New well-posedness results for perfectly matched layers

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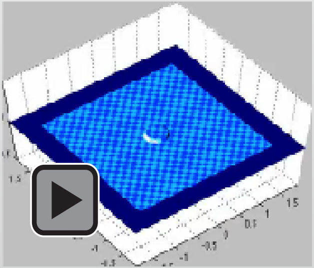
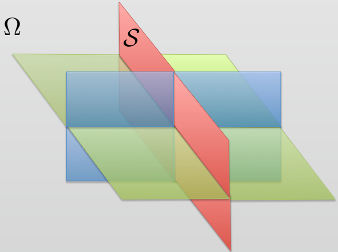
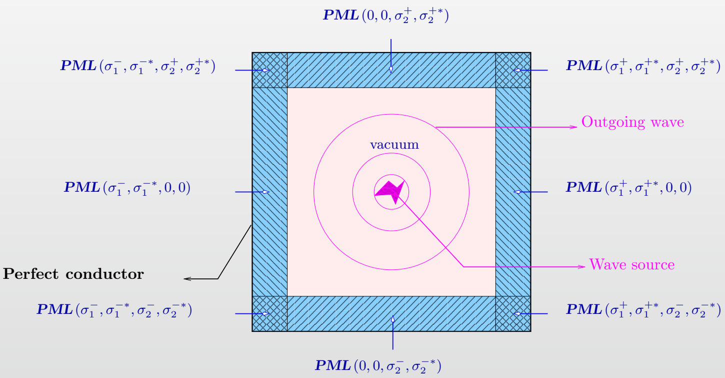




Outline

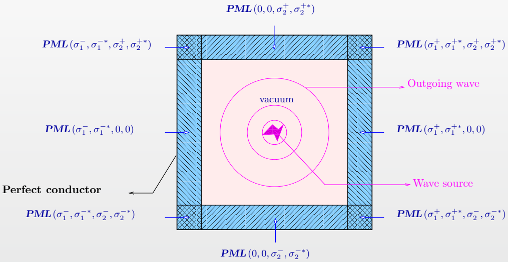
Perfectly matched layers

Bérenger, 1994 1996, Maxwell 2D and 3D.



Properties : Perfect matching, exponential decay.

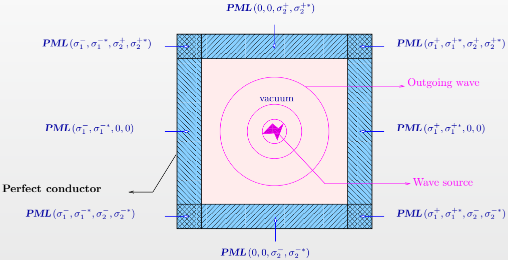
Construction



$$L\mathcal{U} := \partial_t \mathcal{U} + A_1 \partial_{x_1} \mathcal{U} + A_2 \partial_{x_2} \mathcal{U} + A_3 \partial_{x_3} \mathcal{U} = F, \; U : \mathbb{R}^3 \to \mathbb{R}^N$$

▸ WP

Construction



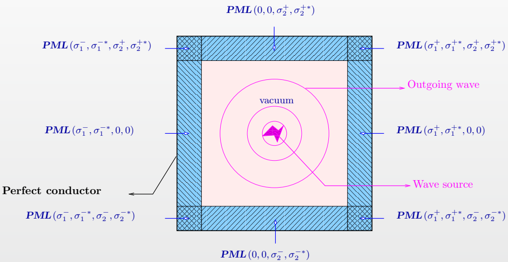
$$L\mathcal{U} := \partial_t\mathcal{U} + A_1\partial_{x_1}\mathcal{U} + A_2\partial_{x_2}\mathcal{U} + A_3\partial_{x_3}\mathcal{U} = F,\ U : \mathbb{R}^3 \to \mathbb{R}^N$$
$$L\,U := \partial_t U + A_1\partial_{x_1} U + A_2\partial_{x_2} U + A_3\partial_{x_3} U = 0$$

Splitting $\downarrow$

$$\begin{cases} \partial_t U^1 + A_1\partial_{x_1}(U^1+U^2+U^3) & = 0 \\ \partial_t U^2 + A_2\partial_{x_2}(U^1+U^2+U^3) & = 0 \\ \partial_t U^3 + A_3\partial_{x_3}(U^1+U^2+U^3) & = 0 \\ U = U^1+U^2+U^3 & \end{cases}$$

WP

Construction



$$L\mathcal{U} := \partial_t\mathcal{U} + A_1\partial_{x_1}\mathcal{U} + A_2\partial_{x_2}\mathcal{U} + A_3\partial_{x_3}\mathcal{U} = F,\ U : \mathbb{R}^3 \to \mathbb{R}^N$$
$$\partial_t U + A_1\partial_{x_1} U + A_2\partial_{x_2} U + A_3\partial_{x_3} U = 0$$

$$\textit{Splitting} \downarrow \textit{Absorption}$$

$$\begin{cases} \partial_t U^1 + A_1\partial_{x_1}(U^1+U^2+U^3) + \sigma_1(x_1)U^1 = 0 \\ \partial_t U^2 + A_2\partial_{x_2}(U^1+U^2+U^3) + \sigma_2(x_2)U^2 = 0 \\ \partial_t U^3 + A_3\partial_{x_3}(U^1+U^2+U^3) + \sigma_3(x_3)U^3 = 0 \\ U = U^1+U^2+U^3 \end{cases}$$

