MASSEY PRODUCTS AND UNIQUENESS OF A_∞-ALGEBRA STRUCTURES

Operations in Highly Structured Homology Theories, Banff, 22–27 May 2016.

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Universidad de Sevilla

Two classical problems

Given a spectrum with a homotopy associative multiplication, does it come from an A_∞-algebra structure? If so, is it unique?

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Kadesihvili'88

Robinson'89

Rezk'98

Tamarkin'98

Lazarev'01

Goerss–Hopkins'04

Angeltveit'08

Roitzheim–Whitehouse'11

…

These questions have been considered by many people.

For spectra, chain complexes, simplicial modules…

For many operads: $\mathtt{A}_\infty$, $\mathtt{E}_\infty$, $\mathtt{L}_\infty$, $\mathtt{G}_\infty$…

Using (variations of) Hochschild cohomology.

The space of A_∞-algebras

$$B\mathcal{A}_\infty \longrightarrow B\mathcal{S}$$

$$\begin{aligned} \mathcal{A}_\infty &= \text{category of } A_\infty\text{-algebras} \\ \mathcal{S} &= \text{category of spectra} \\ B\mathcal{M} &= \text{classifying space of a model category } \mathcal{M} \\ &= \text{nerve of the category of weak equivalences in } \mathcal{M} \end{aligned}$$

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$$B\mathcal{A}_\infty = \lim B\mathcal{A}_n \to \cdots \to B\mathcal{A}_{n+1} \longrightarrow B\mathcal{A}_n \to \cdots \to B\mathcal{A}_1 = B\mathcal{S}$$

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The space of A_∞-algebras

A fixed base point $R \in B\mathcal{A}_\infty$ allows for the construction of the Bousfield–Kan'72 FRINGED SPECTRAL SEQUENCE of the tower,

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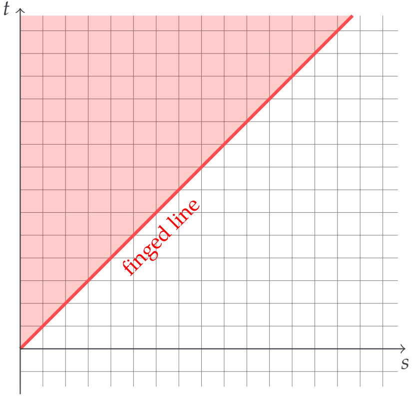
Bousfield–Kan's fringed spectral sequence

$$E_2^{s,t} \Longrightarrow \pi_{t-s}(B\mathcal{A}_\infty, R)$$



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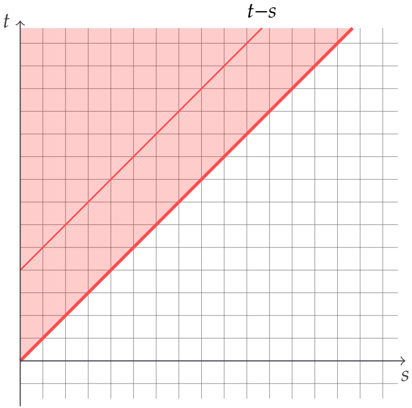

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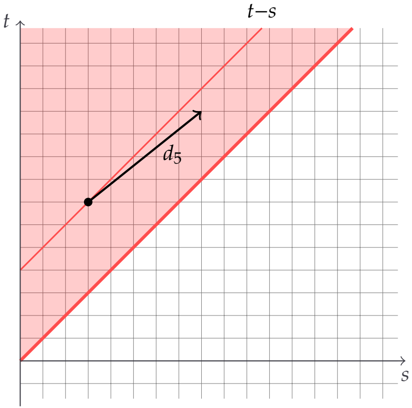

