I'm going to talk about 3 interconnected stories: one in topology, one in algebra, & one in both camps. Today, my focus is more introductory, so I might not get as far as one might like.

The context in which I'll be working is equivariant (stable) homotopy, but I'll assume little beyond what you know about G-spaces.

<u>Def</u> An <u>operad</u> is a collection of objects $\mathcal{O}_0, \mathcal{O}_1, \ldots$ s.t. $\mathcal{O}_n$ has an action of Σ_n & we have maps $\mathcal{O}_k \times \mathcal{O}_{n_1} \times \cdots \times \mathcal{O}_{n_k} \xrightarrow{\circ} \mathcal{O}_{n_1 + \cdots + n_k}$ that behave like compositions, and $\exists$ "identity" in $\mathcal{O}_1$.

The equivariance here is subtle, but if you think of these as operations & of the Σ_n action as permuting the input, you won't be led astray.

<u>Example</u> ① $\mathrm{Asso}_n = \Sigma_n$ w/ obvious action. $\circ$ is block composition.

② $\mathrm{Com}_n = *$.

③ If X is a space (spectrum), $\mathrm{End}(X)_n = \mathrm{Map}(X^n, X)$, and $\circ$ is comp.

③ lets us have actions of operads.

<u>Def</u> If $\mathcal{O}$ is an operad, an $\mathcal{O}$-algebra structure on X is a map of operads $\mathcal{O} \to \mathrm{End}(X)$.

If we unpack this, then we get an alternative description:

Space X + maps $m_n : \mathcal{O}_n \underset{\Sigma_n}{\times} X^n \to X$ s.t.

$$\begin{tikzcd}
\mathcal{O}_k \underset{\Sigma_k}{\times} \left(\mathcal{O}_{k_1} \underset{\Sigma_{k_1}}{\times} X^{k_1} \times \cdots \times \mathcal{O}_{n_k} \underset{\Sigma_{n_k}}{\times} X^{n_k}\right) \cong \mathcal{O}_k \underset{\Sigma_k}{\times} \left(\mathcal{O}_{n_1} \times \cdots \times \mathcal{O}_{n_k} \underset{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}{\times} X^{n_1 + \cdots + n_k}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{O}_{n_1 + \cdots + n_k} \underset{\Sigma_{n_1 + \cdots + n_k}}{\times} X^{n_1 + \cdots + n_k} \arrow[d] \\
\mathcal{O}_k \underset{\Sigma_k}{\times} X^k \arrow[r] & X
\end{tikzcd}$$

<u>Cor</u>: Given any Σ_n-space Y and a Σ_n-equivariant map $Y \to \mathcal{O}_n$, get a "twisted multiplicati[on]"

$$Y \underset{\Sigma_n}{\times} X^n \to X \quad \text{via} \quad Y \underset{\Sigma_n}{\times} X^n \to \mathcal{O}_n \underset{\Sigma_n}{\times} X^n \to X.$$

These are natural in X (and Y).

<u>Ex</u>: ① An Asso space is an associative monoid.

② A Comm space is a commutative monoid.

③ A comm Green functor is a Comm algebra in Mackey functors.

④ Neither Mackey nor Tambara functors are easily $\mathcal{O}$-algebras

Here topology arises: orbits are badly behaved & not homotopical.

Def $\mathcal{O}$ is an E_∞ operad if $\mathcal{O}_n$ is a free, contract. Σ_n-space: $\mathcal{O}_n \simeq E\Sigma_n$.

Ex: If $\mathcal{U}$ is an ∞ dim'l inner product space, then

$$\mathcal{L}(\mathcal{U})_n = \mathcal{L}(\mathcal{U}^{\oplus n}, \mathcal{U}) \quad \& \quad \mathcal{D}(\mathcal{U})_n = \mathrm{Emb}(\amalg^n D, D) \text{ are } E_\infty\text{-operads.}$$

structuring O-space of fibrant spectra.

Equivariantly, this is very harsh! Very harsh. This is Aaron's question.

