Local Statistics for Particle Systems with Repulsive Interaction

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Banff Workshop “Beta Ensembles: Universality, Integrability, and Asymptotics”

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Topics

1. From β-Ensembles to Repulsive Particle Systems.
2. How to tackle the new Interaction?
3. Arbitrary $\beta > 0$: Bulk Correlations.
4. $\beta = 2$:
   - Correlations under Unfolding.
   - Empirical Spacings.
   - Edge Correlations.
   - Fluctuations and Deviations of the Largest Particle.
5. Central Ideas of the Method.

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  - Spacings between energy levels of excited heavy nuclei.
  - Quantum chaology (Bohigas-Giannoni-Schmit conjecture).
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$$P_{N,Q,\beta}(x) := \frac{1}{Z_{N,Q,\beta}} \prod_{i<j} |x_i - x_j|^\beta \, e^{-N \sum_{j=1}^N Q(x_j)}.$$

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Question: Why not change the interaction potential?

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- Define repulsive particle system on $\mathbb{R}^N$

$$P^h_{N,Q,\beta}(x) := \frac{1}{Z^h_{N,Q,\beta}} \prod_{i<j} |x_i - x_j|^\beta \, e^{-\sum_{i<j} h(x_i - x_j)} e^{-N \sum_{j=1}^N Q(x_j)}$$

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  with additional (smooth) pair potential h.
- Particles instead of eigenvalues!

How to tackle the new Interaction? Basic Idea I

Assumptions (to be cont.): h symmetric around 0, negative-definite.

Choose Gaussian process $(f(t))_{t\in\mathbb{R}}$ with mean 0 and $\mathbb{E}f(t)f(s) = -h(t-s)$.

Then

$$\sum_{j=1}^{N} f(x_j) \sim \mathcal{N}\left(0, -\sum_{i,j} h(x_i - x_j)\right) \quad \text{for all } x_1, \dots, x_N \in \mathbb{R} \text{ and}$$

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$$\mathbb{E}\exp\left(\sum_{j=1}^{N} f(x_j)\right) = \exp\left(-\frac{1}{2}\sum_{i,j} h(x_i - x_j)\right)$$
$$= \exp\big(-\sum_{i<j} h(x_i - x_j) - \frac{N}{2}h(0)\big).$$

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$$\prod_{i<j} |x_i - x_j|^\beta \, e^{-\sum_{i<j} h(x_i - x_j)} e^{-\sum_{j=1}^N NQ(x_j)} \propto \mathbb{E} \prod_{i<j} |x_i - x_j|^\beta \, e^{-\sum_{j=1}^N NQ(x_j) + f(x_j)}.$$

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- Allows for Fourier techniques.
- Can split $h = h^+ - h^-$ with positive-definite functions $h^\pm$.
- Complex analysis (Vitali's Theorem): Suffices to consider $h_z := zh^+ - h^-$ with negative z.

Repulsive Particles: Summary Assumptions

$$P^h_{N,Q,\beta}(x) = \frac{1}{Z^h_{N,Q,\beta}} \prod_{i<j} \left|x_i - x_j\right|^\beta e^{-\sum_{i<j} h(x_i - x_j)} e^{-N\sum_{j=1}^N Q(x_j)}.$$

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- Assumptions on h: symmetric around zero, Schwartz function, real analytic.
- Assumptions on potential Q: symmetric around zero, real analytic and for given h sufficiently uniformly convex: $\min_{t\in\mathbb{R}} Q''(t) > C(h) \geq 0$ (ensures uniqueness of equilibrium measure).

Global Correlations

$\rho^{h,k}_{N,Q,\beta}(x_1,\dots,x_k) := \int_{\mathbb{R}^{N-k}} P^h_{N,Q,\beta}(x)dx_{k+1}\dots dx_N$: k-th correlation function.

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Theorem (Götze-V., AoP. '14)

There exists $\mu_{Q,\beta}^h$ probability measure with compact connected support and positive density on the interior, s.th.

$$\rho_{N,Q,\beta}^{h,k}(dt) \to \left(\mu_{Q,\beta}^h\right)^{\otimes k} \text{ weakly, as } N \to \infty.$$

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- Somewhat related interactions occur for multi-matrix models (Figalli,Guionnet'14).