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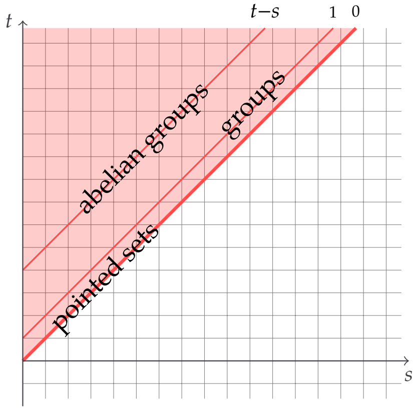
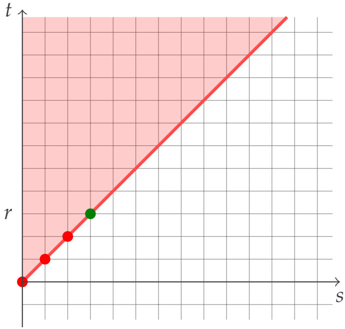


The fringed line and uniqueness

$E_r^{s,s}$ = weak equivalence classes of A_{r+1}-algebras which extend to A_{r+s}-algebras and restrict to the same A_r-algebra as R, $s \leq r$.



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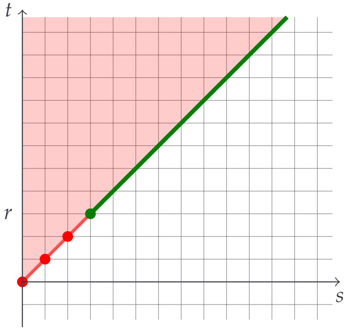

If the green line vanishes, the A_r-algebra underlying R extends uniquely to an A_n-algebra for all $n \geq r$.

The fringed line and uniqueness

The obstruction to A_∞-uniqueness is the $\lim^1$ in the Milnor s.e.s.

$$\lim_n{}^1 \pi_1(B\mathcal{A}_n, R) \hookrightarrow \pi_0(B\mathcal{A}_\infty, R) \twoheadrightarrow \lim_n \pi_0(B\mathcal{A}_n, R)$$

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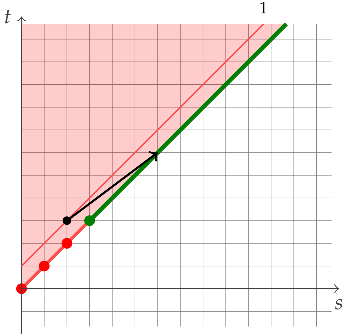
which vanishes provided $\lim_n^1 E_n^{s,s+1} = 0$ for all $s \geq 0$,



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Corollary (Kadeishvili'88)

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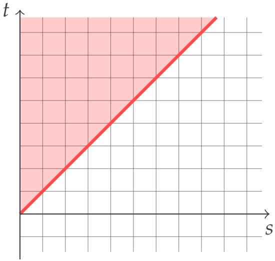
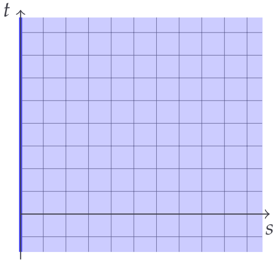
What about existence? We could even be unable to choose a base point in $B\mathcal{A}_\infty$ with given algebra $\pi_* R$.

Below the fringed line and existence (Angeltveit'08 and '11)

$$E_2^{s,t} \Rightarrow \pi_{t-s}(B\mathcal{A}_\infty, R)$$

$$HH^{s,-t}(\pi_* R) \Rightarrow HH^{s-t}(R)$$





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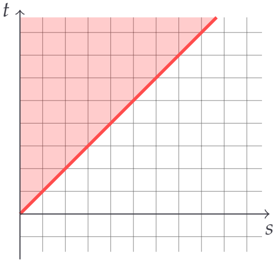

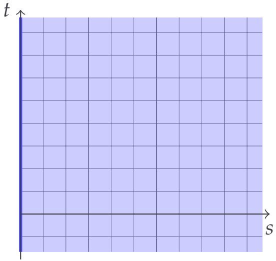

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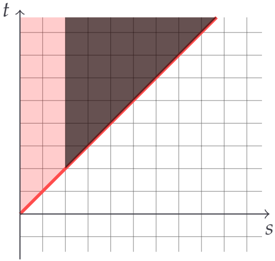

