Asymptotics for the number of standard Young tableaux of skew shape

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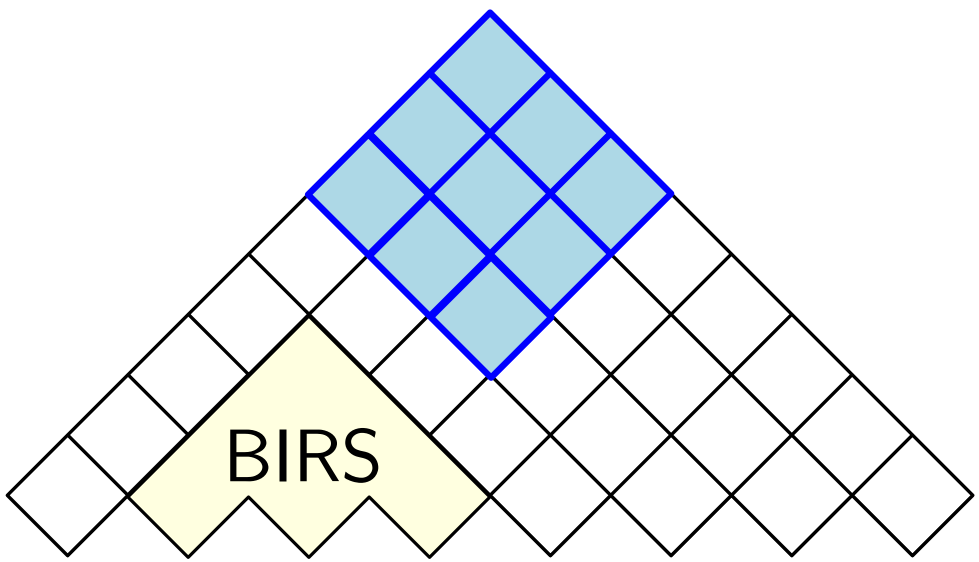

October 26, 2016

Linear extensions of posets

$\mathcal{P}$ be a ranked poset with n elements,
a **linear extension** is a linear order or permutation of the elements compatible with the order of $\mathcal{P}$.

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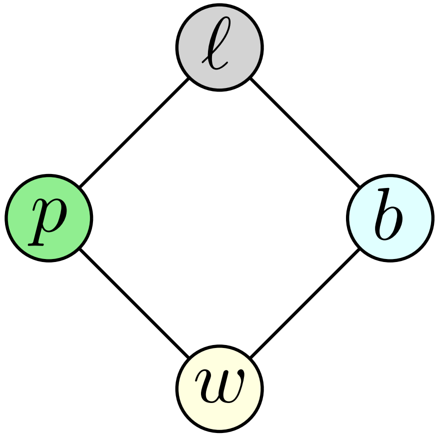

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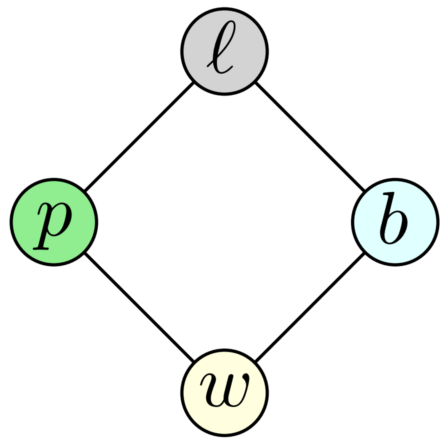

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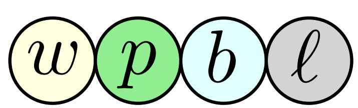

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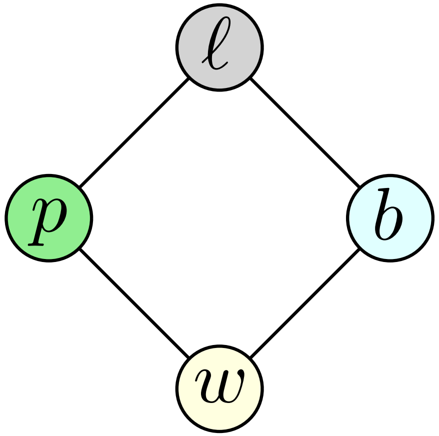

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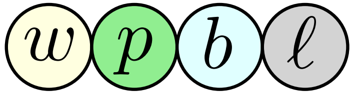

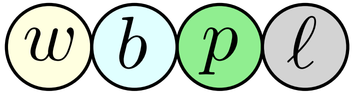

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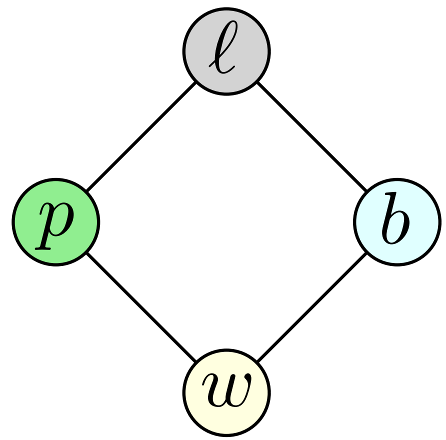

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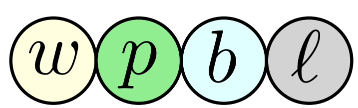
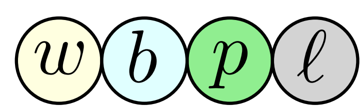
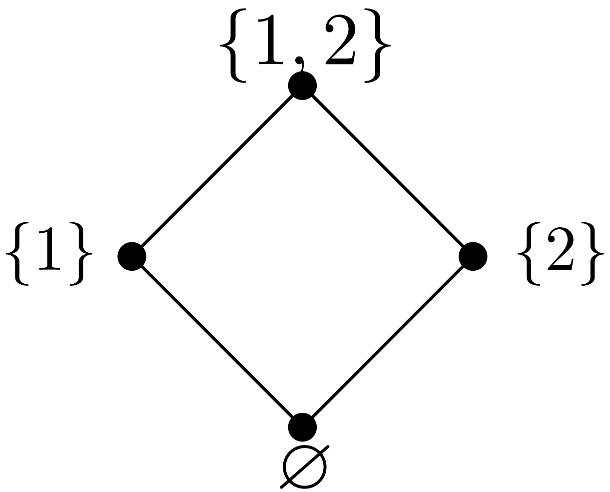


