Fields of Character Values in Finite Groups

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BIRS

New Perspectives in Representation Theory of Finite Groups

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$\mathbb{F} \subseteq \mathbb{C}$, G a finite group
$x \in G$, $\chi \in \mathrm{Irr}(G)$.

Definition

If $\chi(x) \in \mathbb{F}$ *for all* $x \in G$, χ *is an* $\mathbb{F}$-**character**.

$\mathrm{Irr}_{\mathbb{F}}(G) := \{\chi \in \mathrm{Irr}(G) : \chi \textit{ is an } \mathbb{F}\textit{-character}\}$.

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Definition

*If $\chi(x) \in \mathbb{F}$ for all $\chi \in \mathrm{Irr}(G)$, then x is an $\mathbb{F}$-**element**.*

$\mathrm{Cl}_{\mathbb{F}}(G) := \{x^G \in \mathrm{Cl}(G) : x$ *is an* $\mathbb{F}$*-element*$\}$ *($\mathbb{F}$-**classes**).*

Note: $x \in G$ is rational if and only if x and x^t are G-conjugate whenever $(t, o(x)) = 1$.

Motivation

Some classical results:

- (Thompson). If every non-linear $\chi \in \mathrm{Irr}(G)$ has degree divisible by p then G has a normal p-complement.
- (Ito-Michler). If $p \nmid \chi(1)$ for every $\chi \in \mathrm{Irr}(G)$ then G has a normal, abelian Sylow p-subgroup.
- If $\#\{\chi(1) : \chi \in \mathrm{Irr}(G)\} \leq 3$ then G is solvable.

Motivation

"$\mathbb{F}$-versions" of classical results:

- (Navarro-Tiep). If every non-linear $\chi \in \mathrm{Irr}_{\mathbb{Q}_p}(G)$ has degree divisible by p then G has a normal p-complement.
- (Dolfi-N-T). If $2 \nmid \chi(1)$ for every $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ then G has a normal, abelian Sylow 2-subgroup.
- (N-Sanus-T). If $\#\{\chi(1) : \chi \in \mathrm{Irr}_{\mathbb{R}}(G)\} \leq 3$ then G is solvable.

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Theorem (Navarro-Tiep, 2008)

Suppose G is a finite group. Then

- $|\mathrm{Irr}_{\mathbb{Q}}(G)| = 1$ *if and only if* $|\mathrm{Cl}_{\mathbb{Q}}(G)| = 1$.
- $|\mathrm{Irr}_{\mathbb{Q}}(G)| = 2$ *if and only if* $|\mathrm{Cl}_{\mathbb{Q}}(G)| = 2$.

Both parts require CFSG.

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Conjecture

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Remark: GAP $\mathrm{SmallGroup}(672, 128)$ has $|\mathrm{Irr}_{\mathbb{Q}}(G)| = 4$ but $|\mathrm{Cl}_{\mathbb{Q}}(G)| = 6$.

The Navarro-Tiep Conjecture

Theorem (R. 2017)

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Proof outline:

1. Assume G has no rational element of order 4 – Use Brauer's character table lemma.
2. If G has a rational element of order 4, the structure of G can be very tightly controlled – Enough control to handle explicitly.

Galois action and Brauer's lemma

The setup:

- Assume $|G| = n$, $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$, and $\zeta^\sigma = \zeta^s$.
- If $x \in G$, define $x^\sigma = x^s$ and $(x^G)^\sigma = (x^\sigma)^G$.
- If $\chi \in \mathrm{Irr}(G)$, define $\chi^\sigma = \sigma^{-1} \circ \chi$.

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Lemma (Brauer)

Let $|G| = n$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. Then σ fixes equal numbers of conjugacy classes and irreducible characters of G.

$|\mathrm{Cl}_{\mathbb{Q}}(G)| = 3$, No rational elements of order 4

- Let $|G| = n$ and $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$.
- $\mathcal{G} = \prod_{p|n} \mathcal{G}_p$, $\mathcal{G}_p \simeq \mathrm{Gal}(\mathbb{Q}_{n_p}/\mathbb{Q})$, fixing p'-roots of unity.
- Choose generators σ_p for $\mathcal{G}_p$ when p odd and let $\mathcal{G}_2 = \langle \sigma_0 \rangle \times \langle \sigma_2 \rangle$, where σ_2 is complex-conjugation.
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- If $|\mathrm{Irr}_{\mathbb{Q}}(G)| \geq 4$ then G has 4 σ-fixed classes; three rational ones and another, y^G.
- Rational elements have orders $1, 2, \ell$ (prime). Replacing y by some power, can assume that $o(y)$ is 4, 2ℓ, or a p-power (p odd), and still non-rational.
- If $o(y)$ is r' then $y^{\sigma_r} = y \ldots$

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- If $o(y)$ is r' then $y^{\sigma_r} = y \ldots$ so $(y^G)^\sigma = y^G$ implies y is rational.
    - E.g. if $o(y) = 4$ then $y^G = (y^G)^\sigma = (y^\sigma)^G = (y^{\sigma_2})^G = (y^{-1})^G$.

