphs and Kunneth formulas, *Comm. Anal. Geom*, To appear.

[11] R. King, Some remarks on characters of symmetric groups, Schur functions, Littlewood-Richardson and Kronecker coefficients. *"Mathematical Foundations of Quantum Information", Sevilla*, http://congreso.us.es/enredo2009/School_files/Sevilla_King.pdf (2009).

[12] D. E. Littlewood., The Kronecker product of symmetric group representations. *J. London Math. Soc.*, **31** (1956), 89-93.

[13] F. D. Murnaghan, The Characters of the Symmetric Group. *Amer. J. Math.*, **59** (1937), 739-753.

[14] B. Rhoades, Ordered set partition statistics and the delta conjecture. *Arxiv:1605.04007* (2016).

[15] M. H. Rosas., The Kronecker product of Schur functions indexed by two-row shapes or hook shapes. *J. Algebraic Combin.*, **14** (2001), 153–173.

[16] R. Stanley, A Symmetric function generalization of the chromatic polynomial of a graph, *Adv. Math.* **111** (1995), 166-194.

[17] R. Stanley, Graph colorings and related symmetric functions: ideas and applications, *Discrete Math* **193** (1998), 267-286.