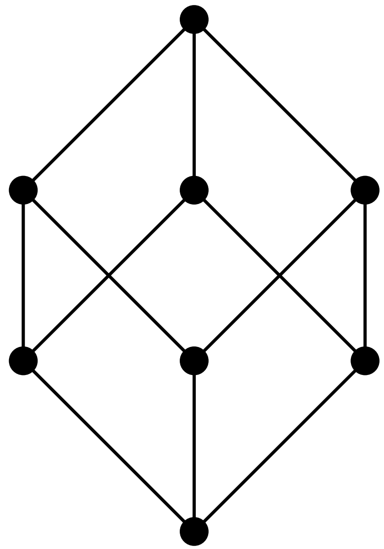


$e(\mathcal{P})$ number of linear extensions of $\mathcal{P}$

Complexity of counting linear extensions

$\mathcal{P} = B_n$ boolean lattice





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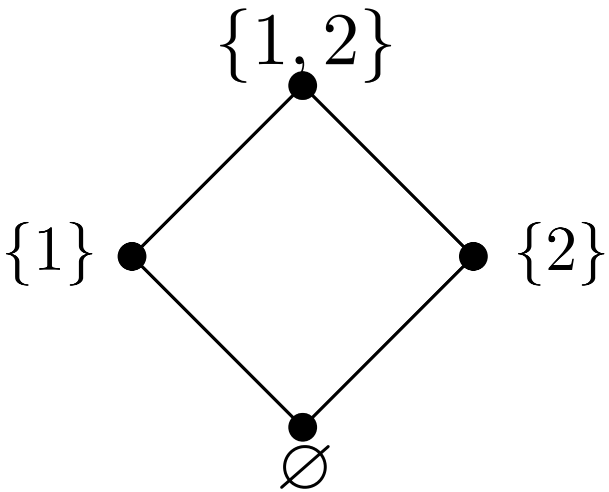

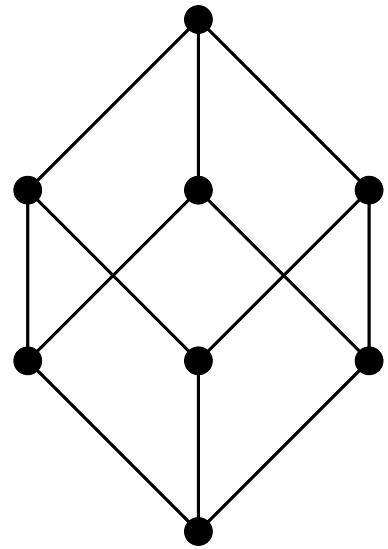

$\dots$

$e(B_n)$ known up to $n = 6$: $1, 2, 48, 1680384, \dots$

Complexity of counting linear extensions

Theorem (Brightwell, Winkler 1991)

For general posets $\mathcal{P}$, counting $e(\mathcal{P})$ is $\#P$-complete.

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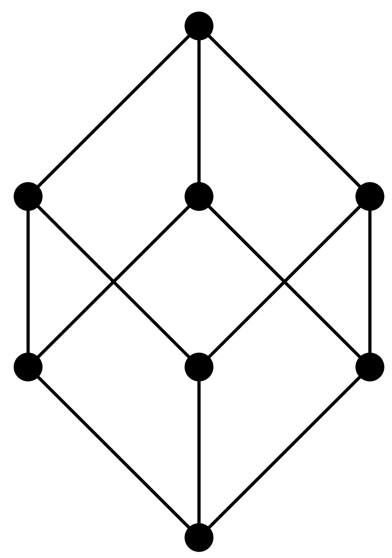
For general posets $\mathcal{P}$, counting $e(\mathcal{P})$ is $\#P$-complete.

- study families of posets $\mathcal{P}$ where $e(\mathcal{P})$ is computable
- find bounds for $e(\mathcal{P})$

General bounds

general bounds: (folklore)

$$e(\mathcal{P}) \leq n!$$



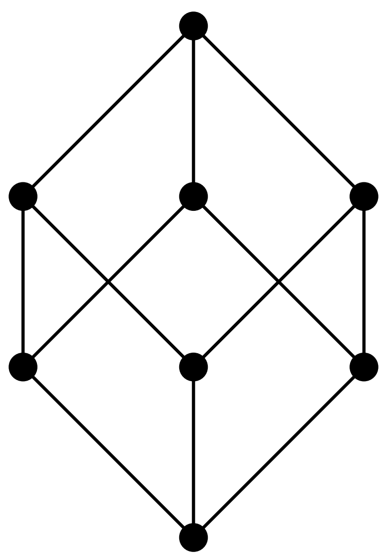
$$48 \leq 8!$$

General bounds

general bounds: (folklore)

$$r_1! \cdots r_\ell! \le e(\mathcal{P}) \le \frac{n!}{c_1! \cdots c_m!}$$

ℓ length of longest chain, m length longest antichain
r_i elements rank i,
$C_1, \ldots, C_m$ decomposition of $\mathcal{P}$ into chains, $c_i = |C_i|$



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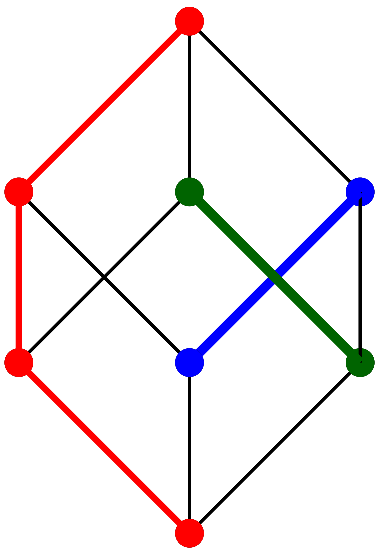

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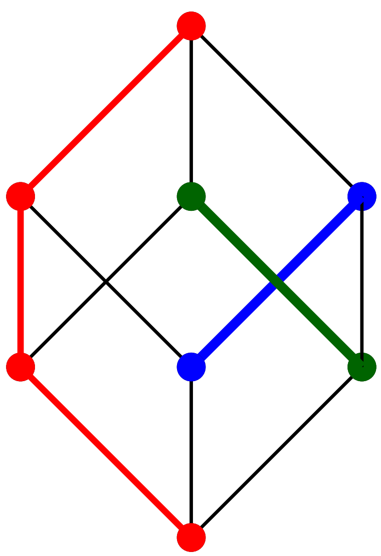
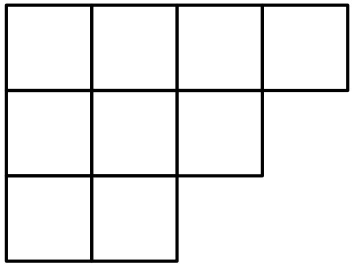


$$36 = 1!3!3!1! \leq 48 \leq \frac{8!}{4!2!2!} = 420$$

Posets from Young diagrams of partitions

λ: partition (straight) shape



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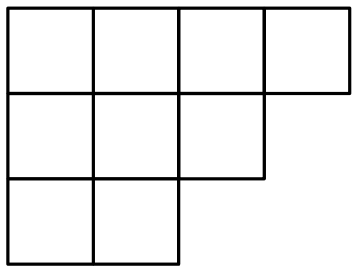