Local Bulk Correlations for arbitrary β: Averaged Vague Convergence

Compare local correlations of $P^h_{N,Q,\beta}$ with those of the Gaussian β-Ensemble $P_{N,G,\beta}$, i.e. $G(t) := t^2$, $\mu_{G,\beta}$ semicircle law, $\rho^k_{N,G,\beta}$ $k-$th correlation function.

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Theorem (V., ECP '13)

Let $0 < \xi \leq 1/2$ and $s_N := N^{-1+\xi}$.

For $k = 1, 2, \dots$, any $a \in \operatorname{supp}(\mu^h_{Q,\beta})^\circ$, any $a' \in \operatorname{supp}(\mu_{G,\beta})^\circ$, any smooth function $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ with compact support

$$\lim_{N\to\infty} \int dt^k f(t)$$

$$\int_{a-s_N}^{a+s_N} \frac{1}{\mu^h_{Q,\beta}(a)^k} \rho^{h,k}_{N,Q,\beta}\left(u + \frac{t_1}{N\mu^h_{Q,\beta}(a)}, \dots, u + \frac{t_k}{N\mu^h_{Q,\beta}(a)}\right) \frac{du}{2s_N}$$

$$-\int_{a'-s_N}^{a'+s_N} \frac{1}{\mu_{G,\beta}(a')^k} \rho^{k}_{N,G,\beta}\left(u' + \frac{t_1}{N\mu_{G,\beta}(a')}, \dots, u' + \frac{t_k}{N\mu_{G,\beta}(a')}\right) \frac{du'}{2s_N}$$

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For $k \geq 1$ we have uniformly on compacts in $t_1, \dots, t_k$ and uniformly in $a \in I$, $I \subset (\mathrm{supp}\mu^h_Q)^\circ$ compact

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Analogous results for β-Ensembles:

- $\beta = 2$: Deift et al.'99; Pastur,Shcherbina'97,'08; Levin,Lubinsky'08–, many more.
- General β: Valko,Virag'09; Bourgade,Erdős,Yau'14; Shcherbina'14; Bekerman,Figalli,Guionnet'15. Compare with Figalli,Guionnet'14.

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- Optimal rate of convergence for $\beta = 2$-Ensembles (KSSV'14):

$$\frac{1}{\mu_Q(a)^k} \rho_{N,Q}^k \left(a + \frac{t_1}{N\mu_Q(a)}, \dots, a + \frac{t_k}{N\mu_Q(a)} \right)$$
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uniformly for $t_1, \dots, t_k = o(N)$.

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- Problem is that $\mu_Q^h(a)$ is not constant.

Local Bulk Correlations for $\beta = 2$: Unfolding

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With $I_N \subset [0, N]$ such that $\frac{1}{N}\text{dist}(I_N, \{0, N\}) \geq c > 0$ for N large enough

Theorem (Schubert-V., EJP '15)

With $\widehat{t_j} := (F_Q^h)^{-1}(t_j/N)$

$$\frac{1}{\prod_{j=1}^k \mu_Q^h(\widehat{t_j})} \rho_{N,Q}^{h,k}(\widehat{t_1}, \ldots, \widehat{t_k}) = \det \left[\frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)} \right]_{1 \leq i,j \leq k} + o(1)$$

uniform for $t_1, \ldots, t_k \in I_N$.

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Theorem (Schubert-V., EJP '15)

With $\widehat{t}_j := (F_Q^h)^{-1}(t_j/N)$ and $h = 0$ ($\beta = 2$-Ensemble),

$$\frac{1}{\prod_{j=1}^k \mu_Q(\widehat{t}_j)} \rho_{N,Q}^k(\widehat{t}_1, \ldots, \widehat{t}_k) = \det \left[\frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)} \right]_{1 \leq i,j \leq k} + \mathcal{O}\left(\frac{1}{N} \right)$$

uniform for $t_1, \ldots, t_k \in I_N$ and any $\varepsilon > 0$.

