Chief series of locally compact groups

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A **locally compact group** is a group G equipped with a locally compact Hausdorff topology, such that $(g, h) \mapsto g^{-1}h$ is continuous (where $G \times G$ carries the product topology).

A **chief factor** K/L of the locally compact group G is a pair of closed normal subgroups $L < K$ such that there are no closed normal subgroups of G lying strictly between K and L.

A **descending chief series** for G is a series of closed normal subgroups $(G_\alpha)_{\alpha \le \beta}$ such that $G = G_0$, $1 = G_\beta$, $G_\lambda = \bigcap_{\alpha < \lambda} G_\alpha$ for each limit ordinal and each factor $G_\alpha / G_{\alpha+1}$ is chief. (Special case: finite chief series.)

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- Every finite group G has a finite chief series.
- Every **pro**finite group has a **descending** chief series with finite chief factors.
- Every connected Lie group has a finite series in which the factors are in the following list:
  1. connected centreless semisimple Lie group;
  2. finite group of prime order;
  3. $\mathbb{R}^n$, $\mathbb{Z}^n$ or $(\mathbb{R}/\mathbb{Z})^n$ for some n.

  We can also make sure all of these are chief factors except for occurrences of $\mathbb{Z}^n$ or $(\mathbb{R}/\mathbb{Z})^n$.
- Every connected locally compact group G has a descending series in which the factors come from Lie quotients of G.

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- Every connected locally compact group G has a descending series in which the factors come from Lie quotients of G.

In all the cases on the previous slide, we know what the factors can look like:

- A finite chief factor is a direct product of copies of a simple group. Finite simple groups have been classified.
- Connected centreless semisimple Lie groups are direct products of finitely many copies of an abstractly simple Lie group. Simple Lie groups have been classified.

So in some very general situations, we get a decomposition of a group G into 'known' groups. Moreover, the non-abelian chief factors we see up to isomorphism are an invariant of G (not dependent on how we constructed the series).

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What about locally compact groups that are not connected or compact? Let us focus on the case when G has a compact generating set. (NB: Every locally compact group is a directed union of compactly generated open subgroups.)

Theorem (Caprace–Monod 2011)

Let G be a compactly generated locally compact group.

(i) Suppose G has no infinite discrete quotient, and there is no cocompact normal subgroup that is connected and soluble. Then there is a cocompact normal subgroup of G with a non-discrete simple quotient.

(ii) Suppose G has no non-trivial compact or discrete normal subgroups. Then every non-trivial closed normal subgroup of G contains a minimal one.

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Theorem 1 (R.–Wesolek)

For every compactly generated locally compact group G, there is an **essentially chief series**, i.e. a finite series

$$G = G_0 > G_1 > G_2 > \cdots > G_n = \{1\}$$

of closed normal subgroups of G, such that each G_i/G_{i+1} is compact, discrete or a chief factor of G.

Theorem 2 (R.–Wesolek)

Let G be a compactly generated locally compact group. Let $(G_i)_{i \in I}$ be a chain of closed normal subgroups of G, let $A = \overline{\bigcup_{i \in I} G_i}$ and let $B = \bigcap_{i \in I} G_i$. Then there exist $i, j \in I$ such that A/G_i and G_j/B each have a compact open G-invariant subgroup.

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Let G be a compactly generated totally disconnected locally compact (t.d.l.c.) group.

Fact

G has an action on a connected graph Γ, called a **Cayley–Abels graph** for G, such that: G acts transitively on vertices; the degree of Γ is finite; and if U is the stabilizer of a vertex, then U is a compact open subgroup of G.

Write $\deg(G)$ for the smallest degree of such a graph for G. Given a group G acting on a graph Γ, define $G\backslash\Gamma$ to have vertex set $\{Gv \mid v \in V\Gamma\}$ and directed edge set $\{Ge \mid e \in E\Gamma\}$. (NB: we allow a loop to be equal to its inverse.) The degree of a vertex is the number of edges coming out of it.

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Lemma 1

If Γ is a Cayley–Abels graph for G and N is a closed normal subgroup of G, then $N\backslash\Gamma$ is a Cayley-Abels graph for G/N.

We have $\deg(N\backslash\Gamma) \leq \deg(\Gamma)$. If equality holds, there is a compact G-invariant subgroup $K = \ker(N \curvearrowright \Gamma)$ of N such that N/K is discrete.

Lemma 2

Let G be a compactly generated t.d.l.c. group and Γ be a Cayley–Abels graph for G. Let $\mathcal{C}$ be a chain of closed normal subgroups of G.

(i) Let $A = \overline{\bigcup_{H \in \mathcal{C}} H}$. Then $\deg(A\backslash\Gamma) = \min\{\deg(H\backslash\Gamma) \mid H \in \mathcal{C}\}$.