$|\mathrm{Cl}_{\mathbb{Q}}(G)| = 3$, Rational elements with order $1, 2, 4$

If G is non-solvable, study $F^*(G) = E(G)F(G)$

- $E(G)$ is the **layer** (product of subnormal, quasisimple subgroups), $F(G)$ the Fitting subgroup.

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- If $E(G) > 1$, show that $S := E(G) = \mathrm{PSL}_2(3^{2f+1})$ or $\mathrm{SL}_2(3^{2f+1})$.
    - Any other quasisimple group has a rational element of order 3 or 5.
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Lemma

If $S = \mathrm{SL}_2(q)$ or $\mathrm{PSL}_2(q)$, $S \lhd G$, and $|G : S|$ is odd then $|\mathrm{Irr}_{\mathbb{Q}}(G)| = |\mathrm{Cl}_{\mathbb{Q}}(G)|$.

The non-solvable case (cont.)

- If $E(G) = 1$, then $F := F^*(G) = O_2(G)$ and $G/Z(F) \leq \mathrm{Aut}(F)$ is non-solvable – so F has more than one involution.

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- Hering's classification: G/F has a rational element of order 3.

$|\mathrm{Cl}_{\mathbb{Q}}(G)| = 3$, Rational elements with order $1, 2, 4$

If G is solvable, can still assume $O_{2'}(G) = 1$

- G has 2-length one (Isaacs-Navarro), so $P \triangleleft G$ ($P \in \mathrm{Syl}_2(G)$)

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- If P has a unique involution:
  - P cyclic – No
  - P generalized quaternion – all the order 4 elements are conjugate in G, so a 2-complement acts non-trivially. So $P = Q_8$.

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- If P has a unique involution:
  - P cyclic – No
  - P generalized quaternion – all the order 4 elements are conjugate in G, so a 2-complement acts non-trivially. So $P = Q_8$.
- Otherwise (Thompson):
  - P homocyclic – No
  - P a **Suzuki** 2-**group** – $\exp(P) = 4$.

Assuming that $O_{2'}(G) = 1$

Theorem (Isaacs-Navarro)

*If $N \triangleleft G$ and N is $\mathbb{Q}$-**free** in G then*

(i) $\mathrm{Irr}_{\mathbb{Q}}(G) = \mathrm{Irr}_{\mathbb{Q}}(G/N)$

(ii) *The map $x \mapsto xN$ induces a bijection $\mathrm{Cl}_{\mathbb{Q}}(G) \to \mathrm{Cl}_{\mathbb{Q}}(G/N)$.*

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Theorem (R. 2017)

Fix a prime p. If $N \lhd G$ contains no non-trivial p-regular $\mathbb{F}$-elements of G then

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(ii) *The map $x \mapsto xN$ induces a bijection $\mathrm{Cl}^{\circ}_{\mathbb{F}}(G) \to \mathrm{Cl}^{\circ}_{\mathbb{F}}(G/N)$.*

Here, $\mathrm{Cl}^{\circ}(G)$ is the set of p-regular classes.

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1. If (G, N) is a minimal counterexample, show that N is minimal-normal.
2. If N is solvable, everything basically follows from Isaacs-Navarro.
3. If N is non-solvable, $N = S_1 \times \cdots \times S_n \simeq S^n$. S contains (rational) involutions and, unless $S = \mathrm{PSL}_2(3^{2f+1})$, rational elements of order 3 or 5.

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4. The critical case, then, is $p = 2$ and $N \simeq S^n$, $S = \mathrm{PSL}_2(3^{2f+1})$.

$\mathrm{Cl}^{\circ}_{\mathbb{F}}(G) \to \mathrm{Cl}^{\circ}_{\mathbb{F}}(G/N)$ is injective

- Need to show: If $xN = yN \in G/N$ are p-regular $\mathbb{F}$-elements, then $x^G = y^G$.
- Assume (G, N, x, y) is a minimal counterexample with $p = 2$ and $N = S_1 \times \cdots \times S_n = S^n$, $S \simeq \mathrm{PSL}_2(3^{2f+1})$.

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Lemma

If $N \lhd G \leq \mathrm{Aut}(N)$, ℓ is a prime dividing $|N|$, and some $g \in G$ has an orbit of length $t := |\mathbb{Q}_\ell : (\mathbb{Q}_\ell \cap \mathbb{F})|$ on the factors of N, then N has an $\mathbb{F}$-element of order ℓ.