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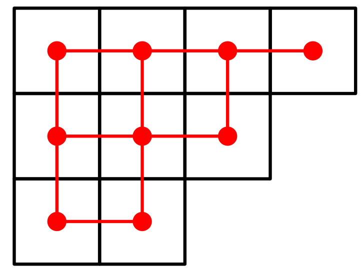

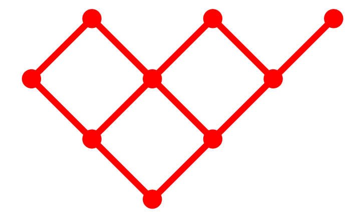

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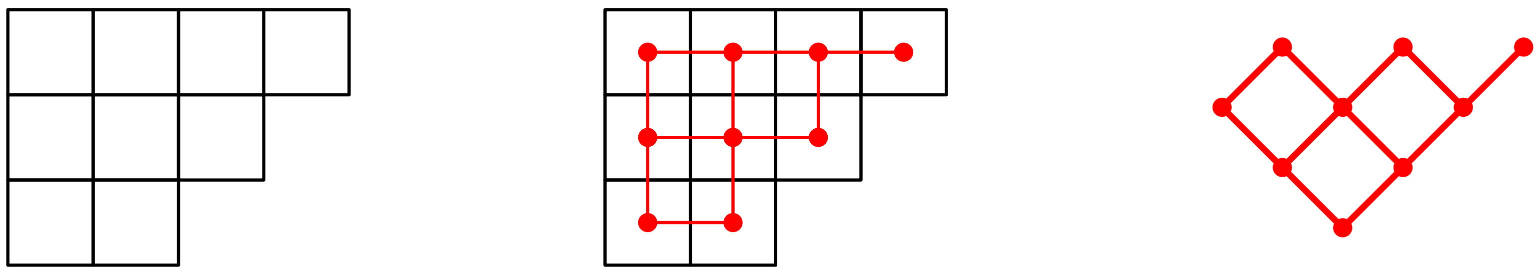
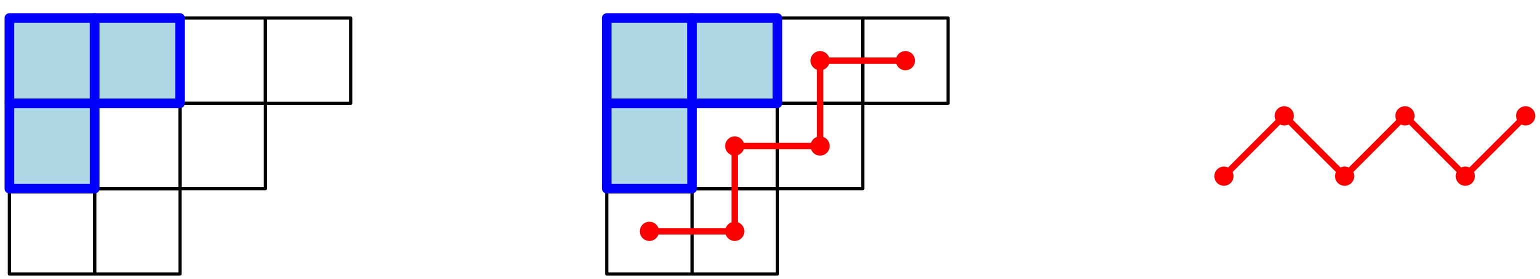


$(4, 3, 2)$

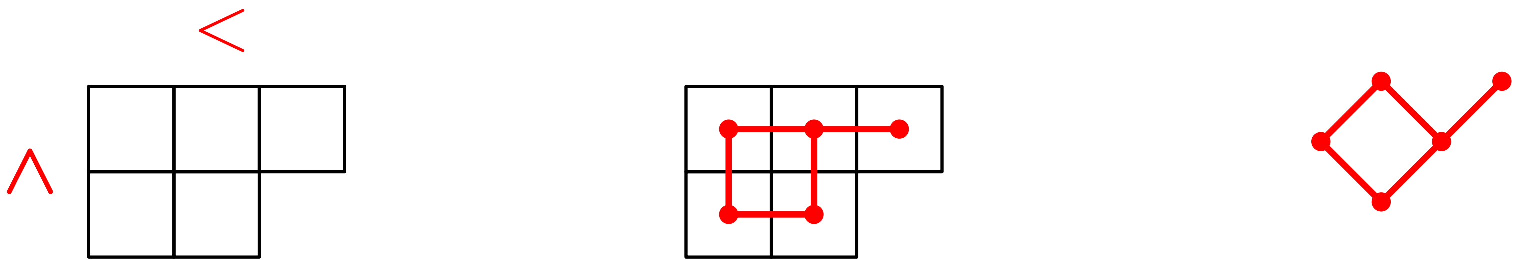
λ/μ: skew shape



$(4, 3, 2)/(2, 1)$

Linear extensions: standard Young tableaux

λ: straight shape



View linear extension in the poset of a Young diagram as a filling with $1, 2, \ldots, n$ increasing in rows and columns.

1	2	3
4	5	

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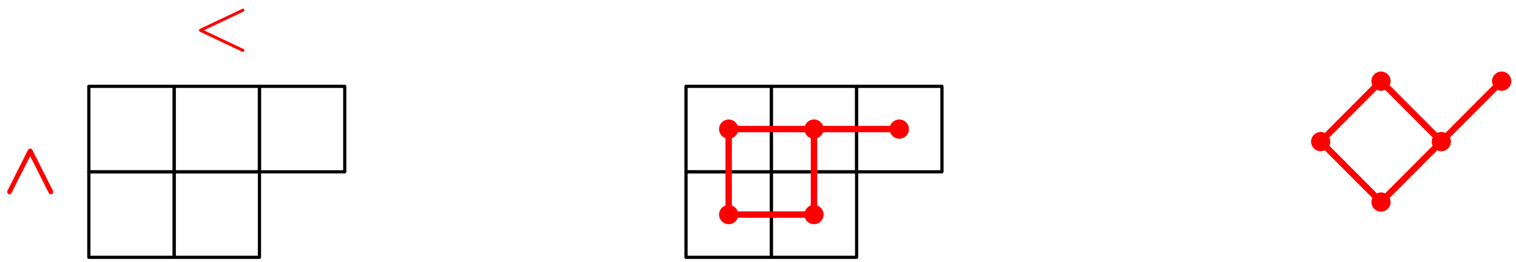