(ii) Let $D = \bigcap_{H \in \mathcal{C}} H$. Then $\deg(D\backslash\Gamma) = \max\{\deg(H\backslash\Gamma) \mid H \in \mathcal{C}\}$.

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Proof of Theorem 1 (t.d.l.c. case):

- Proceed by induction on $\deg(G)$. Let Γ be a Cayley–Abels graph of smallest degree.
- By Lemma 2(i) plus Zorn's lemma, there is a closed normal subgroup A that is maximal amongst closed normal subgroups such that $\deg(A\backslash\Gamma) = \deg(\Gamma)$. Break up A using Lemma 1.
- By Lemma 2(ii) plus Zorn's lemma, there is a minimal non-trivial closed normal subgroup D/A of G/A; in other words, D/A is a chief factor of G.
- By the maximality of A, we have $\deg(D\backslash\Gamma) < \deg(A\backslash\Gamma)$, so $\deg(G/D) < \deg(G)$. By induction, G/D has an essentially chief series, which we can now extend to an essentially chief series for G.

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From now on we assume G is a locally compact second-countable (l.c.s.c.) group. Any chief factor of G is then a *characteristically simple* l.c.s.c. group.

Theorem (R.–Wesolek)

Let G be a non-trivial characteristically simple l.c.s.c. group. Then G has at least one of the following three structures:

(i) (semisimple type) $G = \overline{\langle \mathcal{S} \rangle}$ where $\mathcal{S}$ is the set of topologically simple closed normal subgroups of G;

(ii) G has 'low topological complexity';

(iii) ('stacking phenomenon') G has a characteristic class of chief factors $\{K_i/L_i \mid i \in I\}$, such that for all $i, j \in I$, there is an automorphism α of G such that $\alpha(K_i) < L_j$ and $\alpha(C_G(K_i/L_i)) < C_G(K_j/L_j)$.

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By a result of Caprace–Monod, every compactly generated characteristically simple l.c.s.c. group is discrete, abelian or of semisimple type. However, a chief factor of a compactly generated group need not itself be compactly generated, and indeed need not be of semisimple type.

Wesolek (2015) defined a large class $\mathcal{E}$ of t.d.l.c.s.c. groups, the **elementary** groups, that are built from profinite and discrete groups via elementary operations. The class admits a well-behaved rank function ξ, taking values in the countable ordinals.

On the next slide I will give characterizations of certain bounds on $\xi(G)$.

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Let G be a non-trivial elementary t.d.l.c.s.c. group.

- $\xi(G) = 2$ if and only if, for every compactly generated subgroup H of G, then H has arbitrarily small compact open normal subgroups.
- $\xi(G)$ is finite if and only if G has a finite normal series

$$G = G_1 > G_2 > \cdots > G_n = \{1\}$$

such that $\xi(G_i/G_{i+1}) = 2$ for all i.

- $\xi(G) \leq \omega + 1$ if and only if, for every compactly generated subgroup H of G, $\xi(H)$ is finite.

The characteristically simple l.c.s.c. groups G of 'low topological complexity' are either connected abelian, or elementary with $\xi(G) = 2$ or $\omega + 1$.

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Here is a general construction to show that (stacking type) chief factors can have high topological complexity and very complex subnormal subgroup structure:

Let T be a regular tree of countably infinite degree with a distinguished end δ. Orient the edges to point towards δ, and define a colouring $c : ET \to X$ ($|X| = \aleph_0$) that restricts to a bijection on the set of in-edges of each vertex. Let G be a transitive closed subgroup of $\mathrm{Sym}(X)$ with compact point stabilizer U.
Now let $E(G, U)$ be the group of automorphisms g of T such that $g.\delta = \delta$; at every vertex, the local action of g (with respect to c) is an element of G; and at all but finitely many vertices, the local action of g is an element of U.

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Let T be a regular tree of countably infinite degree with a distinguished end δ. Orient the edges to point towards δ, and define a colouring $c : ET \to X$ ($|X| = \aleph_0$) that restricts to a bijection on the set of in-edges of each vertex. Let G be a transitive closed subgroup of $\mathrm{Sym}(X)$ with compact point stabilizer U.

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The group $E := E(G, U)$ is then a t.d.l.c.s.c. group, if we give it the topology so that $E(U, U)$ is open and carries the permutation topology on VT. It is compactly generated if G is.

Let $P := P(G, U)$ be the subgroup of elements of E that also fix a vertex of the tree. Then $E = P \rtimes \mathbb{Z}$ and $M = \overline{[P, P]}$ is a chief factor of E. But M is certainly not a quasi-product of simple groups: P has normal factors $P_n/P_{n+1} \cong \bigoplus_{\mathbb{N}}(G, U)$, where P_n is the fixator of a horosphere around δ, so M will pick up at least the derived group of each of these factors.

In this way, we obtain a *characteristically simple* group M whose subnormal structure is at least as complicated as that of $\overline{[G, G]}$, a group which was *not* assumed to be characteristically simple.

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