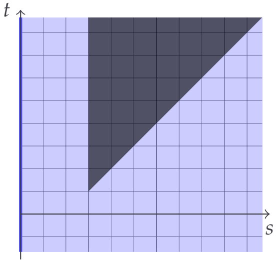

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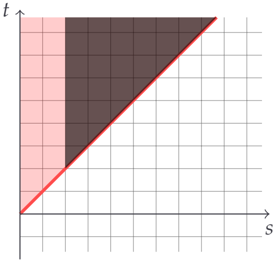

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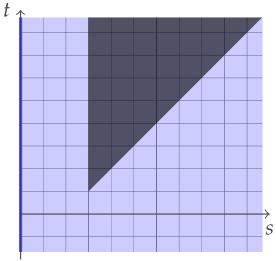

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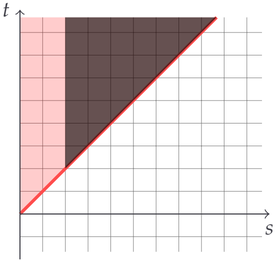

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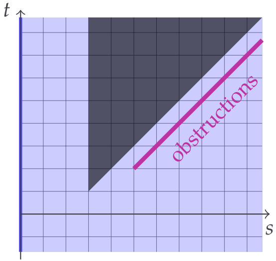

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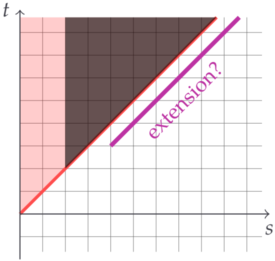

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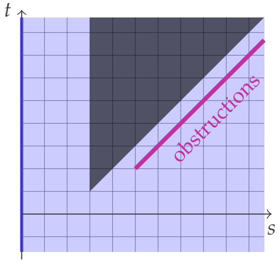

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Extending the fringed spectral sequence

Bousfield'89 defined for the tower of the totalization of a cosimplicial space:

- an EXTENSION of the fringed spectral sequence, given a global base point;
- TRUNCATED spectral sequences, given an intermediate base point;
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Our tower is not naturally like this. We proceed in a different way, suitable for explicit computations beyond the second page.

Extending the fringed spectral sequence

$\mathcal{S} = Hk$-module spectra, k a field (in order to stay safe).

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\downarrow & & \downarrow & \text{(Rezk'96)} & \downarrow & \text{pulling back} & & \downarrow X \\
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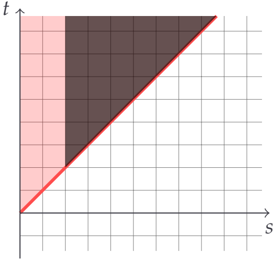
\end{array}$$

$$\begin{aligned}
\mathtt{A}_n &= \text{operad for } A_n\text{-algebras} \\
\mathtt{End}_X &= \text{the endomorphism operad of a spectrum } X \\
\mathtt{Q}^{\mathtt{P}} &= \mathrm{Map}(\mathtt{P}, \mathtt{Q}) \\
&= \text{the space of maps } \mathtt{P} \to \mathtt{Q} \text{ in the category of (non-}\Sigma\text{) operads}
\end{aligned}$$

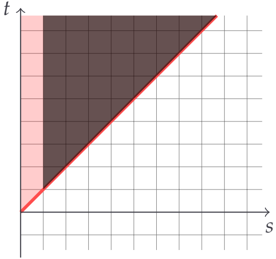
Extending the fringed spectral sequence

The spectral sequences of these towers substantially overlap.

S.s. of $\{B\mathcal{A}_n\}_{n\geq 1}$



S.s. of $\{\mathtt{End}_X^{\mathtt{A}_n}\}_{n\geq 2}$



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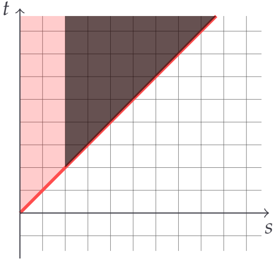

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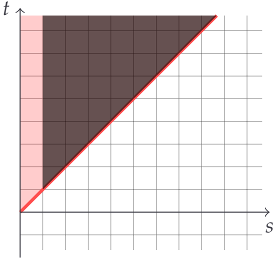

We can take advantage of the homotopy theory of $\mathtt{A}_\infty$.

