Rational discrete first degree cohomology for totally disconnected locally compact groups

Ilaria Castellano

University of Southampton

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Aim of the talk

Castellano I., Th. Weigel. *Rational discrete cohomology for totally disconnected locally compact groups.* Journal of Algebra, 2016.

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Theorem (M.J. Dunwoody, 1977)

Let R be a commutative ring with unit. For any group G, $\mathrm{cd}_R(G) \leq 1$ if, and only if, G is isomorphic to the fundamental group $\pi(\mathcal{G}, \Lambda)$ of a graph of finite groups with no R-torsion.

Rational discrete $\mathbb{Q}[G]$-modules

Let G be a t.d.l.c. group and $\mathbb{Q}$ the field of rationals.

Definition

A $\mathbb{Q}[G]$-module M is said to be *discrete* if the pointwise stabilizers are open subgroups of G.

Denote by ${}_{\mathbb{Q}[G]}\mathbf{dis}$ the full subcategory of ${}_{\mathbb{Q}[G]}\mathbf{mod}$ whose objects are the discrete modules.

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Proposition (I.C, Th. Weigel)

A discrete $\mathbb{Q}[G]$-module M is projective in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ if, and only if, M is a direct summand of a permutation $\mathbb{Q}[G]$-module with compact open stabilizers.

Rational discrete cohomology for t.d.l.c. groups

For $M \in \mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis})$ denote by

$$\mathrm{dExt}^{k}_{\mathbb{Q}[G]}(M, \_) = \mathcal{R}^{k}\mathrm{Hom}_{{}_{\mathbb{Q}[G]}\mathbf{dis}}(M, \_)$$

the right derived functors of $\mathrm{Hom}_{\mathbb{Q}[G]}(M, \_)$ in ${}_{\mathbb{Q}[G]}\mathbf{dis}$.

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Thus the k^{th} *discrete cohomology group* of G with coefficients in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ is defined by

$$\mathrm{dH}^k(\mathbb{Q}[G], \_) = \mathrm{dExt}^k_{\mathbb{Q}[G]}(\mathbb{Q}, \_), \qquad k \geq 0,$$

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Fact

For any compact open subgroup K of a t.d.l.c. group G there is a natural isomorphism

$$\mathrm{dH}^1(G, M) \cong \mathrm{Der}_K(G, M)/\mathrm{PDer}_K(G, M), \quad M \in \mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis}).$$

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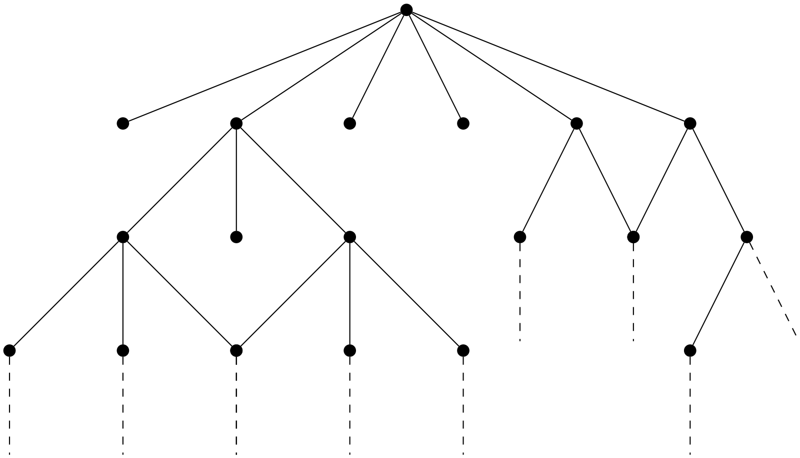
(i) $V(\Gamma) = G/\mathcal{O}$ is the set of vertices;

(ii) $E(\Gamma) = \{(g\mathcal{O}, gs\mathcal{O}), (gs\mathcal{O}, g\mathcal{O}) | g \in G,\ s \in S\}$ is the set of edges.

Definition

Let Γ be a locally finite connected graph. The *number* $e(\Gamma)$ *of ends of* Γ is defined to be the least upper bound (possibly ∞) of the number of infinite connected components that can be obtained by removing finitely many edges.

