Asymptotic syzygies of Stanley-Reisner rings of iterated subdivisions

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Outline

1. Motivation
2. The simplex case
   - Barycentric subdivisions
   - Edgewise subdivisions
3. Asymptotic behavior

Notation

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- projective dimension of A:
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  - For r sufficiently large

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  - Moreover,

$$\lim_{r \to \infty} \frac{\#\{i \ : \ \beta_{i,i+j}(A^{(r)}) \neq 0\}}{\text{pdim } A^{(r)}} = 1.$$

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Ein/Erman/Lazarsfeld:

- Considered Cohen-Macaulay algebras A:
  - For $1 \leq j \leq \dim A - 1$ and r sufficiently large

$$\beta_{i,i+j}(A^{(r)}) \neq 0 \text{ for all } i \in [c_j, d_j]$$

    with endpoints c_j, d_j depending on j.

  - For $1 \leq j \leq \dim A - 1$

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Theorem (Sturmfels)

Let $\mathcal{A} = \{(i_1, \ldots, i_n) \in \mathbb{N}_0^n \ : \ i_1 + \cdots + i_n = r\}$, $A = \mathbb{K}[x_1, \ldots, x_n]$ *and* $A^{(r)} \cong \mathbb{K}[x_{i_1,\ldots,i_n} \ : \ (i_1, \ldots, i_n) \in \mathcal{A}]/I_r$.

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Given a term order $\preceq$, *there exists a regular triangulation* Δ *of the point set* $\mathcal{A} \subseteq \mathbb{R}^n$ *such that*

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Here: Δ *is the projection of the lower hull of the "lifted" point set* $\mathcal{A}$.

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Theorem

If I *is a homogeneous ideal in* $A = \mathbb{K}[x_1, \ldots, x_n]$. *Then*

$$\beta_{i,j}(A/I) \leq \beta_{i,j}(A/\mathrm{in}_{\preceq}(I)).$$

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2.) What happens asymptotically? I.e., for $r \to \infty$ study

$$\frac{\#\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\Delta(r)]) \neq 0\}}{\operatorname{pdim} \mathbb{K}[\Delta(r)]}.$$

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Δ simplicial complex

The barycentric subdivision of Δ is the simplicial complex $\mathrm{sd}(\Delta)$ on vertex set $\Delta \setminus \{\emptyset\}$, whose faces are chains

$$\emptyset \neq A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_r$$

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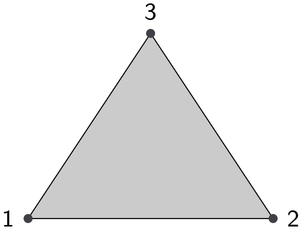
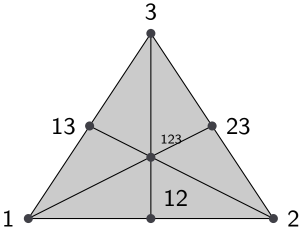
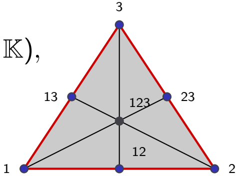




Hochster's formula

Let Δ be a simplicial complex on vertex set $[n] := \{1, 2, \ldots, n\}$ and let $\mathbb{K}[\Delta]$ be the Stanley-Reisner ring of Δ. Then:

$$\beta_{i,i+j}(\mathbb{K}[\Delta]) = \sum_{\substack{W \subseteq [n] \\ \#W = i+j}} \dim_{\mathbb{K}} \widetilde{H}_{j-1}(\Delta_W; \mathbb{K}),$$



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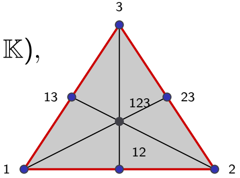

where

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In particular,

$$\beta_{i,i+j}(\mathbb{K}[\Delta]) \neq 0.$$

$\Leftrightarrow$ There exists $W \subseteq [n]$, $\#W = i + j$ and $\widetilde{H}_{j-1}(\Delta_W; \mathbb{K}) \neq 0$.

The Castelnuovo-Mumford regularity of $\mathbb{K}[\mathrm{sd}(\Delta)]$

$$\operatorname{reg} \mathbb{K}[\mathrm{sd}(\Delta)] = \begin{cases} \dim \Delta, & \text{if } \widetilde{H}_{\dim \Delta}(\Delta; \mathbb{K}) = 0 \\ \dim \Delta + 1, & \text{if } \widetilde{H}_{\dim \Delta}(\Delta; \mathbb{K}) \neq 0 \end{cases}$$

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$\Rightarrow$ $\widetilde{H}_{d-1}(\mathrm{sd}(\Delta);\mathbb{K}) = 0$

$\Rightarrow$ $\widetilde{H}_{d-1}(\mathrm{sd}(\Delta)_W;\mathbb{K}) = 0$ for subsets W of the vertices of $\mathrm{sd}(\Delta)$