From now on, we work with the second one.

Where do classical obstructions come from?

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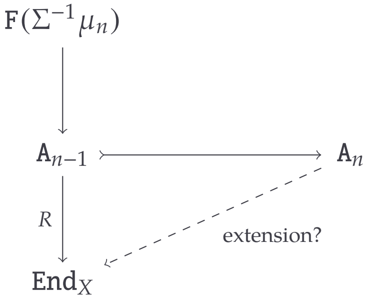

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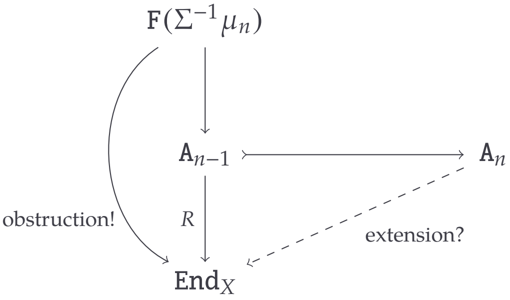

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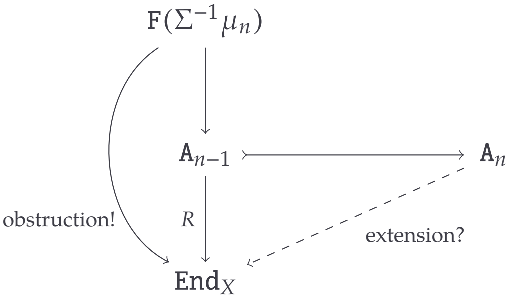

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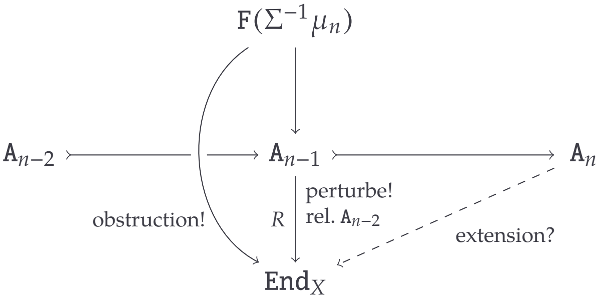

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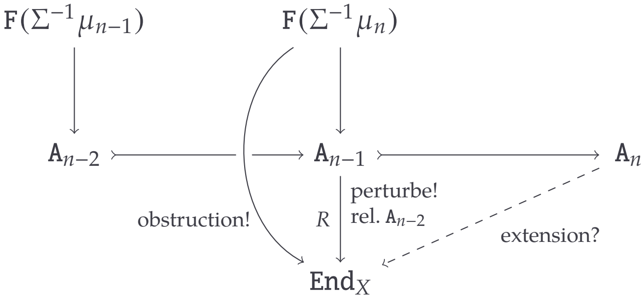

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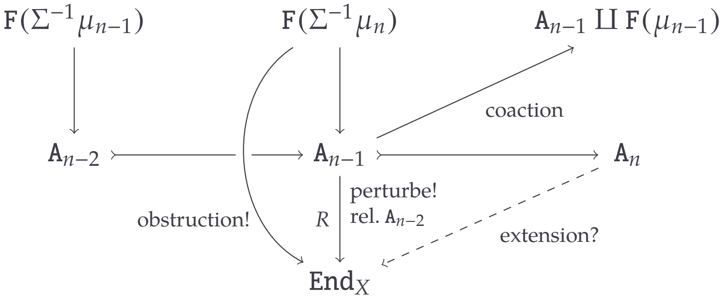