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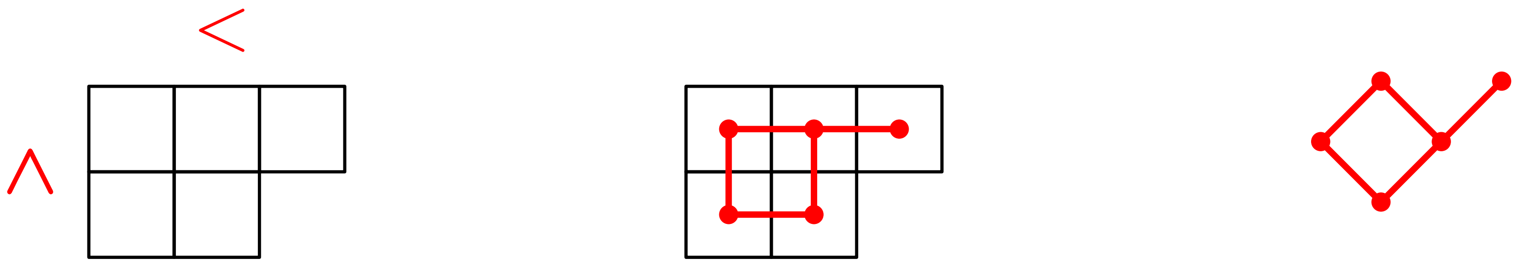

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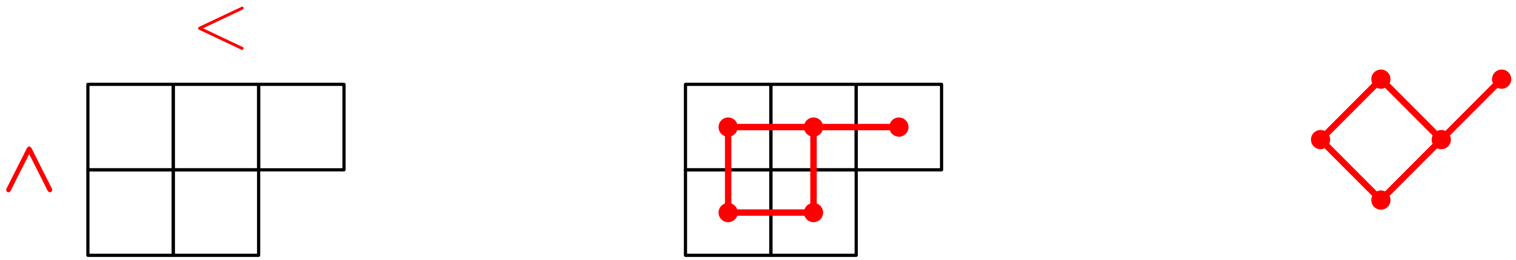

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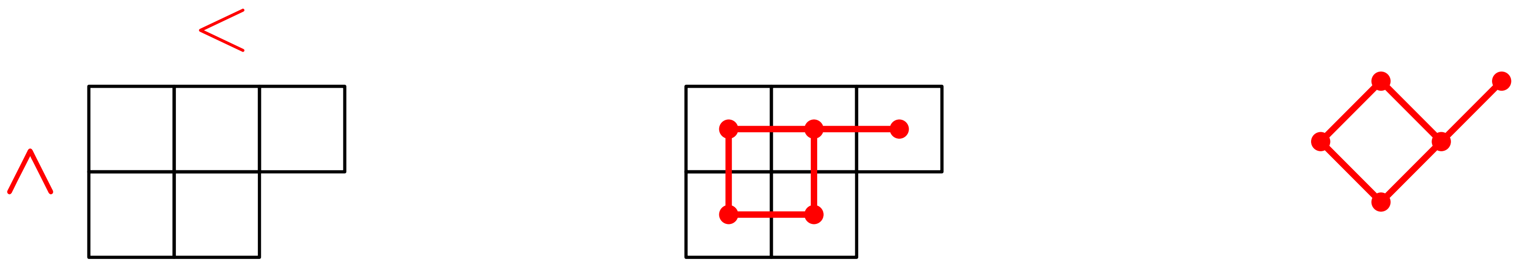

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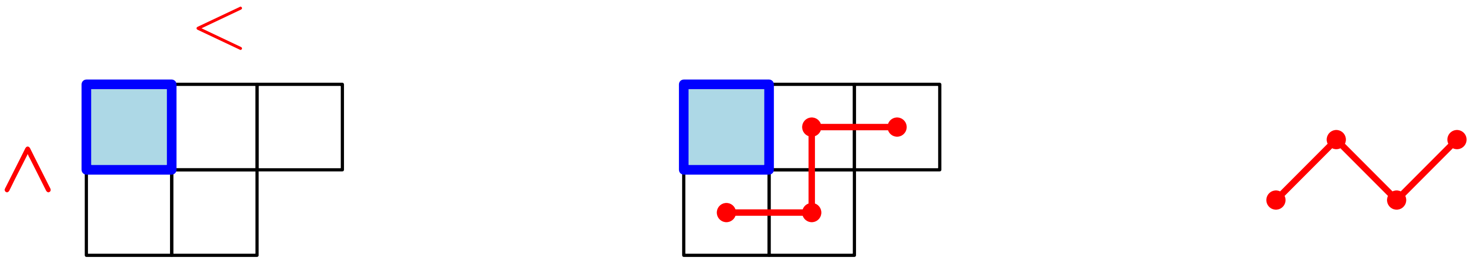
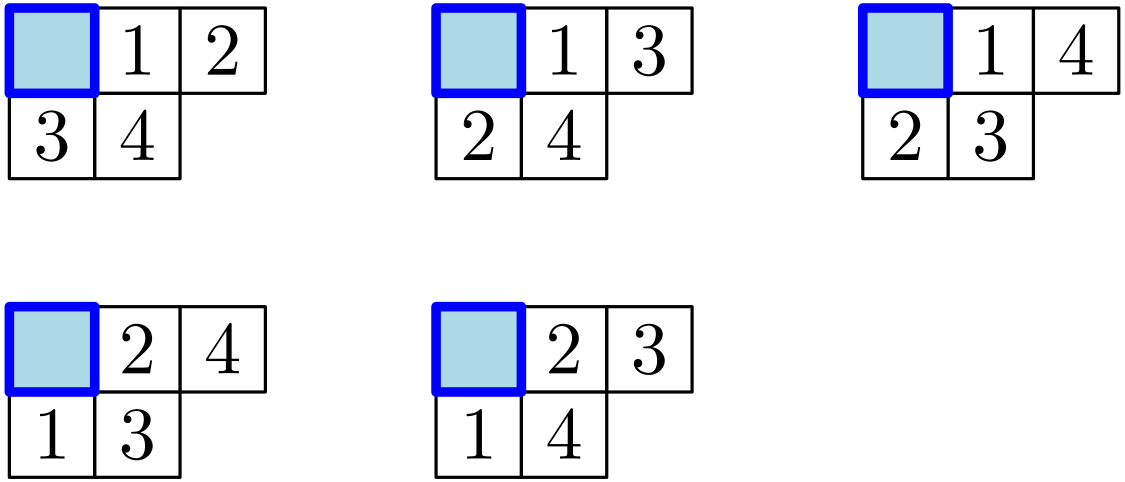
Such fillings are called **Standard Young tableaux (SYT)**