Nearest Neighbor Spacings

- Let $x_1 \leq x_2 \leq \cdots \leq x_N$ and I_N interval, nearest neighbor spacings in I_N:

$$\sigma(I_N, x) := \sum_{x_j, x_{j+1} \in I_N} \delta_{x_{j+1} - x_j}.$$

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- Classical result (Deift et al. 99): $a \in \operatorname{supp}\mu_Q^\circ$, $t_N \to \infty$ and $t_N/N \to 0$, s given, localized scaling $x_{\text{loc}} := N\mu_Q(a)x$,

$$\lim_{N\to\infty} \mathbb{E}_{N,Q} \frac{1}{2t_N\mu_Q(a)} \int_0^s d\sigma\big((a - t_N, a + t_N), x_{\text{loc}}\big) = G(s),$$

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$$\lim_{N\to\infty} \mathbb{E}_{N,Q} \frac{1}{2t_N\mu_Q(a)} \int_0^s d\sigma\big((a - t_N, a + t_N), x_{\text{loc}}\big) = G(s),$$

with G being the distribution function of the Gaudin distribution,

$$G(s) := \sum_{k\geq 2} \frac{(-1)^k}{(k-1)!} \int_{[0,s]^{k-1}} \det\left[(S(z_i - z_j))_{1\leq i,j\leq k}\right]\big|_{z_1=0} dz_2 \ldots dz_k,$$

where S is the sine kernel.

Nearest Neighbor Spacings: Kolmogorov Distance

- More recently (Tao'13; Erdős,Yau'15; Bekerman,Figalli,Guionnet'15; cf. Figalli,Guionnet'14) for any index set I excluding edge indices

$$\lim_{N\to\infty} \mathbb{E}_{N,Q} \frac{1}{|I|} \sum_{i\in I} g\Big(\mu_Q(q_i) N(x_{i+1} - x_i)\Big) = \int g(s) dG(s)$$

with q_i being the i/N-quantile of μ_Q, g test function.

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However, only a tiny fraction of all spacings is considered!

Nearest Neighbor Spacings: New Results

$$\widehat{\sigma}(I_N, x) := \frac{1}{\int_0^\infty d\sigma(I_N, x)} \sigma(I_N, x), \qquad (x_{\mathrm{unf}})_i := NF_Q^h(x_i).$$

Theorem (Schubert-V., EJP'15)

$$\lim_{N\to\infty} \mathbb{E}_{N,Q}^h \left(\sup_{s\in\mathbb{R}} \left| \int_0^s d\widehat{\sigma}([0,N], x_{\mathrm{unf}}) - G(s) \right| \right) = 0.$$

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Let h negative-definite. If $I_N \subset [0,N]$ with $\frac{1}{N}\mathrm{dist}(I_N, \{0,N\}) \geq c > 0$ for N large enough, then for any $\varepsilon > 0$

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$$\lim_{N \to \infty} \mathbb{E}_{N,Q}^h \left(\sup_{s \in \mathbb{R}} \left| \int_0^s d\widehat{\sigma}([0, N], x_{\text{unf}}) - G(s) \right| \right) = 0.$$

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If h arbitrary, then error is $o(1)$.

Edge Correlations for Repulsive Particles

$$P^h_{N,Q}(x) = \frac{1}{Z^h_{N,Q}} \prod_{i<j} |x_i - x_j|^2 \, e^{-\sum_{i<j} h\left(x_i - x_j\right) - N \sum_{j=1}^N Q(x_j)}.$$

Theorem (Kriecherbauer-V.,'15)

Let $\mathrm{supp}\mu^h_Q = [-b, b]$. There exists $c^* > 0$ such that for $q < 0 < p$ and $t \in [q, pN^{4/15}]^k$

$$\left(\frac{N^{1/3}}{c^*}\right)^k \rho^{h,k}_{N,Q}\left(b + \frac{t_1}{c^* N^{2/3}}, \ldots, b + \frac{t_k}{c^* N^{2/3}}\right) = \det\left[K_{\mathrm{Ai}}(t_i, t_j)\right]_{1 \le i,j \le k} + o(1).$$

$$P_{N,Q}^h(x) = \frac{1}{Z_{N,Q}^h} \prod_{i<j} |x_i - x_j|^2 \, e^{-\sum_{i<j} h(x_i - x_j) - N \sum_{j=1}^N Q(x_j)}.$$

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If h is negative-definite ($\widehat{h} \le 0$), then replace $o(1)$ by $\mathcal{O}(N^{-1/3+\varepsilon})$, $\varepsilon > 0$.