WP

Outline

Well-posedness for the homogeneous operator

$$L(\partial_t, \partial_x)U := \partial_t U + \sum A_j \partial_j U = 0$$

$$L(0,k) = \sum k_j A_j, \quad \hat{U}(t) = e^{-iL(0,k)t}\,\hat{U}_0$$

Cauchy problem **strongly** **well-posed** (Maxwell symmetric hyperbolic)

$$\|U(t,.)\|_{L^2(\mathbb{R}^2)} \leq K e^{\alpha\, t} \|U^0\|_{L^2(\mathbb{R}^2)}$$

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$$\widetilde{L}_1(\partial_t, \partial_x)\widetilde{U} := \{\partial_t U^j + A_j \partial_j (U^1 + U^2 + U^3)\}_{j=1,\ldots 3} = 0$$

$$\widetilde{L}_1(0,k) = \begin{pmatrix} k_1 A_1 & k_1 A_1 & k_1 A_1 \\ k_2 A_2 & k_2 A_2 & k_2 A_2 \\ k_3 A_3 & k_3 A_3 & k_3 A_3 \end{pmatrix}, \quad \widetilde{U}(t) = e^{-i\widetilde{L}(0,k)t}\,\widetilde{U}_0$$

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Cauchy problem only weakly well-posed

$$\|(U^1(t,.), U^2(t,.), U^3(t,.))\|_{L^2(\mathbb{R}^2)} \leq K(1+t)e^{\alpha t}\|U^0\|_{H^1(\mathbb{R}^2)}$$

Tools: Gårding and Kreiss

Heinz-Otto Kreiss
12-2015

Bérenger's model is only weakly well-posed

PMLs were originally introduced for Maxwell's equations by Bérenger [8]. Well-posedness and stability of the B´erenger PML has been the topic of numerous works. For example Abarbanel and Gottlieb [1] showed that B´erenger's "split-field" PML was only weakly well-posed and that it supported linearly growing modes. Similar results were also obtained via Fourier and energy techniques by B´ecache and Joly in [6]. The issue of weak well-posedness led to the development of various well-posed

Appelö-Hagström-Kreiss(Gunilla), 2006

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Appelö-Hagström-Kreiss(Gunilla), 2006

[6] : From [9] we know that the corresponding Cauchy problem is weakly well-posed but not strongly well-posed: there is necessarily a loss of regularity, at least for some initial data.

Closer look

2-D Maxwell

$$
\begin{aligned}
\varepsilon_0 \partial_t E_x &= \partial_y H \\
\varepsilon_0 \partial_t E_y &= -\partial_x H \\
\mu_0 \partial_t H &= \partial_y E_x - \partial_x E_y
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \hat{E}_x}{\partial t} &= \frac{i\omega_2}{\varepsilon_0}(\hat{H}_x + \hat{H}_y) \\
\frac{\partial \hat{E}_y}{\partial t} &= -\frac{i\omega_1}{\varepsilon_0}(\hat{H}_x + \hat{H}_y) \\
\frac{\partial \hat{H}_x}{\partial t} &= -\frac{i\omega_1}{\mu_0}\hat{E}_y \\
\frac{\partial \hat{H}_y}{\partial t} &= \frac{i\omega_2}{\mu_0}\hat{E}_x,
\end{aligned}
$$

Numerical experiments

Prediction $\|(H_x, H_y)(t,.)\|_{L^2(\mathbb{R}^2)} \leq K(1+t)e^{\alpha t}\|U^0\|_{H^1(\mathbb{R}^2)}$

Initial data $\mathbf{E}^0 = \mathbf{a}(x,y)\, e^{2\pi i\,\omega\,\mathbf{v}\cdot(x,y)}$, $\mathbf{v} = \dfrac{1}{\sqrt{2}}\left(1,-1\right)$, $\omega = 5\times 2^n,\ 0 \leq n \leq 5.$

Maxwell system: $\|(\mathbf{E}, H)\|_{L^2_{t,x}}$, Bérenger system $\|(\mathbf{E}, H_x, H_y, H)\|_{L^2_{t,x}}$.
Normalized by $\|(E,H)\|_{L^2_x}$ at initial time.

Frequency	10	20	40	80	160
Maxwell	0.1702	0.1703	0.1703	0.1703	0.1703
Berenger	0.2121	0.3012	0.5247	1.0036	1.9546

Table : L^2 norm as a function of the frequency. General case

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Frequency	10	20	40	80	160
Maxwell	0.1269	0.1132	0.1162	0.1226	0.1266
Berenger	0.0642	0.0568	0.0581	0.0613	0.0633

Table : L^2 norm as a function of the frequency. $\mathrm{div}(E_0) = 0$

Numerical experiments

- For solenoidal initial data ($\operatorname{div} \mathbf{E} = 0$),

$$\|(\mathbf{E}, H)\|_{L^2((0,T)\times\Omega)} = C(T)\|(\mathbf{E}, H)(0)\|_{L^2(\Omega)}$$

$$\|(H_x, H_y)\|_{L^2((0,T)\times\Omega)} = C(T)\|(\mathbf{E}, H)(0)\|_{L^2(\Omega)}$$

- For non solenoidal initial data ($\operatorname{div} \mathbf{E} \neq 0$)

$$\|(\mathbf{E}, H)\|_{L^2((0,T)\times\Omega)} = C(T)\|(\mathbf{E}, H)(0)\|_{L^2(\Omega)}$$

$$\|(H_x, H_y)\|_{L^2((0,T)\times\Omega)} \simeq C(T)\omega\|(\mathbf{E}, H)(0)\|_{L^2(\Omega)}$$

$\sigma = 0$, STRONG WELL-POSEDNESS FOR PHYSICAL SOLUTIONS.