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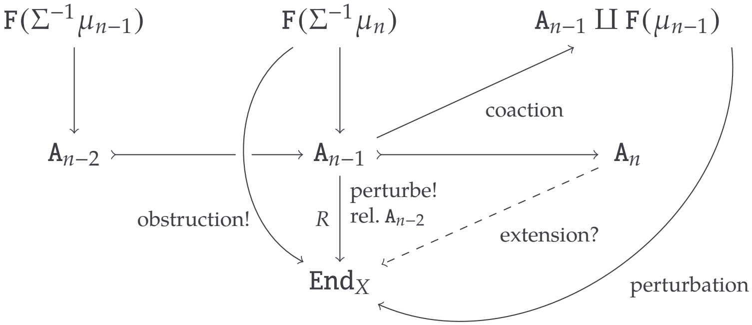

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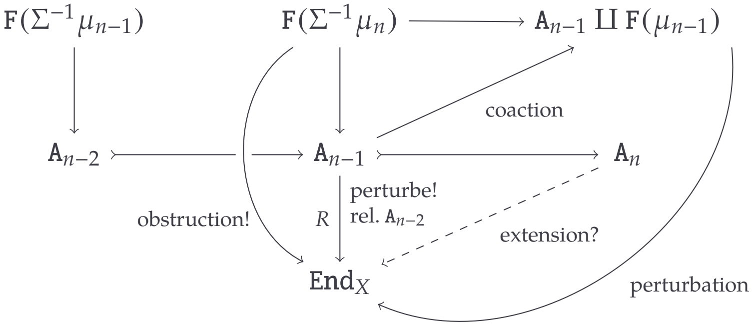

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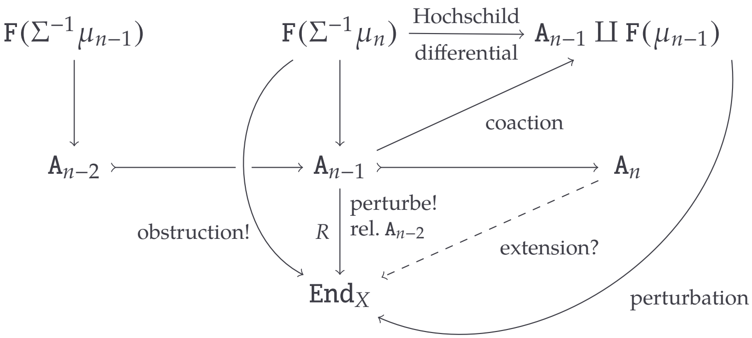

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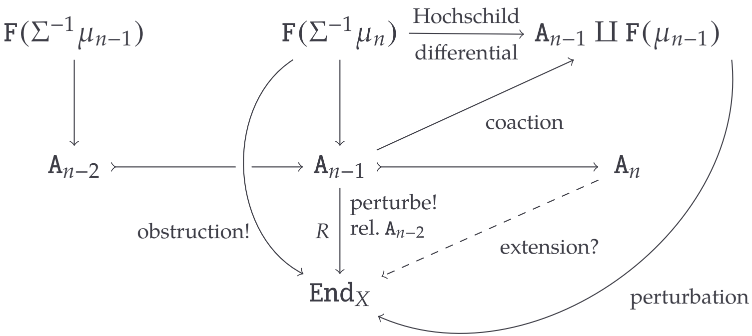

The obstruction is in (over a field, $X = \pi_* R$) for $n \geq 4$,

$$\mathtt{End}_X(n)^{3-n} = \underbrace{\mathtt{Hom}(X^{\otimes n}, X)^{3-n}}_{\text{Hochschild cplx.}} \rightsquigarrow HH^{n,3-n}(\pi_* R).$$

Where do new obstructions come from?

Proposition

For $1 \leq s \leq m \leq r$, there is a linear $\mathtt{A}_m$-bimodule $\mathtt{B}_{m,r,s}$ and a cofiber sequence rel. $\mathtt{A}_m$

$$\mathtt{F}_{\mathtt{A}_m}(\Sigma^{-1}_{\mathtt{A}_m}\mathtt{B}_{m,r,s}) \longrightarrow \mathtt{A}_r \rightarrowtail \mathtt{A}_{r+s}.$$

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Given an operad $\mathtt{P} = \{\mathtt{P}(n)\}_{n\geq 0}$, a **linear P-module** $\mathtt{B}$ is a sequence $\mathtt{B} = \{\mathtt{B}(n)\}_{n\geq 0}$ equipped with maps, $1 \leq i \leq s$,

$$\mathtt{P}(s) \otimes \mathtt{B}(t) \xrightarrow{\circ_i} \mathtt{B}(s+t-1) \xleftarrow{\circ_i} \mathtt{B}(s) \otimes \mathtt{P}(t)$$

satisfying the obvious associativity and unitality laws, e.g. $\mathtt{B} = \mathtt{P}$.