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Edge universality for β-Ensembles: Ramirez,Rider,Virag'11; Krishnapur,Rider,Virag'15; Bourgade,Erdős,Yau'14; Bekerman,Figalli,Guionnet'15; cf. Figalli,Guionnet'14

Theorem (Kriecherbauer-V.,'15)

For any (fixed) $s \in \mathbb{R}$,

$$P_{N,Q}^{h}\left((x_{\max} - b)\, c^{*} N^{2/3} \leq s\right) = F_{\mathrm{TW}}(s) + o(1).$$

where F_{TW} is the distribution function of the ($\beta = 2$) Tracy-Widom distribution. If h is negative-definite ($\widehat{h} \leq 0$), then replace $o(1)$ by $\mathcal{O}(N^{-1/3+\varepsilon})$ for any $\varepsilon > 0$.

Asymptotics for the largest Particle: Fluctuations and Deviations

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- Large deviations: $s \to \infty$ and $t = b + \frac{s}{c^* N^{2/3}} > b$. Non-universal!

Moderate deviations in logarithmic form:

Theorem (Kriecherbauer-V.,'15)

$$\frac{\log P^h_{N,Q}\left((x_{\max} - b)c^* N^{2/3} > s\right)}{s^{3/2}} = -\frac{4}{3} + o(1)$$

with $o(1)$ uniform in $s \in [1, o(N^{2/3})]$.

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with $o(1)$ uniform in $s \in [1, o(N^{2/3})]$. For h negative definite, replace $o(1)$ by $-\frac{\log(16\pi s^{3/2})}{s^{3/2}} + \mathcal{O}\left(\frac{s}{N^{2/3}}\right) + \mathcal{O}\left(\frac{1}{s^3}\right)$.

Uses known right-tail asymptotics for Tracy-Widom distribution:

$$1 - F_{\mathrm{TW}}(s) = \frac{1}{16\pi}\frac{e^{-\frac{4}{3}s^{3/2}}}{s^{3/2}}\left(1 + \mathcal{O}\left(\frac{1}{s^{3/2}}\right)\right), \quad \text{for } s \to \infty.$$

Theorem (Kriecherbauer-V.,'15)

$$\frac{P^h_{N,Q}\left((x_{\max} - b)c^* N^{2/3} > s\right)}{(1 - F_{\mathrm{TW}}(s))} = \exp\left(s^{3/2} \sum_{j=1}^{\infty} d_j \left(\frac{s}{N^{2/3}}\right)^j\right)(1 + o(1))$$

uniformly for $s = o(N^{2/3})$ with coefficients d_j depending on Q and h.

Moderate Deviations: Non-Logarithmic Form

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uniformly for $s=o(N^{2/3})$ with coefficients d_j depending on Q and h.

- To be compared with result in classical probability (Cramér'38) for sums of i.i.d. r.v. X_i with mean μ and variance σ^2:

$$\frac{P\left(\frac{\sum_{i=1}^N X_i-\mu}{\sigma\sqrt{N}}>s\right)}{(1-\Phi(s))}=\exp\left(s^2\sum_{j=1}^{\infty}a_j\left(\frac{s}{\sqrt{N}}\right)^j\right)(1+o(1)).$$

Moderate Deviations: Non-Logarithmic Form

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- Describes the transition from universality to non-universality!

Theorem (Kriecherbauer-V.,'15)

Let $M > b$ and $0 < \varepsilon < 2/3$. Then uniformly for $t \in [b + N^{-\varepsilon}, M]$ (large deviations),

$$\frac{\log P_{N,Q}^{h}\left(x_{\max} > t\right)}{N} = -\eta_{Q,h}(t) - \frac{\log\left(N(t-b)^{3/2}\right)}{N} + \mathcal{O}\left(\frac{1}{N}\right),$$

where the $\mathcal{O}$ term is uniform in N and in $t \in (b + N^{-2/3}, T)$ and the rate function is

$$\eta_{Q,h}(t) = -2\int \log|t-s|\, d\mu_Q^h(s) + \int h(t-s) d\mu_Q^h(s) + Q(t) - C.$$

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- Without rate in Borot,Guionnet,Kozlowski'15. Non-logarithmic form for $\beta = 2$-Ensembles in Eichelsbacher,Kriecherbauer,Schüler'16.

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- Without rate in Borot,Guionnet,Kozlowski'15. Non-logarithmic form for $\beta = 2$-Ensembles in Eichelsbacher,Kriecherbauer,Schüler'16.
- Conjecture new source of non-universality.