Proof (Abarbanel-Gottlieb)

Abarbanel/Gottlieb's solution ▸ Calculs Maple

$$\hat{E}_x(0) = \hat{e}_0, \quad \hat{E}_y(0) = \hat{g}_0$$
$$\hat{H}_x(0) = \hat{h}_0 - \ell_0, \quad \hat{H}_y(0) = \ell_0$$

Cauchy data

$$\hat{E}_x = \frac{\omega_1^2}{\omega_1^2 + \omega_2^2}\hat{e}_0 + \frac{\omega_1\omega_2}{\omega_1^2 + \omega_2^2}\hat{g}_0 + \frac{i\omega_2}{\varepsilon_0 c\sqrt{\omega_1^2 + \omega_2^2}}(\hat{h}_0 \sin \nu t - \beta \cos \nu t)$$

$$\hat{E}_y = \frac{\omega_1\omega_2}{\omega_1^2 + \omega_2^2}\hat{e}_0 + \frac{\omega_2^2}{\omega_1^2 + \omega_2^2}\hat{g}_0 - \frac{i\omega_1}{\varepsilon_0 c\sqrt{\omega_1^2 + \omega_2^2}}(\hat{h}_0 \sin \nu t - \beta \cos \nu t)$$

$$\hat{H}_x = \frac{\omega_2^2\hat{h}_0 - (\omega_1^2 + \omega_2^2)\hat{\ell}_0}{\omega_1^2 + \omega_2^2} - \frac{i\omega_1\omega_2(\omega_1\hat{e}_0 + \omega_2\hat{g}_0)}{(\omega_1^2 + \omega_2^2)\mu_0}t + \frac{\omega_1^2}{\omega_1^2 + \omega_2^2}(\hat{h}_0 \cos \nu t + \omega \sin \nu t)$$

$$\hat{H}_y = -\frac{\omega_2^2\hat{h}_0 - (\omega_1^2 + \omega_2^2)\hat{\ell}_0}{\omega_1^2 + \omega_2^2} + \frac{i\omega_1\omega_2(\omega_1\hat{e}_0 + \omega_2\hat{g}_0)}{(\omega_1^2 + \omega_2^2)\mu_0}t$$

$$\omega_2^2 \ldots$$

Proof (Abarbanel-Gottlieb). Continue

$$\hat{E}_x = \frac{\omega_1^2}{\omega_1^2+\omega_2^2}\hat{e}_0 + \frac{\omega_1\omega_2}{\omega_1^2+\omega_2^2}\hat{g}_0 + \frac{i\omega_2}{\varepsilon_0 c\sqrt{\omega_1^2+\omega_2^2}}(\hat{h}_0 \sin \nu t - \beta \cos \nu t)$$

$$\hat{E}_y = \frac{\omega_1\omega_2}{\omega_1^2+\omega_2^2}\hat{e}_0 + \frac{\omega_2^2}{\omega_1^2+\omega_2^2}\hat{g}_0 - \frac{i\omega_1}{\varepsilon_0 c\sqrt{\omega_1^2+\omega_2^2}}(\hat{h}_0 \sin \nu t - \beta \cos \nu t)$$

$$\hat{H}_x = \frac{\omega_2^2\hat{h}_0 - (\omega_1^2+\omega_2^2)\hat{\ell}_0}{\omega_1^2+\omega_2^2} - \frac{i\omega_1\omega_2(\omega_1\hat{e}_0+\omega_2\hat{g}_0)}{(\omega_1^2+\omega_2^2)\mu_0}t + \frac{\omega_1^2}{\omega_1^2+\omega_2^2}(\hat{h}_0 \cos \nu t + \omega \sin \nu t)$$

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$$\hat{E}_x(0) = \hat{e}_0, \quad \hat{E}_y(0) = \hat{g}_0$$

$$\hat{H}_x(0) = \hat{h}_0 - \ell_0, \quad \hat{H}_y(0) = \ell_0$$

$$\widehat{\mathrm{div}(\mathbf{E})}(\omega, t=0)$$

Proof (Abarbanel-Gottlieb). Continue

For a physical solution,

$$\operatorname{div}(\mathbf{E}) = 0$$

$$\hat{E}_x = \frac{\omega_1^2}{\omega_1^2 + \omega_2^2}\hat{e}_0 + \frac{\omega_1\omega_2}{\omega_1^2 + \omega_2^2}\hat{g}_0 + \frac{i\omega_2}{\varepsilon_0 c\sqrt{\omega_1^2 + \omega_2^2}}(\hat{h}_0 \sin \nu t - \beta \cos \nu t)$$

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$$\hat{H}_x = \frac{\omega_2^2\hat{h}_0 - (\omega_1^2 + \omega_2^2)\hat{l}_0}{\omega_1^2 + \omega_2^2} - \frac{i\omega_1\omega_2(\omega_1\hat{e}_0 + \omega_2\hat{g}_0)}{(\omega_1^2 + \omega_2^2)\mu_0}t + \frac{\omega_1^2}{\omega_1^2 + \omega_2^2}(\hat{h}_0 \cos \nu t + \omega \sin \nu t)$$

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Numerical experiments

- For solenoidal initial data ($\operatorname{div} \mathbf{E} = 0$),

$$\|(\mathbf{E}, B_z)\|_{L^2((0,T)\times\Omega)} = C(T)\|(\mathbf{E}, B_z)\|_{L^2(\Omega)}$$

$$\|(B_{zx}, B_{zy})\|_{L^2((0,T)\times\Omega)} = C(T)\|(\mathbf{E}, B_z)\|_{L^2(\Omega)}$$

- For non solenoidal initial data ($\operatorname{div} \mathbf{E} \neq 0$)

$$\|(\mathbf{E}, B_z)\|_{L^2((0,T)\times\Omega)} = C(T)\|(\mathbf{E}, B_z)\|_{L^2(\Omega)}$$

$$\|(B_{zx}, B_{zy})\|_{L^2((0,T)\times\Omega)} \simeq C(T)k\|(\mathbf{E}, B_z)\|_{L^2(\Omega)}$$

$\sigma = 0$, STRONG WELL-POSEDNESS FOR PHYSICAL SOLUTIONS.