The category of linear $\mathtt{P}$-modules is a pointed stable $\mathcal{S}$-model category and there is a Quillen pair

$$\text{linear } \mathtt{P}\text{-modules} \overset{\mathtt{F}_\mathtt{P}}{\rightleftarrows} \mathtt{P} \downarrow \text{Operads}.$$

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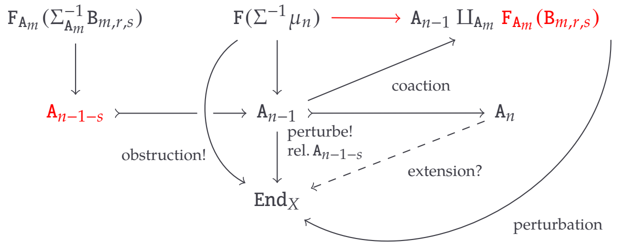
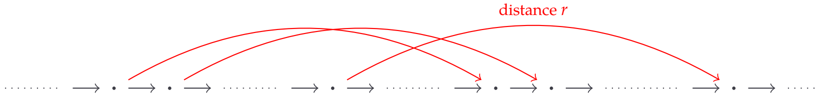
Tanking $1 \leq s \leq \frac{n-1}{2}$ and $r = n-1-s$,



The extended spectral sequence

The E_{r+1} terms of the spectral sequence of a pointed tower depend on the fibers of the r-fold composites,



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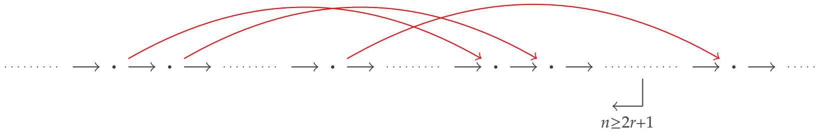

For $\{\mathtt{End}_X^{\mathtt{A}_n}\}_{n\geq 2}$, $R \in \mathtt{End}_X^{\mathtt{A}_\infty}$, and $n \geq 2r+1$, these fibers are the following mapping spaces rel. $\mathtt{A}_m$,

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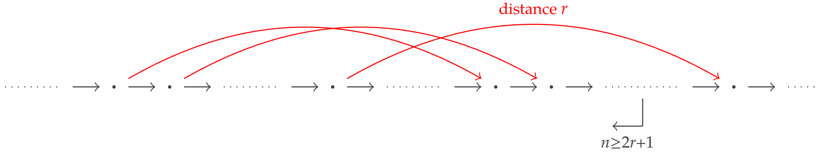

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which are deloopings of the following mapping Hk-module spectra in the model category of linear $\mathtt{A}_m$-bimodules

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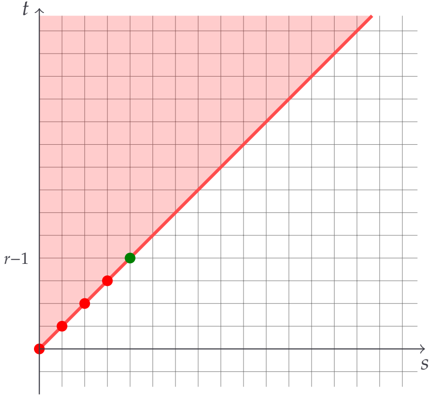

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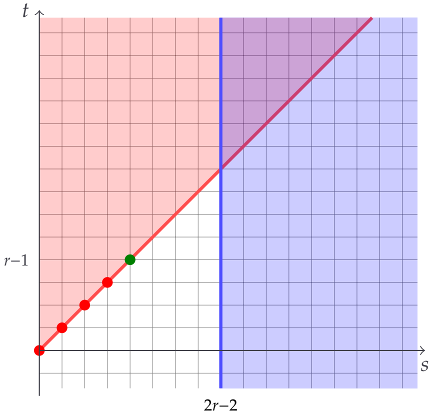