Ideas of the Method: Determining the Equilibrium Measure 1

$$P^h_{N,Q,\beta}(x) = \frac{1}{Z^h_{N,Q,\beta}} \prod_{i<j} |x_j - x_i|^\beta \, e^{-\sum_{i<j} h(x_i - x_j) - N \sum_{j=1}^N Q(x_j)}.$$

- Idea: Compare $P^h_{N,Q,\beta}$ with β-Ensemble $P_{N,V,\beta}$ that has the same limiting measure.

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- Idea: Compare $P^h_{N,Q,\beta}$ with β-Ensemble $P_{N,V,\beta}$ that has the same limiting measure.
- Let μ probability measure on $\mathbb{R}$. Hoeffding type decomposition

$$\sum_{i<j} h(x_i - x_j) = c_N + N\sum_{j=1}^N h_\mu(x_j) - \mathcal{U}_\mu(x),$$

  where c_N is a constant, $h_\mu(t) := \int h(t-s)d\mu(s)$ and $\mathcal{U}_\mu$ is the quadratic statistic

$$\mathcal{U}_\mu(x) := -\frac{1}{2}\left(\sum_{i,j=1}^N h(x_i - x_j) - h_\mu(x_i) - h_\mu(x_j) + h_{\mu\mu}\right).$$

Ideas of the Method: Determining the Equilibrium Measure 2

Now, with $V_\mu(t) := Q(t) + h_\mu(t)$

$$P^h_{N,Q,\beta}(x) = \frac{1}{Z^h_{N,Q,\beta}} \prod_{i<j} |x_i - x_j|^\beta e^{-\sum_{i<j} h(x_i - x_j) - N \sum_{j=1}^N Q(x_j)}$$

$$= \frac{e^{-c_N} Z_{N,V_\mu,\beta}}{Z^h_{N,Q,\beta}} \frac{1}{Z_{N,V_\mu,\beta}} \prod_{i<j} |x_i - x_j|^\beta e^{-N \sum_{j=1}^N V_\mu(x_j) + \mathcal{U}_\mu(x)}$$

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- Choose μ such that $\mathcal{U}_\mu$ is small under $P_{N,V_\mu,\beta}$. But $\mathcal{U}_\mu$ is centered under μ! Thus

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- Solve this implicit problem by applying a fixed point theorem: get $\mu^h_{Q,\beta}$. $V := Q + h_{\mu^h_{Q,\beta}}$.

Ideas of the Method: Determining the Equilibrium Measure 2

Now, with $V_\mu(t) := Q(t) + h_\mu(t)$

$$P^h_{N,Q,\beta}(x) = \frac{1}{Z^h_{N,Q,\beta}} \prod_{i<j} |x_i - x_j|^\beta e^{-\sum_{i<j} h(x_i - x_j) - N\sum_{j=1}^N Q(x_j)}$$

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- Solve this implicit problem by applying a fixed point theorem: get $\mu^h_{Q,\beta}$. $V := Q + h_{\mu^h_{Q,\beta}}$.
- Show that $\mathcal{U} = \mathcal{U}_{\mu^h_{Q,\beta}}$ is indeed just a perturbation. Here convexity used (concentration inequalities).

- Linearization trick: Recall that for $-h$ positive definite ($-\widehat{h} \geq 0$)

$$\exp(-\frac{1}{2}\sum_{i,j} h(x_i - x_j)) = \mathbb{E}\exp(\sum_{j=1}^{N} f(x_j)).$$

Moreover

Ideas of the Method: Linearization

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Moreover

$$\exp(\mathcal{U}(x)) = \mathbb{E}\exp(\sum_{j=1}^{N} f(x_j) - N\int f\, d\mu_{Q,\beta}^{h}).$$

- Now add f to the potential: ensembles with densities proportional to

$$\prod_{i<j} |x_i - x_j|^{\beta}\, e^{-\sum_{j=1}^{N} NV(x_j)+f(x_j)}.$$

Perturbation f does not change global or local asymptotics.

Summary

$$P^h_{N,Q,\beta}(x) = \frac{1}{Z^h_{N,Q,\beta}} \prod_{i<j} |x_i - x_j|^\beta e^{-\sum_{i<j} h(x_i - x_j) - N \sum_{j=1}^N Q(x_j)}$$

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Thank you for your attention!