Well-posedness for the full operator

▸ Equations

$$\widetilde{L}(\partial_t, \partial_x)\widetilde{U} := \{\partial_t U^j + A_j \partial_j (U^1 + U^2 + U^3) + \sigma_j U^j\}_{j=1,\dots 3} = 0$$

THEOREM

1. The Cauchy problem for L_1 is weakly well posed if and only if for each $\xi \in \mathbb{R}^d$, the eigenvalues of $L_1(0,\xi)$ are real.
2. The Cauchy problem for L_1 is strongly well posed if and only if for each $\xi \in \mathbb{R}^d$, the eigenvalues of $L_1(0,\xi)$ are real and $L_1(0,\xi)$ is uniformly diagonalisable, there is an invertible $S(\xi)$ satisfying,

$$S(\xi)^{-1} L_1(0,\xi)\, S(\xi) = \mathrm{diagonal}, \qquad S\,,\, S^{-1} \in L^\infty(\mathbb{R}^d_\xi)\,.$$

3. If $\mathcal{B}$ has constant coefficients, then the Cauchy problem for $L = L_1 + B$ is weakly well posed if and only if there exists $M \geq 0$ such that for any $\xi \in \mathbb{R}^d$, $\det L(\tau,\xi) = 0 \implies |\Im\tau| \leq M$.

Results for constant absorption

$$\widetilde{L}(\partial_t,\partial_x)\widetilde{U} := \{\partial_t U^j + A_j\partial_j(U^1+U^2+U^3) + \sigma_j U^j\}_{j=1,\ldots 3} = 0$$

THEOREM(HRP,Confluentes Mathematicii 2011. Generalizing several papers) Suppose that $\tau = 0$ is an isolated root of constant multiplicity m of $\det L_1(\tau,\xi) = 0$.

1. If the Cauchy problem for L_1 is strongly well posed, then for arbitrary constant absorptions $\sigma_j \in \mathbb{C}$, the Cauchy problem for $\widetilde{L}_1 + B$ is weakly well posed.
2. If the Cauchy problem for L_1 is strongly well posed, and if there is a $\xi \neq 0$ such that $\ker L(0,\xi) \neq \bigcap_{\xi_j\neq 0} \ker A_j$, then $\widetilde{L}_1(0,\xi)$ is not diagonalizable. Therefore the Cauchy problem for $\widetilde{L}$ is not strongly well posed.
3. If the Cauchy problem for L is strongly well posed and for all ξ,

   well posed. This condition holds if $L_1(0,\partial_x)$ is elliptic, that is $\det L_1(0,\xi) \neq 0$ for all real ξ.

Results for constant absorption

$$\widetilde{L}(\partial_t, \partial_x)\widetilde{U} := \{\partial_t U^j + A_j \partial_j (U^1 + U^2 + U^3) + \sigma_j U^j\}_{j=1,\dots 3} = 0$$

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1. If the Cauchy problem for L_1 is strongly well posed, then for arbitrary constant absorptions $\sigma_j \in \mathbb{C}$, the Cauchy problem for $\widetilde{L}_1 + B$ is weakly well posed.
2. If the Cauchy problem for L_1 is strongly well posed, and if there is a $\xi \neq 0$ such that $\ker L(0, \xi) \neq \bigcap_{\xi_j \neq 0} \ker A_j$, then $\widetilde{L}_1(0, \xi)$ is not diagonalizable. Therefore the Cauchy problem for $\widetilde{L}$ is not strongly well posed.
3. If the Cauchy problem for L is strongly well posed and for all ξ, $\ker L_1(0, \xi) = \bigcap_{\xi_j \neq 0} \ker A_j$, then the Cauchy problem for $\widetilde{L}$ is strongly well posed. This condition holds if $L_1(0, \partial_x)$ is elliptic, that is $\det L_1(0, \xi) \neq 0$ for all real ξ.

Results for constant absorption

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2. If the Cauchy problem for L_1 is strongly well posed, and if there is a $\xi \neq 0$ such that $\ker L(0, \xi) \neq \bigcap_{\xi_j \neq 0} \ker A_j$, then $\widetilde{L}_1(0, \xi)$ is not diagonalizable. Therefore the Cauchy problem for $\widetilde{L}$ is not strongly well posed.
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Results for constant absorption

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2. If the Cauchy problem for L_1 is strongly well posed, and if there is a $\xi \neq 0$ such that $\ker L(0, \xi) \neq \bigcap_{\xi_j \neq 0} \ker A_j$, then $\widetilde{L}_1(0, \xi)$ is not diagonalizable. Therefore the Cauchy problem for $\widetilde{L}$ is not strongly well posed.
3. If the Cauchy problem for L is strongly well posed and for all ξ, $\ker L_1(0, \xi) = \bigcap_{\xi_j \neq 0} \ker A_j$, then the Cauchy problem for $\widetilde{L}$ is strongly well posed. This condition holds if $L_1(0, \partial_x)$ is elliptic, that is $\det L_1(0, \xi) \neq 0$ for all real ξ.

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1. If the Cauchy problem for L_1 is strongly well posed, then for arbitrary constant absorptions $\sigma_j \in \mathbb{C}$, the Cauchy problem for $\widetilde{L}_1 + B$ is weakly well posed.
2. If the Cauchy problem for L_1 is strongly well posed, and if there is a $\xi \neq 0$ such that $\ker L(0, \xi) \neq \bigcap_{\xi_j \neq 0} \ker A_j$, then $\widetilde{L}_1(0, \xi)$ is not diagonalizable. Therefore the Cauchy problem for $\widetilde{L}$ is not strongly well posed.
3. If the Cauchy problem for L is strongly well posed and for all ξ, $\ker L_1(0, \xi) = \bigcap_{\xi_j \neq 0} \ker A_j$, then the Cauchy problem for $\widetilde{L}$ is strongly well posed. This condition holds if $L_1(0, \partial_x)$ is elliptic, that is $\det L_1(0, \xi) \neq 0$ for all real ξ.