Cutting up graphs



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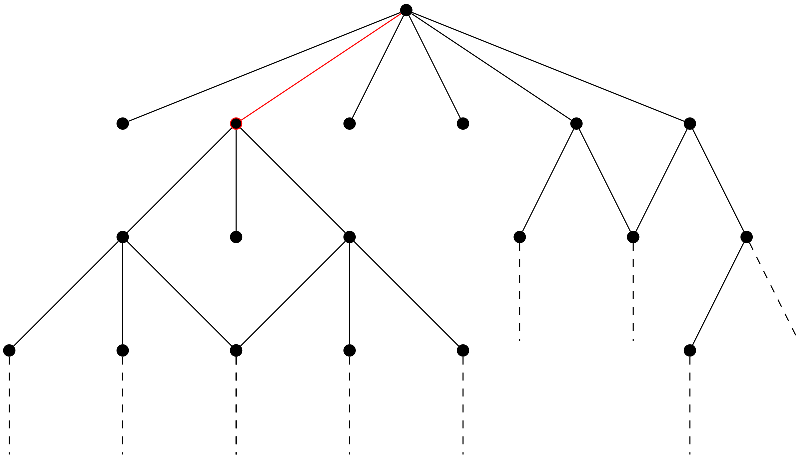

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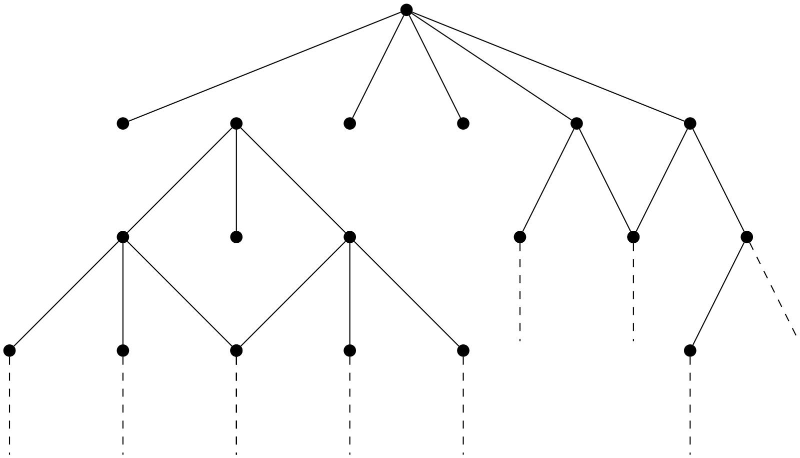

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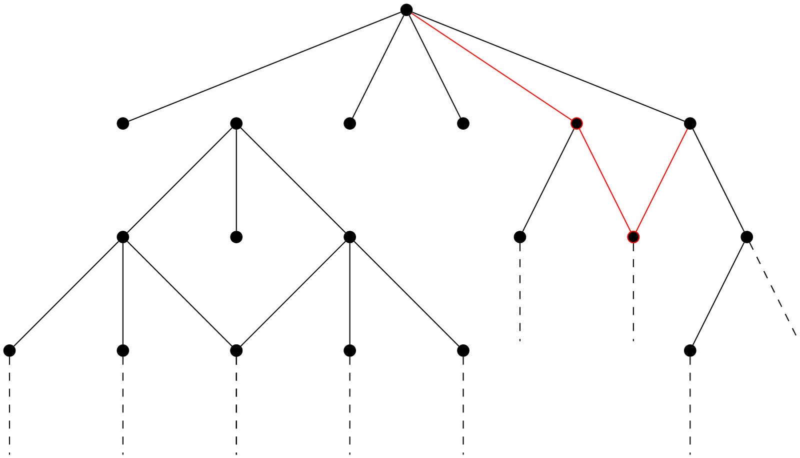

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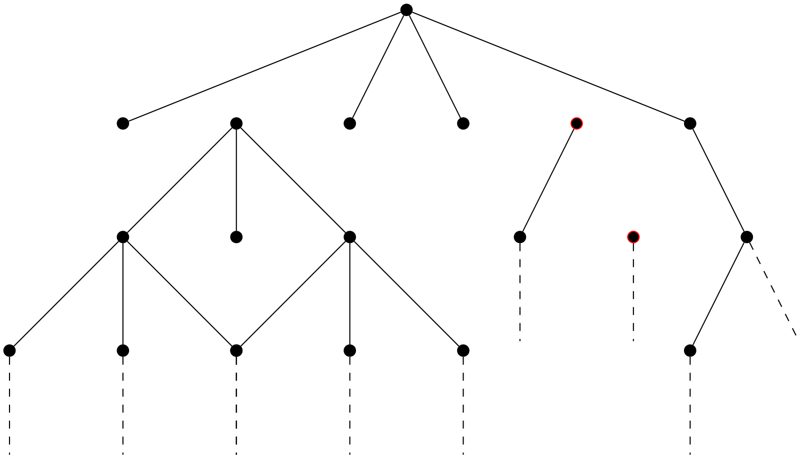