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- Choose $x \in S$ with order ℓ.
- Construct an element $v = (v_1, \ldots, v_n)$ of N where:
  - $v_1 = x$;
  - v_i is either 1, or a power of x twisted by outer automorphisms, depending on action of g.

$\mathrm{Cl}^\circ_{\mathbb{F}}(G) \to \mathrm{Cl}^\circ_{\mathbb{F}}(G/N)$ is injective

Lemma

Assume $N \lhd G \leq \mathrm{Aut}(N)$ that N° is $\mathbb{F}$-free in G but that G° contains an $\mathbb{F}$-element of prime order ℓ. Then some $g \in G$ has an orbit of length $t = |\mathbb{Q}_\ell : (\mathbb{Q}_\ell \cap \mathbb{F})|$ on the factors of N.

- $\mathrm{Aut}(N) = \mathrm{Aut}(S) \wr \mathrm{Sym}(n)$ – the key case is when the $\mathbb{F}$-element has non-trivial image in $\mathrm{Sym}(n)$.
- In $\mathrm{Sym}(n)$, if x is an $\mathbb{F}$-element of order ℓ and $x^g = x^a$ then explicitly compare the cycle decompositions of x^g and x^a.

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Lemma

Assume $N \lhd G \leq \mathrm{Aut}(N)$ *that* N° *is* $\mathbb{F}$*-free in* G *but that* G° *contains an* $\mathbb{F}$*-element of prime order* ℓ. *Then some* $g \in G$ *has an orbit of length* $t = |\mathbb{Q}_\ell : (\mathbb{Q}_\ell \cap \mathbb{F})|$ *on the factors of* N.

- $x, y \in G$ are p-regular $\mathbb{F}$-elements, N° is $\mathbb{F}$-free in G, $xN = yN \in G/N$.
- Let $\pi = \pi(N)$. Then $x_\pi, y_\pi \in C_G(N)$. In fact, can show that $x_\pi = y_\pi$.
- Schur-Zassenhaus: $x_{\pi'}$ and $y_{\pi'}$ are N-conjugate.
- Conclude that $x^G = y^G$.

What about the other half of Navarro-Tiep Conjecture?

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Theorem (R. 2017)

Let G *be a finite non-solvable group and assume that* $|\mathrm{Irr}_{\mathbb{Q}}(G)| = 3$. *Let* $M := O^{2'}(G)$ *and let* $N := O_{2'}(M)$. *Then* $M/N =: S$ *is quasisimple, and* $|\mathrm{Irr}_{\mathbb{Q}}(S)| \leq 3$.

What about the other half of Navarro-Tiep Conjecture?

Conjecture (Navarro-Tiep)

$|\mathrm{Cl}_{\mathbb{Q}}(G)| = 3$ *if and only if* $|\mathrm{Irr}_{\mathbb{Q}}(G)| = 3$.

Theorem (R. 2017)

Let G be a finite non-solvable group and assume that $|\mathrm{Irr}_{\mathbb{Q}}(G)| = 3$. Let $M := O^{2'}(G)$ and let $N := O_{2'}(M)$. Then $M/N =: S$ is quasisimple, and $|\mathrm{Irr}_{\mathbb{Q}}(S)| \leq 3$.

Remarks:

1. The possibilities for S are:
   - $PSL_2(3^{2f+1})$
   - $\mathrm{SL}_2(3^{2f+1})$
   - $\mathrm{SL}_2(2^n)$
   - $\mathrm{PSL}_2(q)$, $q \equiv \pm 5 \pmod{24}$
   - ${}^2B_2(2^{2n+1})$
2. If G is solvable with $|\mathrm{Irr}_{\mathbb{Q}}(G)| = 3$ then G has 2-length one (Tent).

What about groups with $|\mathrm{Irr}_{\mathbb{Q}}(G)| = 3$?

Theorem (R. 2017)

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Theorem (R. 2017)

Assume G is non-solvable and $|\mathrm{Irr}_{\mathbb{Q}}(G)| = 3$. With the previous notation, $|\mathrm{Cl}_{\mathbb{Q}}(G)| = 3$, except possibly when $S = \mathrm{PSL}_2(3^{2f+1})$.

What would it take to finish the N-T Conjecture?

1. G non-solvable: $M/N = \mathrm{PSL}_2(3^{2f+1})$
   - Assume N contains non-trivial rational elements.
   - Can handle the case when N is minimal-normal in M.
   - In other cases, need stronger tools to relate rational elements of G with those in quotients by subgroups of N.

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2. G solvable:
   - If $O_{2'}(G)$ contains non-trivial rational elements, have similar problems.
   - Even if $O_{2'}(G) = 1$ (so $P \triangleleft G$) need a way to control the number of classes of involutions in P.

Thank you!