APPLIES TO MAXWELL

1: Seidenberg-Tarski Theorem (on the roots of the characteristic

Outline

The result

$$U = E + iH,$$

$$\mathcal{H} := \Big\{ \widetilde{U} = (U^1, U^2, U^3) \in H^2(\mathbb{R}^3 ; \mathbb{C}^3)^3 \} \ : \ U^1_1 = 0, \quad U^2_2 = 0, \quad U^3_3 = 0 \Big\}.$$

The result

$U = E + iH,$

$$\mathcal{H} := \left\{ \widetilde{U} = (U^1, U^2, U^3) \in H^2(\mathbb{R}^3\,;\,\mathbb{C}^3)^3 \right\} \;:\; U^1_1 = 0, \quad U^2_2 = 0, \quad U^3_3 = 0 \Big\}.$$

THEOREM If σ_j, for $j = 1, 2, 3$, belong to $W^{2,\infty}(\mathbb{R})$, then for any $\widetilde{U}_0 = (U^1_0, U^2_0, U^3_0)$ in $\mathcal{H}$ there is a unique solution $\widetilde{U}$ in $L^2(0, T; \mathcal{H})$ of the split Cauchy problem with initial value $\widetilde{U}_0$. Furthermore there is a $C_1 > 0$ independent of $\widetilde{U}_0$ so that for all positive time t,

$$\|\widetilde{U}(t, \cdot)\|_{(L^2(\mathbb{R}^3))^9} \;\leq\; C_1 e^{C_1\, t} \left\| \widetilde{U}_0 \right\|_{(H^2(\mathbb{R}^3))^9}.$$

- 2D estimates: JLLions-Metral-Vacus
- Full proof in 2D with the Yee scheme : Sabrina Petit thesis.
- 3D : HPR.

Elements of proof

- Get estimates on a larger vector $\mathbb{V}$ for which a strongly hyperbolic problem holds.
- Semi-discretize in space and obtain similar discrete estimates
- Pass to the limit.
- Uniqueness goes through the estimates.

Estimates

$$\mathbb{V} \;:=\; \Big(U\,,\,V^i, V^{i,j}\,,\,W^j\,,\,U^j\,,\,W\,,\,Z^j\Big)\;\in\;\mathbb{C}^{54}.$$

$$U \;:=\; U^1+U^2+U^3, \qquad V^j \;:=\; \partial_j U, \qquad V^{i,j} \;:=\; \partial_{ij} U,$$

$$W \;:=\; \textstyle\sum_k \sigma_k(x_k)U^k, W^j \;:=\; \partial_j W,$$

$$Z \;:=\; \textstyle\sum_k \partial_k(W_k+\sigma_k(x_k)U_k), \qquad Z^j \;:=\; \partial_j Z,$$

$$\partial_t \mathbb{V} + P(\partial)\mathbb{V} + \mathbb{B}(\sigma, D\sigma, D^2\sigma)\mathbb{V} = 0$$

LemmaThis problem is strongly well-posed (symmetrizable).

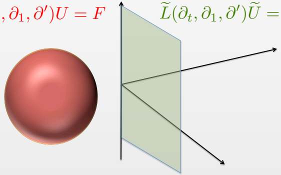
Estimates

$$P(\partial) = \begin{pmatrix} I_4 \otimes L(0,\partial) & 0_{4,6} \otimes 0_{3,3} & 0_{4,3} \otimes 0_{3,3} & 0_{4,3} \otimes 0_{3,3} & 0_{4,4} \otimes 0_{3,3} \\ 0_{6,4} \otimes 0_{3,3} & I_6 \otimes L(0,\partial) & (I_6 \otimes L(0,\partial))M & 0_{6,3} \otimes 0_{3,3} & 0_{6,4} \otimes 0_{3,3} \\ 0_{3,4} \otimes 0_{3,3} & 0_{3,6} \otimes 0_{3,3} & 0_{3,3} \otimes 0_{3,3} & 0_{3,3} \otimes 0_{3,3} & 0_{3,4} \otimes 0_{3,3} \\ 0_{3,4} \otimes 0_{3,3} & 0_{3,6} \otimes 0_{3,3} & 0_{3,3} \otimes 0_{3,3} & 0_{3,3} \otimes 0_{3,3} & 0_{3,4} \otimes 0_{3,3} \\ 0_{4,4} \otimes 0_{3,3} & 0_{4,6} \otimes 0_{3,3} & 0_{4,3} \otimes 0_{3,3} & 0_{4,3} \otimes 0_{3,3} & 0_{4,4} \otimes 0_{3,3} \end{pmatrix}.$$

$$M := \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_1 \\ 0 & A_2 & 0 \\ 0 & 0 & A_2 \\ 0 & 0 & A_3 \end{pmatrix}.$$

Outline

Transmission Problem, One Absorption

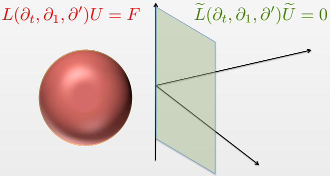


THEOREM

1. If $\sigma(x_1) = constant \times 1_{x_1 > 0}$ and $\tilde{L}(\partial)$ is hyperbolic, non degenerate with respect to x_1, then the constant coefficient transmission problem is weakly well posed.
2. If $\sigma(0) = 0$, $\sigma(x_1) \in W^{1,\infty}(\mathbb{R})$, $\tilde{L}(\partial)$ is hyperbolic for some constant σ , non degenerate with respect to x_1, then the transmission problem is weakly well posed.