Let $f^{\lambda} := e(\lambda)$

Standard Young tableaux skew shape

λ/μ: skew shape

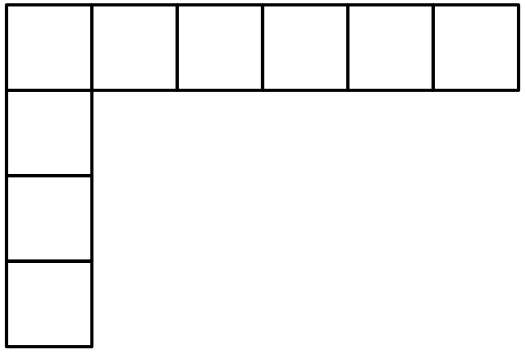




Let $f^{\lambda/\mu} := e(\lambda/\mu)$

number of SYT of straight shape

Example: hooks



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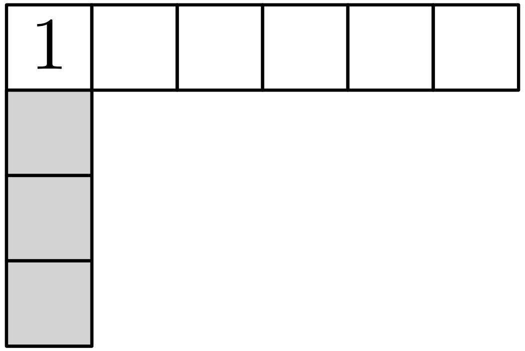
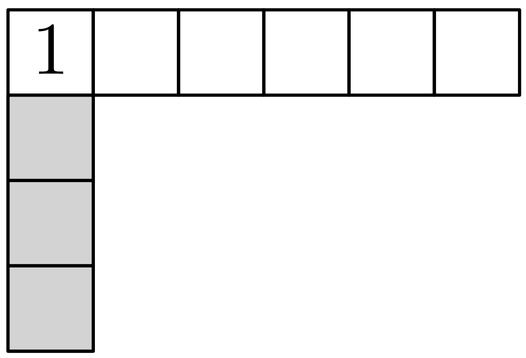


$$f^{(6,1,1,1)} = \binom{8}{3}$$

number of SYT of straight shape

Example: hooks



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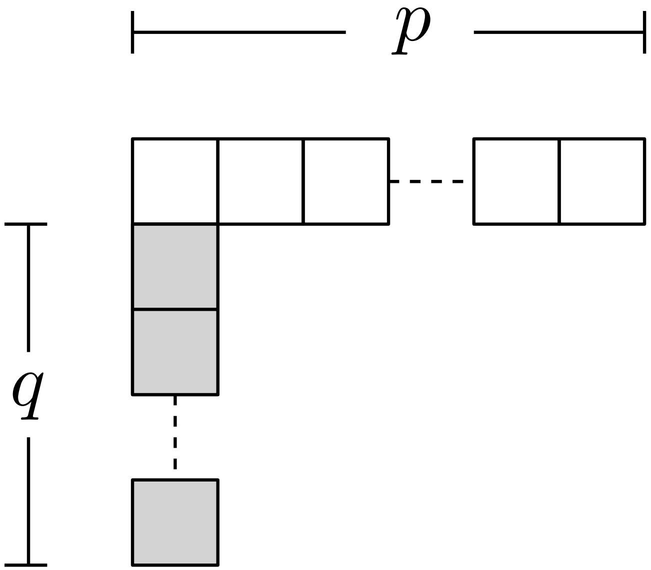

$$f^{(p,1^q)} = \binom{p+q-1}{q}$$

number of SYT of $2 \times n$ rectangle

Example: $2 \times n$ rectangle

1	2
3	4

1	3
2	4

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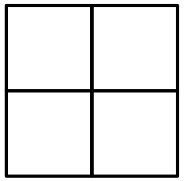

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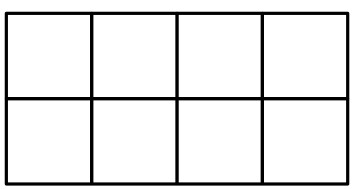
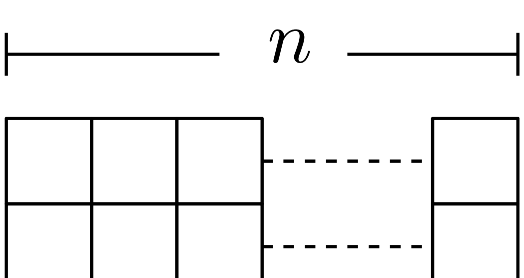


$$f^{(4,4)} = 14$$



$$f^{(n,n)} = \frac{1}{n+1}\binom{2n}{n}$$

Hook-length formula

Theorem (Frame-Robinson-Thrall 1954)

$$f^\lambda = n! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)},$$

$h(i,j) = \lambda_i - i + \lambda'_j - j + 1$ is the **hook-length** of (i,j)

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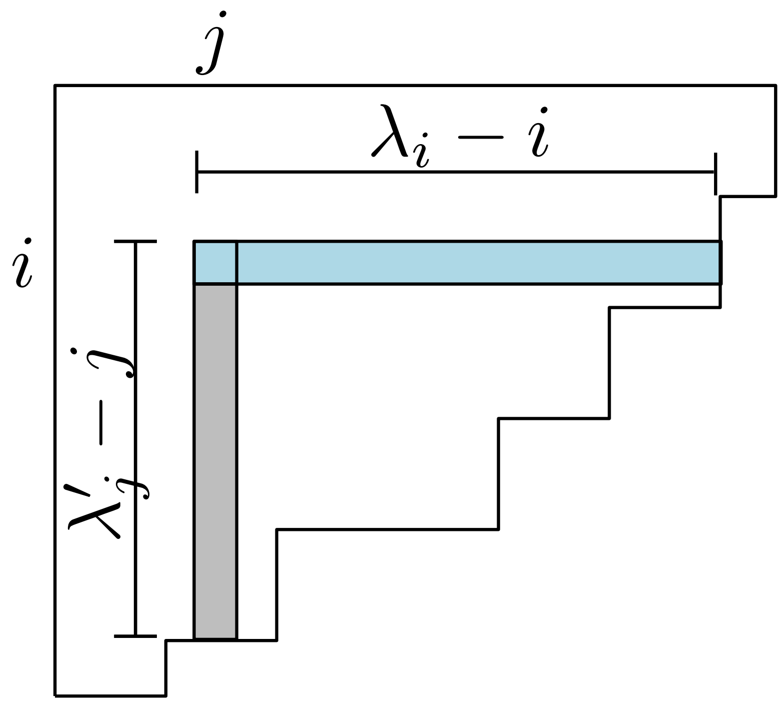