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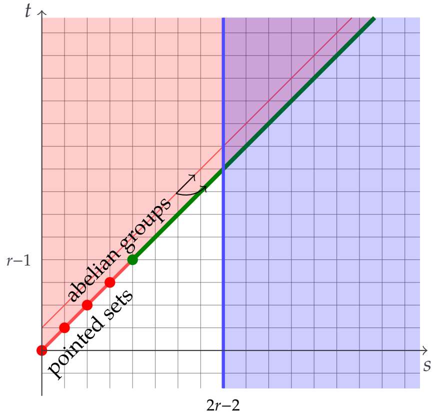

It consists of k-modules in the blue region and in $t - s \geq 2$.

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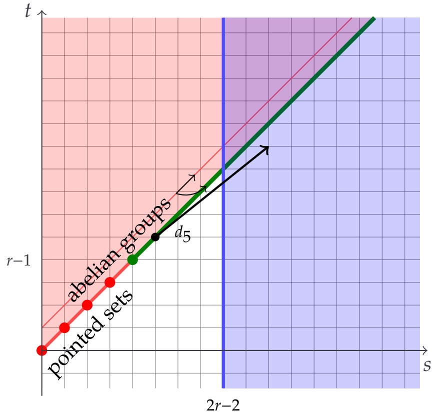

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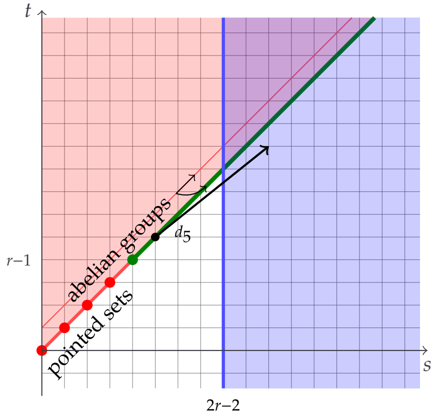

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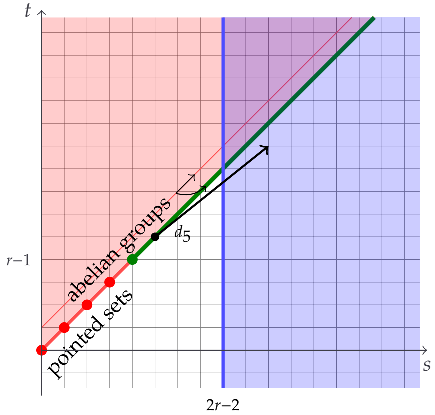

It consists of k-modules in the blue region and in $t - s \geq 2$. If R is an A_{2r-1}-algebra, the spectral sequence is defined up to E_r. The second page is $E_2^{s,t} = HH^{s+2,-t}(\pi_* R)$ for $s \geq 1$ where defined.

Theorem

For $1 \leq s < r$, given an A_{r+s}-algebra R, there is an obstruction in $E_{s+1}^{r+s-1,r+s-2}$ vanishing iff the A_r-algebra underlying R extends to an A_{r+s+1}-algebra.

For $s = 1$, we recover the classical obstruction in Hochschild cohomology $E_2^{r,r-1} = HH^{r+2,1-r}(\pi_* R)$. The best obstruction is in $E_r^{2r-2,2r-3}$, for $s = r - 1$.

The first non-trivial obstruction $(r, s) = (3, 1)$

$E_2^{1,1}$ = weak equivalence classes of A_3-algebras R which extend to A_4-algebras with fixed homology algebra $\pi_* R$.

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The classifying class is called UNIVERSAL MASSEY PRODUCT or UNIVERSAL TODA BRACKET[1],

$$\{m_3\} \in E_2^{11} = HH^{3,-1}(\pi_* R),$$

since, given $x, y, z \in \pi_* R$ with $xy = 0 = yz$,

$$m_3(x,y,z) \in \langle x, y, z \rangle.$$

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Take $(\pi_* R, d = 0, m_2, m_3, m_4)$ to be a minimal model for (R, d, m_2, m_3, m_4).