Proof. For **1** Verify the criterion of R. Hersh.
For **2** the problem can be nearly conjugated to the constant coefficient case. □

Transmission Problem with Discontinuous Absorption



$$\partial_t E = \sum C_j \partial_j B - \mathbf{j},$$
$$\partial_t B = -\sum C_j \partial_j E.$$

6 unknowns

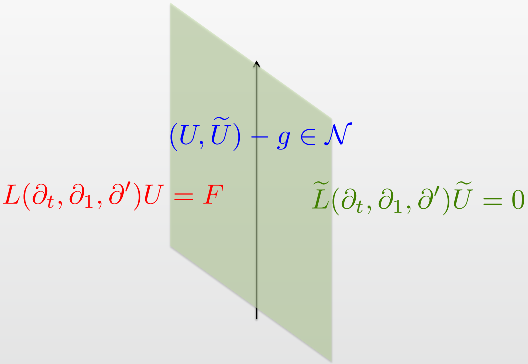
$$\partial_t E^j + \sigma_j(x_j)E^j = C_j \partial_j B,$$
$$\partial_t B^j + \sigma_j(x_j)E^j = -\sum C_j \partial_j E.$$
$$E = E^1 + E^2 + E^3$$
$$B = B^1 + B^2 + B^3$$

18 unknowns

Transmission conditions at $x_1 = 0$: $[C_1 E] = 0$, $[C_1 B] = 0$.

$$C_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow [(E_2, E_3)] = 0, \quad [(B_2, B_3)] = 0.$$

Hersh Theorem



$$G_L^{\pm}(\tau, \eta) = \{V(x_1) \text{ solution of } L(\tau, \partial_1, i\eta)V = 0, V \to 0 \text{ when } x_1 \to \pm\infty\}$$

$$\dot{G}_L^{\pm}(\tau, \eta) = \{\text{trace at } x_1 = 0 \text{ of elements in } G_L^{\pm}(\tau, \eta)\}$$

Uniqueness $\iff \forall(\tau, \eta), \Re\tau > 0, (\dot{G}_L^{-}(\tau, \eta), \dot{G}_{\widetilde{L}}^{+}(\tau, \eta)) \cap \mathcal{N} = \{0\}$

Well-posedness $\iff \forall(\tau, \eta), \Re\tau > 0, (\dot{G}_L^{-}(\tau, \eta), \dot{G}_{\widetilde{L}}^{+}(\tau, \eta)) \oplus \mathcal{N} = \{0\}$

Outline

Elliptic generator

$$L_1(\partial_t, \partial_x)U := \partial_t U + \sum A_j \partial_j U = 0$$

$$L_1(0, k) = \sum k_j A_j, \quad U(t) = e^{-iL(0,k)t}\, U_0$$

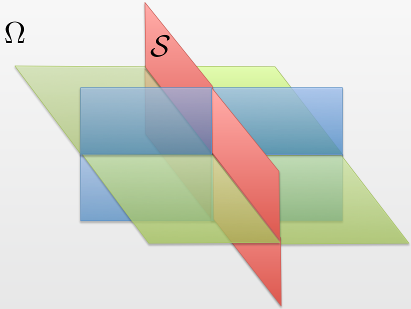
If the Cauchy problem for L_1 is strongly well posed and for all ξ, $\ker L_1(0, \xi) = \bigcap_{\xi_j \neq 0} \ker A_j$, then the Cauchy problem for $\widetilde{L}$ is strongly well posed. This condition holds if $L_1(0, \partial_x)$ is elliptic, that is $\det L_1(0, \xi) \neq 0$ for all real ξ.

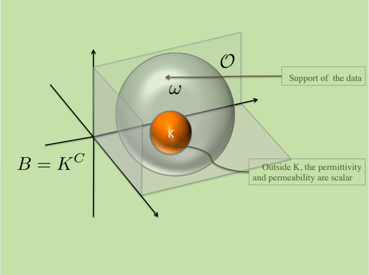
Warm-up for the $3 - D$ Bérenger-Maxwell problem

L. Halpern & J. Rauch, *Bérenger/Maxwell with Discontinous Absorptions: Existence, Perfection, and No Loss.* Séminaire Laurent Schwartz-2012-2013, Exp. No. 10.

Outline

Transmission Problem





$$\begin{cases} \varepsilon\partial_t E = \sum C_j\partial_j B - \mathbf{j}, \\ \mu\partial_t B = -\sum C_j\partial_j E. \end{cases} \qquad \begin{cases} \varepsilon(\partial_t E^j + \sigma_j(x_j)E^j) = C_j\partial_j(\sum B_k), \\ \mu(\partial_t B^j + \sigma_j(x_j)E^j) = -\sum C_j\partial_j(\sum E_k). \end{cases}$$

$$(E,B) = \begin{cases} (E,B) \text{ in } \mathcal{O}, \\ (\sum E_k, \sum B_k) \text{ in } \Omega\setminus\mathcal{O}. \end{cases}$$