Def $\mathcal{O}$ is an N_∞-operad if $\mathcal{O}_n \simeq E\mathcal{F}_n$, $\mathcal{F}_n$ is a family of s.g. Γ of $G\times\Sigma_n$ s.t. $\Gamma\cap\{e\}\times\Sigma_n = \{e\}$ & $G\times\{e\} \in \mathcal{F}_n$.

Here $E\mathcal{F}_n$ is like $E\Sigma_n$: $\quad (E\mathcal{F}_n)^\Gamma \simeq \begin{cases} * & \Gamma\in\mathcal{F}_n \\ \emptyset & \Gamma\notin\mathcal{F}_n \end{cases}$.

Ex: If $\mathcal{U}$ is a G-universe, then $\mathcal{L}(\mathcal{U})$ and $\mathcal{D}(\mathcal{U})$ are both N_∞ operads. We'll come back.

Since $E\mathcal{F}_n$ is a universal space, for any $G\times\Sigma_n$-space Y, $\mathrm{Map}^{G\times\Sigma_n}(Y, E\mathcal{F}_n) \simeq \begin{cases} * & \mathrm{Stab}(y)\in\mathcal{F}_n\ \forall y \\ \emptyset & \text{otherwise} \end{cases}$

So there is a unique (up to homotopy) map.

Prop If $\Gamma \leq G\times\Sigma_n$ has $\Gamma\cap\Sigma_n = \{e\}$, then $\exists\, H\leq G$ & $f: H\to\Sigma_n$ w/ $\Gamma = \Gamma_f = \{(h, f(h)) \mid h\in H\}$.

H-set structure on $\{1,\dots,n\}$.

Pf: $\Gamma \longrightarrow G$ is an injection. $\square$ People call these "graph subgroups".

Def An H-set T is admissible for $\mathcal{O}$ if $\Gamma_T \in \mathcal{F}_{|T|}$.

The collection of admissibles gives a sub-functor of $\mathrm{Orb}_G \longrightarrow \mathrm{Sym}$. $G/H \longmapsto \mathrm{Set}^H$

Prop: The admissible H-sets are closed under

① finite limits ② disjoint unions ③ "self-induction": G/H admissible, T an admissible H-set, then $G\times_H T$ is an admissible G-set.

① & ③ gives "stability under pullback".

Pf Exercise! All of these follow from the composition on $\mathcal{O}$. $\square$

What does this buy us?

Example (Key!) $\quad G\times\Sigma_n/\Gamma_T \underset{\Sigma_n}{\times} X^n \cong G\underset{H}{\times}(\mathrm{Map}(T, i_H^* X))$. (similar for spectra)

Thm If X is an $\mathcal{O}$-algebra, then for any admissible T, we have a ("unique") map

$$\mathrm{Map}(T, X) \longrightarrow X, \text{ and these are coherently compatible.}$$

Cor: If G/H is admissible, then we have a map $X^H \longrightarrow X^G$, a "transfer" $\Rightarrow \pi_k(X)$ is a Mackey functor

Thm If E is an $\mathcal{O}$-algebra in spectra, then for any admissible T, we have a ("unique") norm map $\quad N^T(E) \longrightarrow E. \quad \Rightarrow \pi_k(E)$ is an incomplete Mackey functor.

From the questions:

<u>Def</u> An <u>indexing system</u> is a full subfunctor $\underline{\mathcal{C}} \subseteq \underline{Set} : Orb_G^{op} \longrightarrow Sym$ s.t.

1. $\underline{\mathcal{C}}(G/H)$ is a sym. monoidal subcat of Set^H.
2. $\underline{\mathcal{C}}(G/H)$ is closed under finite limits
3. $\underline{\mathcal{C}}$ is closed under self-induction.

The collection of such is a poset $\mathcal{I}$.

<u>Thm</u>: The assignment $\mathcal{O} \longmapsto \mathcal{C}_{\mathcal{O}}$ gives a fully faithful embedding $hoN_\infty \longrightarrow \mathcal{I}$.

<u>Thm</u> (Gutiérrez-White) This is essentially surjective.