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If $\frac{1}{2} \in k$, the obstruction to extending an A_4-algebra to an A_5-algebra is

$$\begin{aligned} HH^{3,-1}(\pi_* R) &\longrightarrow HH^{5,-2}(\pi_* R) \\ \{m_3\} &\mapsto \frac{1}{2}[\{m_3\}, \{m_3\}]. \end{aligned}$$

Beyond the second page

Theorem

Recall that $E_2^{s,t} = HH^{s+2,-t}(\pi_* R)$ for $s > 0$. We have

$$d_2 = \pm[\{m_3\}, -]\colon HH^{s+2,-t}(\pi_* R) \longrightarrow HH^{s+4,-t-1}(\pi_* R).$$

Beyond the second page

The EULER CLASS $\{\delta\} \in HH^{1,0}(\pi_* R)$, $\delta(x) = |x| \cdot x$, satisfies

$$\{m_3\} \cdot x = [\{m_3\}, \{\delta\} \cdot x] + \{\delta\} \cdot [\{m_3\}, x].$$

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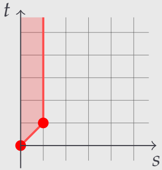
Proposition

If the following map is an isomorphism for $s \geq 2$, then E_3 is concentrated in $s = 0, 1$,

$$\begin{aligned} HH^{s,t}(\pi_* R) &\longrightarrow HH^{s+3,t-1}(\pi_* R) \\ x &\mapsto \{m_3\} \cdot x, \end{aligned}$$



A sufficient condition for existence and uniqueness

Theorem

Suppose $\frac{1}{2} \in k$. Let R be an A_4-algebra with universal Massey product $\{m_3\} \in HH^{3,-1}(\pi_* R)$ such that

$$\begin{aligned} HH^{s,t}(\pi_* R) &\longrightarrow HH^{s+3,t-1}(\pi_* R) \\ x &\mapsto \{m_3\} \cdot x, \end{aligned}$$

is an isomorphism for $s \geq 2$. If

$$\frac{1}{2}[\{m_3\}, \{m_3\}] = 0,$$

then there exists a unique A_∞-algebra with this universal Massey product, up to weak equivalence. Otherwise there is none.

Why do we care about this?

Amiot'07 classified 1-Calabi–Yau triangulated categories of finite type by certain A_4-algebras R such that the category of f.g. projective π_*R-modules has exact triangles

$$X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{q} \Sigma X, \qquad 1_{\Sigma X} \in \langle q, i, f \rangle.$$

By the axioms of triangulated categories, multiplication by the universal Massey product is an isomorphism in the required range. The previous theorem characterizes the existence and uniqueness of models.

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Consider the minimal A_4 algebra ($d = 0$) with $m_4 = 0$ given by the algebra

$$R = \frac{k\langle \epsilon, t^{\pm 1} \rangle}{(\epsilon^2, \epsilon t + t\epsilon)}, \qquad |\epsilon| = 0, \quad |t| = 1,$$

where m_3 is the $k\langle t^{\pm 1} \rangle$-trilinear map defined by

$$m_3(\epsilon, \epsilon, \epsilon) = t^{-1}.$$

Then

$$HH^{*,*}(\pi_* R) = k[\epsilon t, t^{\pm 2}, f, \{\delta\}]$$

where $|f| = (1, -1)$ is given by the $k\langle t^{\pm 1} \rangle$-linear map with

$$f(\epsilon) = t^{-1},$$

$$m_3 = f^3 t^2,$$

$$\dim HH^{n, 2-n}(\pi_* R) = 2, \qquad n \geq 1.$$

MASSEY PRODUCTS AND UNIQUENESS OF A_∞-ALGEBRA STRUCTURES

Operations in Highly Structured Homology Theories, Banff, 22–27 May 2016.

Fernando Muro

Universidad de Sevilla