The result

THEOREM $\exists C, \lambda_0$, depending on $\overline{\omega}$. If $\lambda > \lambda_0$, $\operatorname{supp} \mathbf{j} \subset [0, \infty[\times \overline{\omega}$, and

$$\forall |\alpha| \le 1, \qquad \boxed{\partial^\alpha_{t,x} \mathbf{j} \ \in \ e^{\lambda t} L^2\big(\mathbb{R}\,;\, L^2(\mathbb{R}^3)\big)}$$

then there are E, B defined on $\mathbb{R}_t \times \mathcal{O}$ and split functions E^j, B^j defined on $\mathbb{R}_t \times \cup \mathcal{O}_\kappa$, supported in $t \ge 0$, so that the total field $\boxed{U = (E, B) \in e^{\lambda t} H^1(\mathbb{R} \times \mathbb{R}^3)}$ and satisfies the Bérenger differential equations. Any solution with $U \in e^{\lambda t} H^1(\mathbb{R} \times \mathbb{R}^3)$ satisfies for $\lambda > \lambda_0$

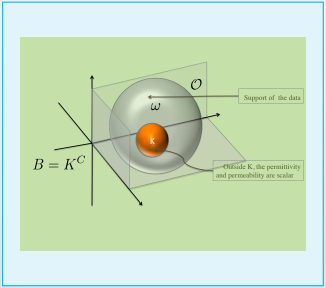
$$\int e^{-2\lambda t} \big\| \lambda U \,,\, \nabla_{t,x} U \,,\, \lambda \nabla_{t,x} U \big|_{\overline{\omega}} \big\|^2_{L^2(\mathbb{R}^3)} \, dt \qquad\qquad \le \ C \int e^{-2\lambda t} \sum_{|\alpha| \le 1} \big\| \partial^\alpha_{t,x} \mathbf{j}(t) \big\|^2_{L^2(\mathbb{R}^3)} \, dt \,. \tag{1}$$

On each octant $\mathcal{O}_\kappa$, the split fields satisfy $E^j_j = B^j_j = 0$ for all j, and

$$\int e^{-2\lambda t} \big\| E^j, B^j \,, \partial_t E^j, \partial_t B^j \big\|^2_{L^2(\mathcal{O}_\kappa)} \, dt \qquad\qquad \le \ C \int e^{-2\lambda t} \sum_{|\alpha| \le 1} \big\| \partial^\alpha_{t,x} \mathbf{j}(t) \big\|^2_{L^2(\mathbb{R}^3)} \, dt \,. \tag{2}$$

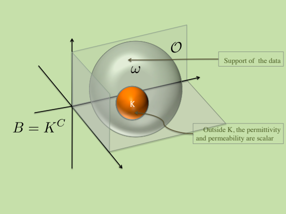
In particular there is uniqueness for such solutions.

The key points



- USE THE DIVERGENCE EQUATION.
- Existence of smooth solutions by the result above for $\sigma \in W^{2,\infty}$.
- Laplace transform+ Paley-Wiener. Passing to the limit needs H^1 estimates. Partition of unity.
  - Standard estimates in $\mathcal{O}$.
  - Estimates in $\mathbb{R}^3 \setminus \bar{\omega}$.
  - Well adapted operator in all of $\mathbb{R}^3$.
- Use the estimates to have weak convergence of a family of solutions with regular σ.
- Uniqueness through the estimates.

The well-adapted operator



Relies on the "tilde" operators of the type

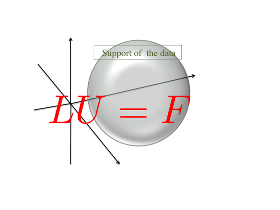
$$\widetilde{\mathrm{div}} u = \sum_j \frac{\tau}{\tau + \sigma_j} \partial_j u_j$$

and algebras like

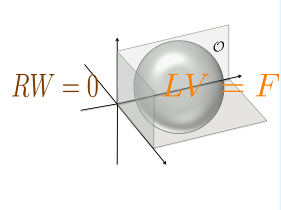
$$\widetilde{\mathrm{div}}\ \widetilde{\mathrm{curl}} = 0, \quad \widetilde{\mathrm{div}}\ \widetilde{\mathrm{grad}} = \widetilde{\Delta}.$$

$$P_E \ := \ \varepsilon\mu \, \frac{\prod_j (\tau + \sigma_j(x_j))}{\tau} - \sum_j \partial_j \, \frac{1}{\varepsilon} \, \frac{(\tau + \sigma_{j+1})(\tau + \sigma_{j+2})}{\tau(\tau + \sigma_j)} \, \partial_j(\varepsilon E) + \ell_E \, .$$

What about perfection ?

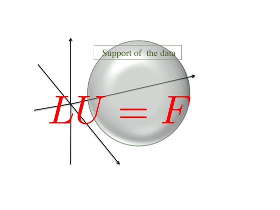


Free-space problem



Transmission problem

What about perfection ?



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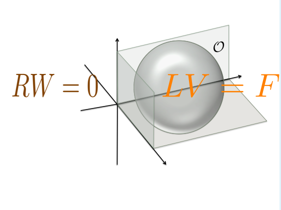
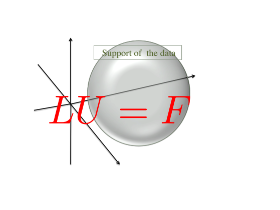


Transmission problem

Perfect matching (Appelo-Hagstrom-Kreiss) is $V = U$ in $\mathcal{O}$.

What about perfection ?



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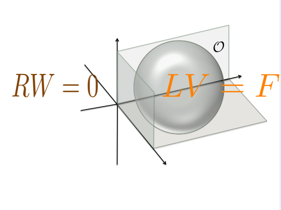


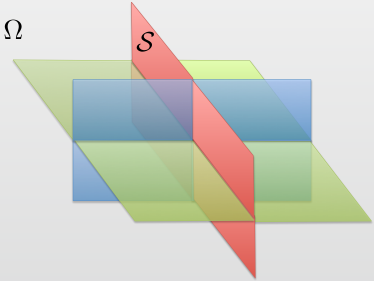
Transmission problem

Perfect matching (Appelo-Hagstrom-Kreiss) is $V = U$ in $\mathcal{O}$.