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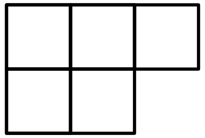

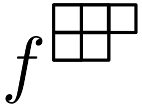

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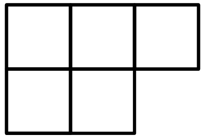

4	3	1
2	1	

$$f^{\yng(3,2)} = \frac{5!}{1^2 \cdot 2 \cdot 3 \cdot 4} = 5$$

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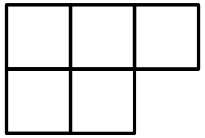

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- probabilistic proof by Greene-Nijenhuis-Wilf 79.
- bijective proof by Novelli-Pak-Stoyanovskii 97.

Asymptotics of large f^λ

From representation theory or the *RSK bijection*:

$$\sum_{\lambda, |\lambda|=n} (f^\lambda)^2 = n!$$

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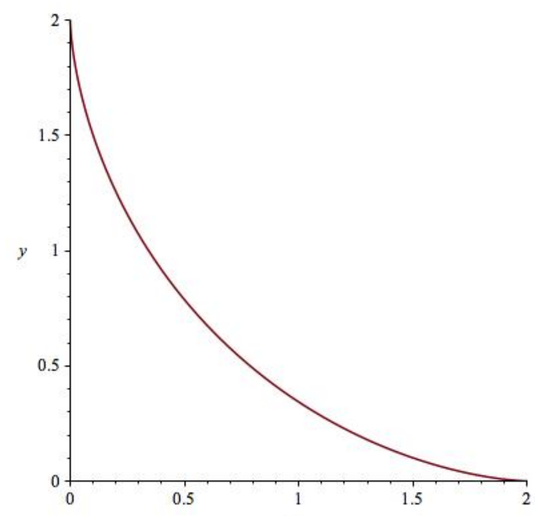

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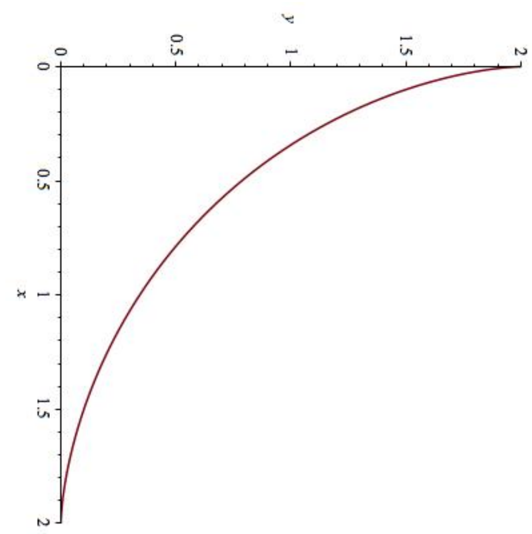


Outline

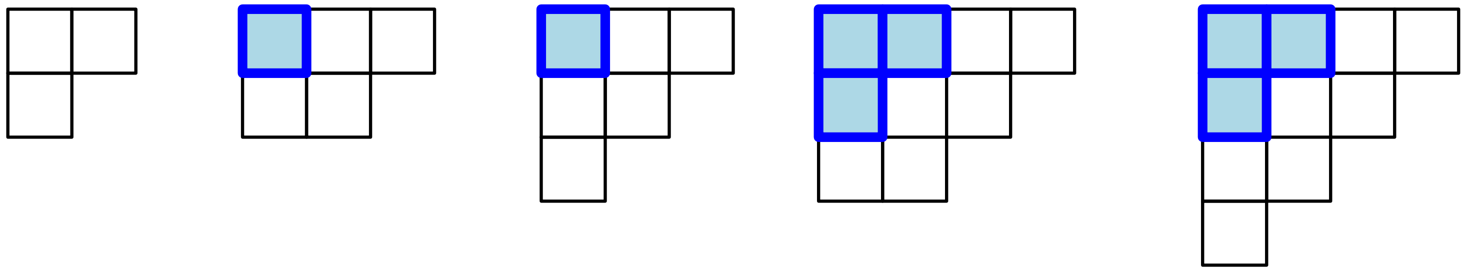
- $f^\lambda = \dfrac{|\lambda|!}{\prod_{u\in\lambda} h(u)}$
- asymptotics

- $f^{\lambda/\mu} = ?$

No product formula for $f^{\lambda/\mu}$

Example: zigzag strip $z(n)$

$\lambda/\mu :$



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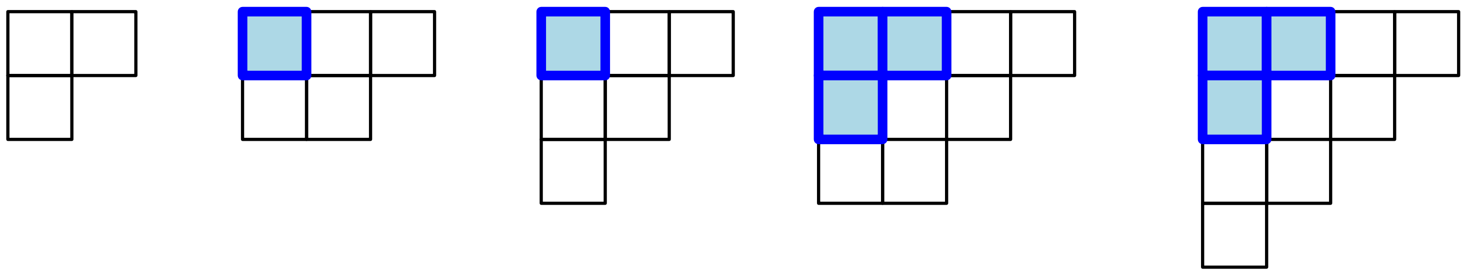
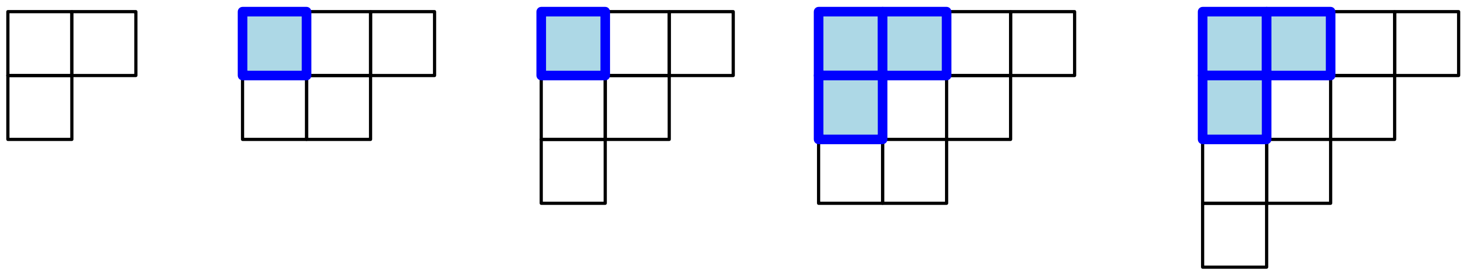