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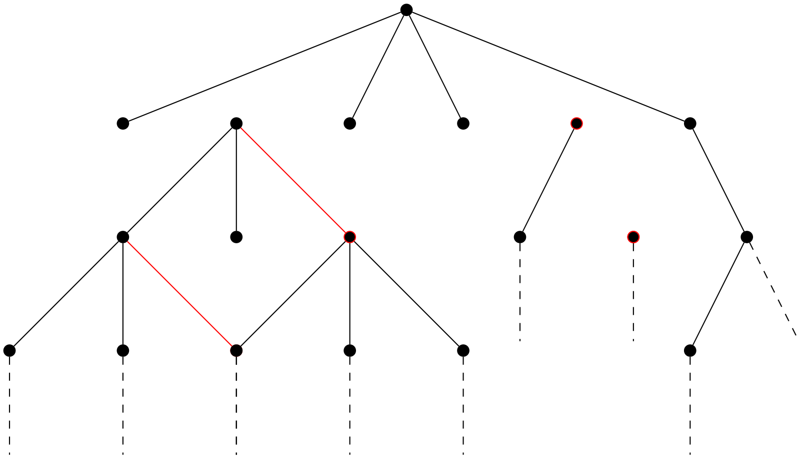

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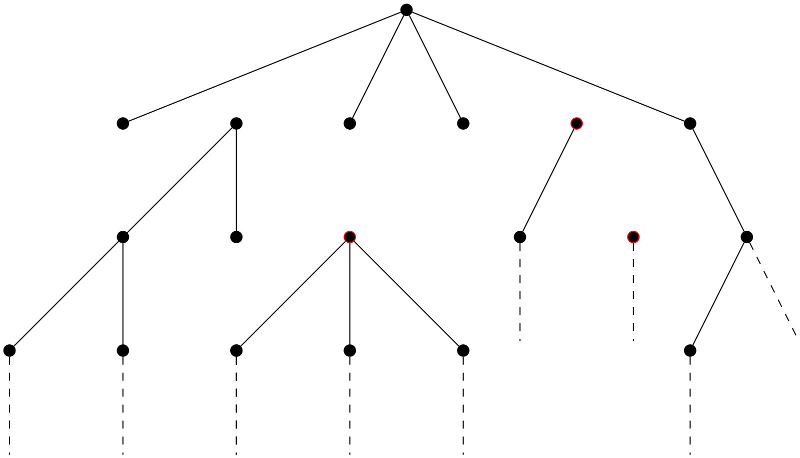

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Group — rough Cayley graph

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Theorem (B. Krön, R.G. Möller, 2008)

Let G be a compactly generated t.d.l.c. group G. The group G has more than one end if, and only if, $G = H *_K J$ (with $K \neq H$ and $K \neq J$) or $G = H*_K^t$ where the subgroups H and J are compactly generated and open, and K is a compact open subgroup.

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Theorem [I.C.]

The conditions in the latter theorem are equivalent to

$$\mathrm{dH}^1(G, \mathrm{Bi}(G)) \neq 0,$$

where $\mathrm{Bi}(G) = \varinjlim(\mathbb{Q}[G/U], \eta_{UV})$ ranging over all compact open subgroups.

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- G is acting on $\mathcal{T}$ without edge inversions;
- the G-action is edge-transitive;
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$$0 \to E(\mathcal{T}) \to V(\mathcal{T}) \to \mathbb{Q} \to 0$$

to compute the first degree cohomology with coefficients in $\mathrm{Bi}(G)$.

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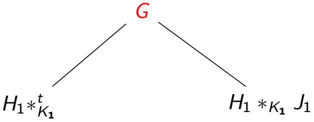

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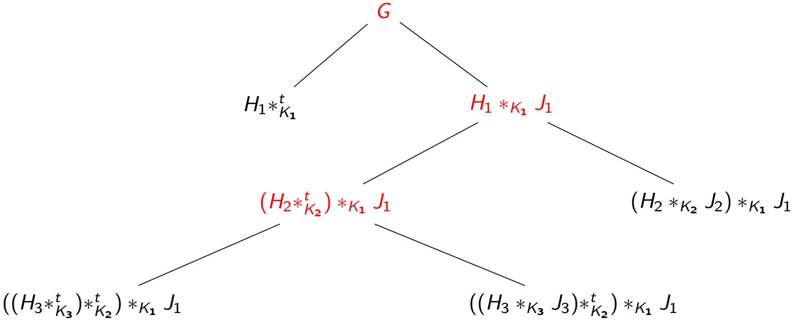