Follows from well-posedness by change of coordinates (Diaz-Joly) thanks to holomorphy.

Conclusions and perspectives

• The first proof of well-posedness for the full 3D Maxwell-Berenger problem with (discontinuous) matrix coefficients.
Hyperbolic Boundary Value Problems with Trihedral Corners to appear in special issue of AIMS for Peter Lax's 90's birthday.



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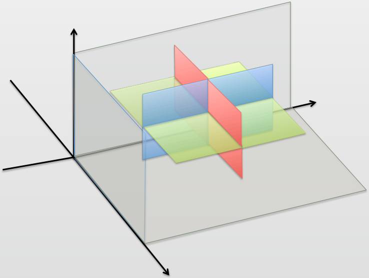

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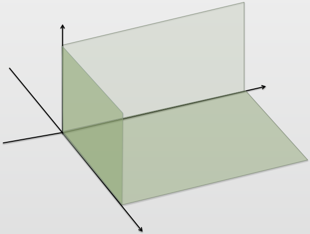
• The boundary value problem



• Maxwell + dissipative boundary conditions : done in AIMS paper.

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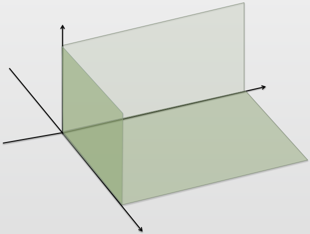
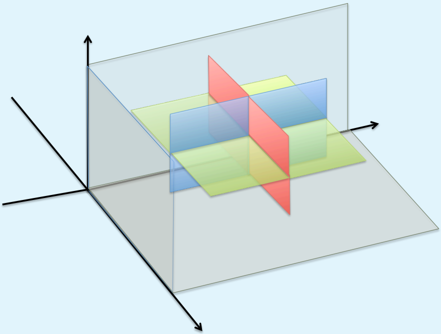


• Maxwell + dissipative boundary conditions : done in AIMS paper.

• Berenger Maxwell : poses real difficulties.

The grail



Thank you for your attention

Calculs pour Maxwell 2D

Calculs Maple

Abarbanel/Gottlieb

$$\text{Maxwell: } \begin{pmatrix} E_x \\ E_y \\ H \end{pmatrix}(t) = e^{tA} \begin{pmatrix} E_x \\ E_y \\ H \end{pmatrix}(0) = Pe^{tD}P^{-1} \begin{pmatrix} E_x \\ E_y \\ H \end{pmatrix}(0)$$

$$\text{Bérenger-Maxwell : } \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix}(t) = e^{tM} \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix}(0) = Pe^{tJ}P^{-1} \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix}(0)$$

$$A := \begin{pmatrix} 0 & 0 & i\omega_2 \\ 0 & 0 & -i\omega_1 \\ i\omega_2 & -i\omega_1 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i|\omega| & 0 \\ 0 & 0 & -i|\omega| \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 0 & i\omega_2 & i\omega_2 \\ 0 & 0 & -i\omega_1 & -i\omega_1 \\ 0 & -i\omega_1 & 0 & 0 \\ i\omega_2 & 0 & 0 & 0 \end{pmatrix}, \; J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i|\omega| & 0 \\ 0 & 0 & 0 & -i|\omega| \end{pmatrix}$$

$$e^{tD} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{i|\omega|t} & 0 \\ 0 & 0 & e^{-i|\omega|t} \end{pmatrix}, \; e^{tJ} = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i|\omega|t} & 0 \\ 0 & 0 & 0 & e^{-i|\omega|t} \end{pmatrix}$$

the factor t will factorize the second component of $P^{-1}U(0)$, $(\omega_1 E_x + \omega_2 E_y)(0) = \text{div}(E)(0)$.

Calculs pour Maxwell 2D

Calculs Maple ▸ Abarbanel/Gottlieb

$$\text{Maxwell: } \begin{pmatrix} E_x \\ E_y \\ H \end{pmatrix}(t) = e^{tA} \begin{pmatrix} E_x \\ E_y \\ H \end{pmatrix}(0) = Pe^{tD}P^{-1} \begin{pmatrix} E_x \\ E_y \\ H \end{pmatrix}(0)$$

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Calculs Maple ▸ Abarbanel/Gottlieb

$$\text{Maxwell: } \begin{pmatrix} E_x \\ E_y \\ H \end{pmatrix}(t) = e^{tA} \begin{pmatrix} E_x \\ E_y \\ H \end{pmatrix}(0) = Pe^{tD}P^{-1} \begin{pmatrix} E_x \\ E_y \\ H \end{pmatrix}(0)$$

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$$A := \begin{pmatrix} 0 & 0 & i\omega_2 \\ 0 & 0 & -i\omega_1 \\ i\omega_2 & -i\omega_1 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i|\omega| & 0 \\ 0 & 0 & -i|\omega| \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 0 & i\omega_2 & i\omega_2 \\ 0 & 0 & -i\omega_1 & -i\omega_1 \\ 0 & -i\omega_1 & 0 & 0 \\ i\omega_2 & 0 & 0 & 0 \end{pmatrix}, \; J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i|\omega| & 0 \\ 0 & 0 & 0 & -i|\omega| \end{pmatrix}$$

$$e^{tD} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{i|\omega|t} & 0 \\ 0 & 0 & e^{-i|\omega|t} \end{pmatrix}, \; e^{tJ} = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i|\omega|t} & 0 \\ 0 & 0 & 0 & e^{-i|\omega|t} \end{pmatrix}$$

the factor t will factorize the second component of $P^{-1}U(0)$, $(\omega_1 E_x + \omega_2 E_y)(0) = \mathrm{div}(E)(0)$.