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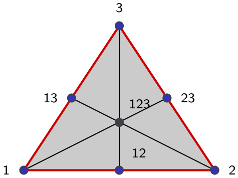

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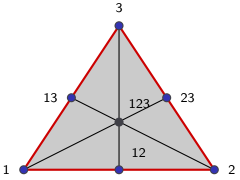

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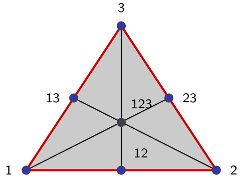
$\Rightarrow$ $\beta_{2^d-2-(d-1),2^d-2}(\mathbb{K}[\mathrm{sd}(\Delta)]) \neq 0$

$\Rightarrow$ $\operatorname{reg} \mathbb{K}[\mathrm{sd}(\Delta)] \geq d - 1$



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$$\widetilde{H}_{d-1}(\mathrm{sd}(\Delta);\mathbb{K}) \neq 0.$$

$\Rightarrow$ $\beta_{\#\{\emptyset\neq F\in\Delta\}-d,\#\{\emptyset\neq F\in\Delta\}}(\mathbb{K}[\mathrm{sd}(\Delta)]) \neq 0$

$\Rightarrow$ $\operatorname{reg}\mathbb{K}[\mathrm{sd}(\Delta)] = d$ $\square$

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$$\beta_{i,i+j}(\mathrm{sd}(\Delta_{d-1})) \begin{cases} \neq 0 \text{ for } j \leq i \leq 2^d - d - 1 - m_{d-j-1}, \\ = 0 \text{ for } 0 \leq i \leq j-1 \text{ and } 2^d - 2d + j < i \leq 2^d - d - 1. \end{cases}$$

$m_j := 2^{a+2}(c + d - j) - 2d + j$, where $(2j - d) = a(d-j) + c$ for $0 \leq c < d - j$.

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(ii) *If $\frac{d}{2} < j \leq d-2$, then*

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(iii) $\beta_{i,i+d-1}(\mathrm{sd}(\Delta_{d-1})) \neq 0$ *if and only if* $i = 2^d - d - 1$.

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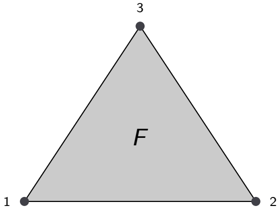

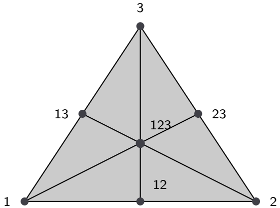

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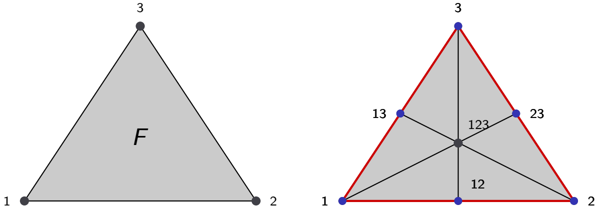

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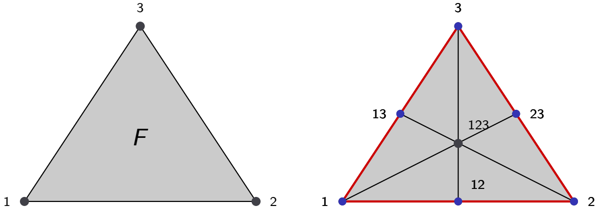
But: $2^{j+1} - 2 - j > j$ is not good enough!



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Idea:

Construct induced subcomplexes that are boundaries of cross polytopes.

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Recall:

The boundary of the j-dimensional cross polytope is the join of j 0-spheres.

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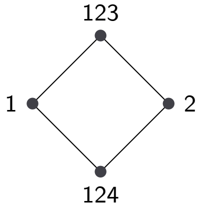

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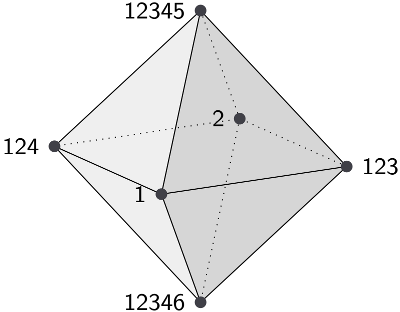
- $j = 3$:



Outline

1. Motivation
2. The simplex case
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   - Edgewise subdivisions
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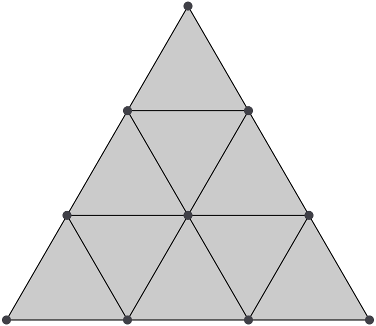
Edgewise subdivisions

Edgewise subdivisions

- Basic idea: Edges are subdivided into r pieces.