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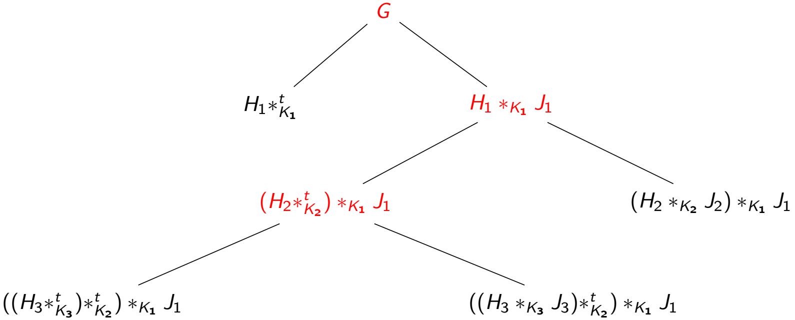

Definition

The group G is *accessible* if it is isomorphic to the fundamental group of a finite graph of groups, with compact edge groups and (at most 1)-ended vertex groups.

Rational discrete cohomological dimension

Theorem [I.C.]

A t.d.l.c. group G is isomorphic to the fundamental group of a finite graph of profinite groups if, and only if, G is compactly presented and $\mathrm{cd}_{\mathbb{Q}}(G) \leq 1$.

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For a t.d.l.c. group G the *rational discrete cohomological dimension*, $\mathrm{cd}_{\mathbb{Q}}(G)$, is the minimum $n \in \mathbb{N} \cup \{\infty\}$ such that there exists a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Q} \to 0,$$

of $\mathbb{Q}$ of length n.

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- G is compact if, and only if, $\mathrm{cd}_{\mathbb{Q}}(G) = 0$.

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Thanks for your attention

References

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Backup slides

The rational discrete standard bimodule $\mathrm{Bi}(G)$

For a t.d.l.c. group G the set of compact open subgroups $\mathfrak{CO}(G)$ of G with the inclusion relation "$\subseteq$" is a directed set.

For $V \subset U$ one has an injective mapping

$$\eta_{U,V} \colon \mathbb{Q}[G/U] \longrightarrow \mathbb{Q}[G/V], \qquad \eta_{U,V}(x\,U) = \frac{1}{|U:V|} \sum_{r \in \mathcal{R}} x\,r\,V, \quad x \in G.$$

By construction, one has for $W \in \mathfrak{CO}(G)$, $W \subseteq V \subseteq U$, that $\eta_{U,W} = \eta_{V,W} \circ \eta_{U,V}$. Let

$$\mathrm{Bi}(G) = \varinjlim_{U \in \mathfrak{CO}(G)} (\mathbb{Q}[G/U], \eta_{U,V}).$$

By definition, $\mathrm{Bi}(G)$ is a discrete left $\mathbb{Q}[G]$-module. Moreover $\mathrm{Bi}(G)$ can be also endowed with a structure of discrete right $\mathbb{Q}[G]$-module.

Properties of the bimodule $\mathrm{Bi}(G)$ in analogy to the group algebra

Proposition (I.C., T. Weigel)

Let G be a t.d.l.c. group.

1. *One has a natural isomorphism* $\theta \colon \mathrm{Bi}(G) \otimes_G \_ \longrightarrow \mathrm{id}_{{}_{\mathbb{Q}[G]}\mathbf{dis}}$.
2. *One has*

$$\mathrm{Hom}_G(\mathbb{Q}, \mathrm{Bi}(G)) \simeq \begin{cases} \mathbb{Q} & \text{if } G \text{ is compact,} \\ 0 & \text{if } G \text{ is not compact.} \end{cases}$$

3. *Let* $M, P \in \mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis})$*, and assume further that* P *is finitely generated and projective. Then there is a natural iso* $\mathrm{Hom}_G(P, \mathrm{Bi}(G)) \otimes_G M \cong \mathrm{Hom}_G(P, M)$ *is an isomorphism.*
4. *If* G *is of type FP, i.e.* $\mathrm{cd}_{\mathbb{Q}}(G) < \infty$ *and* G *is of type* FP_n *for all* n*, then*

$$\mathrm{cd}_{\mathbb{Q}}(G) = max\{\, n \in \mathbb{N} \mid \mathrm{dH}^n(G, \mathrm{Bi}(G)) \neq 0 \,\}.$$

Graph of t.d.l.c. groups

Definition

Let Λ be a connected graph. A *graph of t.d.l.c. groups* $(\mathcal{A}, \Lambda)$ consists of the following data:

(i) a t.d.l.c. group $\mathcal{A}_v$ for every vertex v of Λ,

(ii) a t.d.l.c. group $\mathcal{A}_e$ for every edge e of Λ such that $\mathcal{A}_e = \mathcal{A}_{\bar{e}}$,

(iii) an open group monomorphism $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{t(e)}$ for every edge e of Λ.