$f^{\lambda/\mu} :$ 2 5 16 61 272 $\ldots$

No product formula for $f^{\lambda/\mu}$

Example: zigzag strip $z(n)$

λ/μ :



$f^{\lambda/\mu}$: $2 \quad 5 \quad 16 \quad 61 \quad 272 \quad \ldots$

Euler numbers E_n

$$E_{2n+1} = f^{z(n)}$$

No product formula for $f^{\lambda/\mu}$

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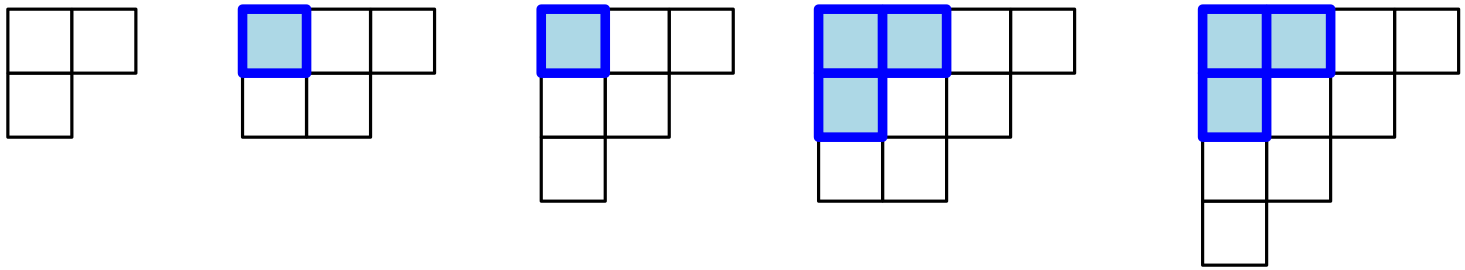

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Recall

$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

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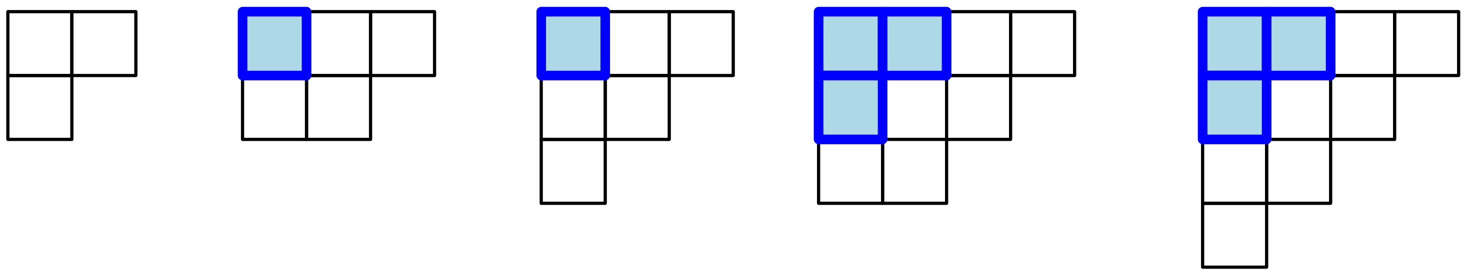

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Euler numbers $E_n \sim \dfrac{2^{n+2} n!}{\pi^{n+1}}(1+o(1)).$

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More about Euler numbers

E_n

1	2	5	16	61	272	$\ldots$

F_{n+1}

2	3	5	8	13	21

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Fact

$$E_n \cdot F_n \geq n!$$

More about Euler numbers

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Fact

$$E_n \cdot F_n \geq n!$$

- note that $\phi > \pi/2$
- inequality comes from bound for $e(\mathcal{P})$ of Sidorenko for zigzag poset.

Alternating formulas for $f^{\lambda/\mu}$

Jacobi-Trudi identity (Feit 1953)

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det\left[\frac{1}{(\lambda_i - \mu_j - i + j)!}\right]_{i,j=1}^{\ell(\lambda)}.$$

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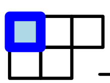
Example

f 

$$= 4! \cdot \det\begin{bmatrix} \frac{1}{2!} & \frac{1}{4!} \\ \frac{1}{1!} & \frac{1}{2!} \end{bmatrix}$$

$$= 4! \cdot \left(\frac{1}{4} - \frac{1}{24}\right) = 5.$$

Positive formulas for $f^{\lambda/\mu}$

Littlewood-Richardson rule

$$f^{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} f^{\nu},$$

where $c^{\lambda}_{\mu,\nu}$ are the **Littlewood-Richardson coefficients**.

Naruse's "hook-length" formula for $f^{\lambda/\mu}$

Theorem (Naruse 2014)

$$f^{\lambda/\mu} = n! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

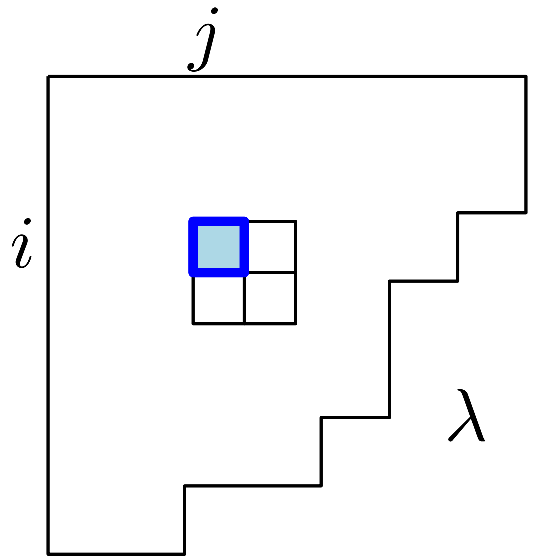
where $\mathcal{E}(\lambda/\mu)$ is the set of **excited diagrams** of λ/μ.