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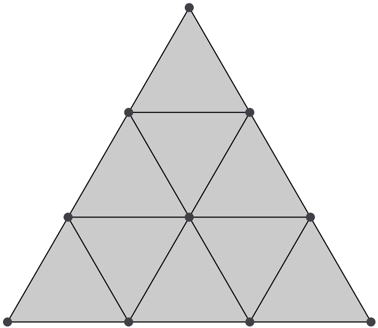

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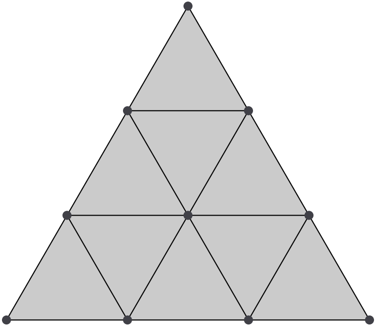

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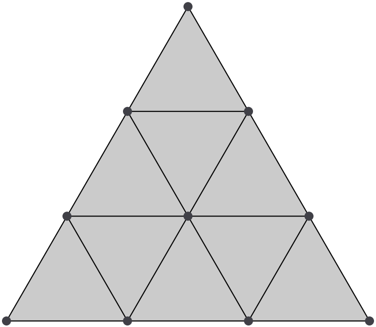

From edgewise subdivisions to Veronese algebras

Theorem (Brun, Römer)

Δ *simplicial complex on vertex set* $[n]$, $r \geq 1$.

Let

$$A^{\langle r \rangle} = \mathbb{K}[x_{i_1,\dots,i_n} \ : \ i_1 + \cdots + i_n = r].$$

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We can apply the results of Ein, Lazarsfeld and Erman if Δ is Cohen-Macaulay.

The edgewise subdivision of the simplex

For edgewise subdivisions ($r \geq d$) we have

Theorem (Conca, J.-K., Welker)

(i) *If* $1 \leq j \leq \frac{d}{2}$, *then*

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$m_j := 2^{a+2}(c+d-j) - 2d + j$, where $(2j-d) = a(d-j) + c$ for $0 \leq c < d-j$.

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(iii) $\beta_{i,i+d-1}(\mathbb{K}[\Delta_{d-1}^{\langle r\rangle}]) \neq 0$ *for* $2^d - d - 1 \leq i \leq \operatorname{pdim} \mathbb{K}[\Delta_{d-1}^{\langle r\rangle}]$.

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The strands in the Betti table go until the very end!

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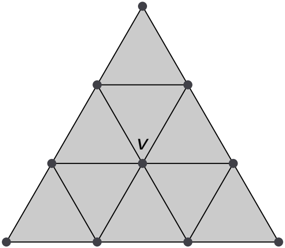

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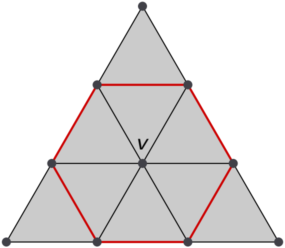

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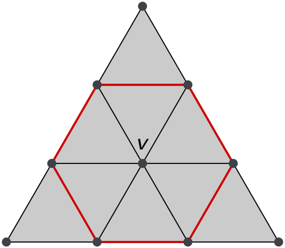

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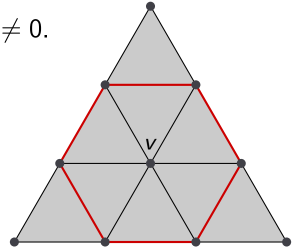
- $\Rightarrow$ If $\beta_{i,i+j}(\mathbb{K}[\mathrm{sd}(\Delta_{d-1})]) \neq 0$, then $\beta_{i,i+j}(\mathbb{K}[\Delta_{d-1}^{\langle r \rangle}]) \neq 0$.
- $\Rightarrow$ We obtain the same lower bounds.



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Then,

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- There exists $F \in \Delta_{d-1}^{\langle r \rangle}$ such that
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- Apply the results for the $(d-1)$-simplex to F and choose A such that $\mathrm{sd}(F)_A \cong \mathbb{S}^{j-1}$ for a fixed j.

The special case $\widetilde{H}_{d-1}(\Delta;\mathbb{K}) \neq 0$

Δ $(d-1)$-dimensional simplicial complex

$\Delta(r)$ r^{th} barycentric or r^{th} edgewise subdivision of Δ Then:

$$\lim_{r\to\infty} \frac{\#\{i \ : \ \beta_{i,i+d}(\mathbb{K}[\Delta(r)]) \neq 0\}}{\operatorname{pdim}\mathbb{K}[\Delta(r)]} = 1 - \frac{f^{\sigma}_{d-1}}{f^{\Delta}_{d-1}},$$

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In particular, any rational number $[0,1)$ can occur as limit.

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Theorem:

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Thank you for your attention!