Fundamental group of a graph of t.d.l.c. groups

Definition

Let F be the free product of the $\mathcal{A}_v$ and the free group generated by the edges $E(\Lambda)$. Let $F(\mathcal{A};\Lambda)$ be the quotient of F by the normal subgroup generated by the elements

$$e\bar{e},\ \ e\alpha_e(c)e^{-1}\alpha_{\bar{e}}^{-1}(c), \quad \forall e \in E(\Lambda),\ c \in \mathcal{A}_e.$$

Given a maximal subtree $\mathcal{T}$ of Λ, the *fundamental group of* $(\mathcal{A},\Lambda)$ *with respect to* $\mathcal{T}$ is defined as follows

$$\pi_1(\mathcal{A},\Lambda,\mathcal{T}) := F(\mathcal{A};\Lambda)/ \ll e|e \in E(\mathcal{T}) \gg_{F(\mathcal{A};\Lambda)}, \tag{1}$$

where $\ll e|e \in E(\mathcal{T}) \gg_{F(\mathcal{A};\Lambda)}$ is the smallest normal subgroup of $F(\mathcal{A};\Lambda)$ containing $E(\mathcal{T})$. The fundamental group is independent of the choice of the maximal subtree up to isomorphism.

Compactly presented t.d.l.c. groups

Definition

A *generalized presentation* of a t.d.l.c. group G is a graph of profinite groups $(\mathcal{A}, \Lambda)$ together with a continuous open surjective group homomorphism

$$\phi : \pi_1(\mathcal{A}, \Lambda, \mathcal{T}) \to G,$$

such that $\phi_{|\mathcal{A}_v}$ is injective for all $v \in \mathcal{V}(\Lambda)$.

Definition

A t.d.l.c. group G is said to be *compactly presented* if there a generalized presentation $\phi\colon \pi(\mathcal{G}, \Lambda) \to G$ satisfying

- $\phi_{|\mathcal{G}_v}$ is injective for all vertex groups G_v,
- $Ker(\phi)$ is finitely generated as normal subgroup of $\pi(\mathcal{G}, \Lambda)$.

Almost invariant functions

Let G be a t.d.l.c. group and $\mathcal{O}$ a compact open subgroup. The set of all functions from $G/\mathcal{O}$ to $\mathbb{Q}$ will be denoted by $(G/\mathcal{O}, \mathbb{Q})$; this is a G-set with

$$g\alpha(x) = \alpha(g^{-1}x) \ \ \forall \alpha \in (G/\mathcal{O}, \mathbb{Q}),\ \forall g \in G,\ \forall x \in G/\mathcal{O}.$$

Definition

We say that two maps $\alpha, \beta \in (G/\mathcal{O}, \mathbb{Q})$ are *almost equal*, $\alpha =_a \beta$, if $\alpha(x) = \beta(x)$ for all but finitely many elements $x \in G/\mathcal{O}$.

Remark

Every element of $\mathbb{Q}[G/\mathcal{O}]$ can be expressed as formal sum $m = \sum_{x \in G/\mathcal{O}} q_x x$ with $q_x \in \mathbb{Q}$ being 0 for almost all $x \in G/\mathcal{O}$. Thus $\mathbb{Q}[G/\mathcal{O}]$ represents the set of all almost zero functions in $(G/\mathcal{O}, \mathbb{Q})$.

Definition

A function $\alpha \in (G/\mathcal{O}, \mathbb{Q})$ is called *almost $(G, \mathcal{O})$-invariant* if $g\alpha =_a \alpha$ for all $g \in G$ and $k\alpha = \alpha$ for all $k \in \mathcal{O}$.

$\mathrm{dH}^1(G, \mathrm{Bi}(G))$ as almost invariant functions

Proposition (I. Castellano)

For every compact open subgroup $\mathcal{O}$ of a t.d.l.c. group G one has

$$\mathrm{dH}^1(G, \mathbb{Q}[G/\mathcal{O}]) \cong \frac{\mathcal{A}Inv_{\mathcal{O}}(G/\mathcal{O}, \mathbb{Q})}{C(G/\mathcal{O}) + \mathbb{Q}[G/\mathcal{O}]^{\mathcal{O}}},$$

where

$$C(G/\mathcal{O}) = \{\alpha \in (G/\mathcal{O}, \mathbb{Q}) | \alpha \textit{ constant}\}.$$