Excited diagrams of λ/μ

Let $S \subseteq \lambda$,

A cell $(i,j) \in S$ is **excited** if

$$(i+1,j),(i,j+1),(i+1,j+1) \in \lambda \setminus S.$$



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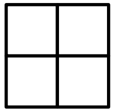

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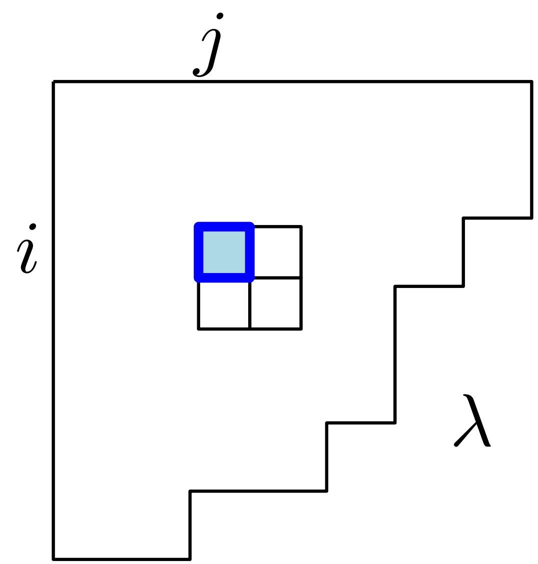

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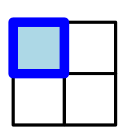

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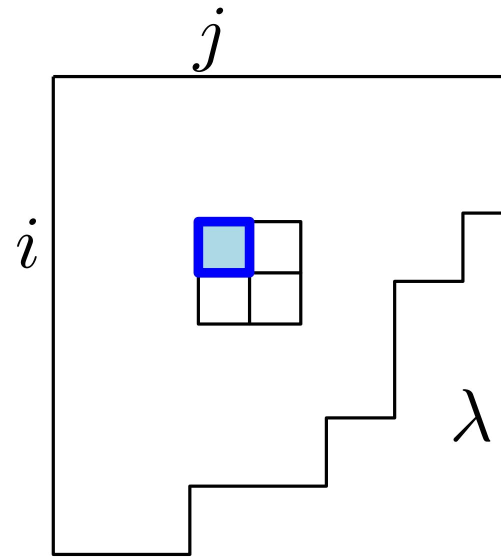

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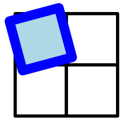

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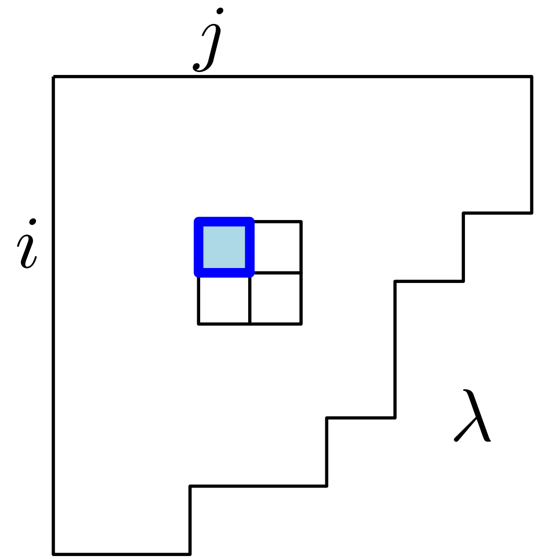

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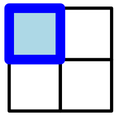

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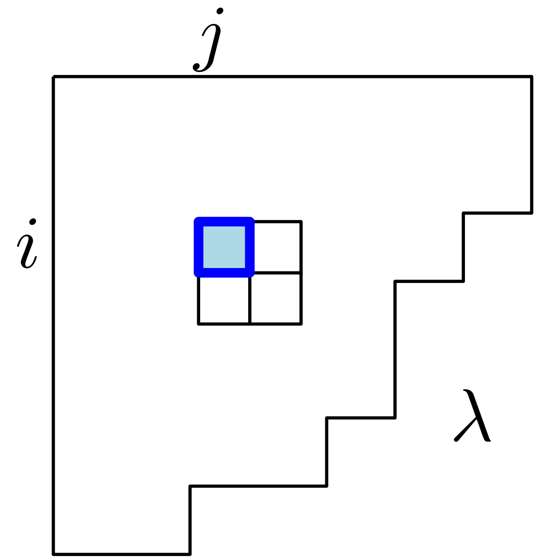

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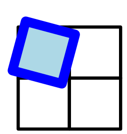

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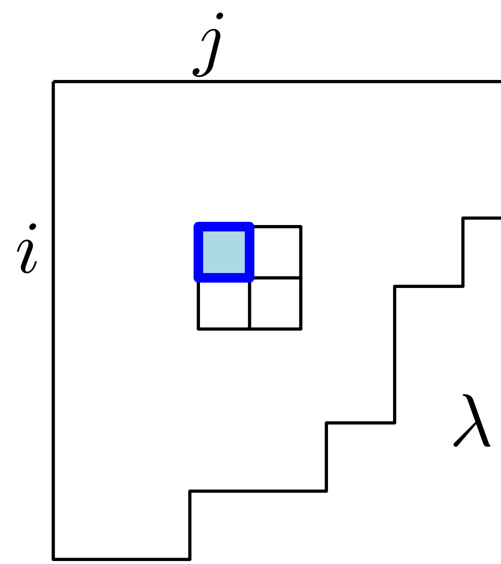

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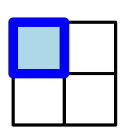

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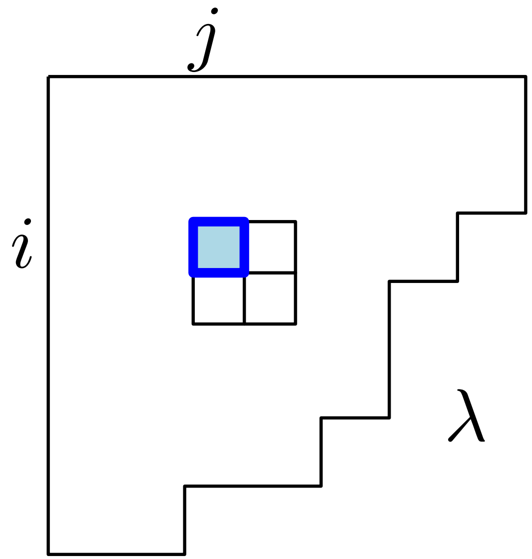

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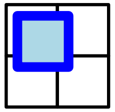

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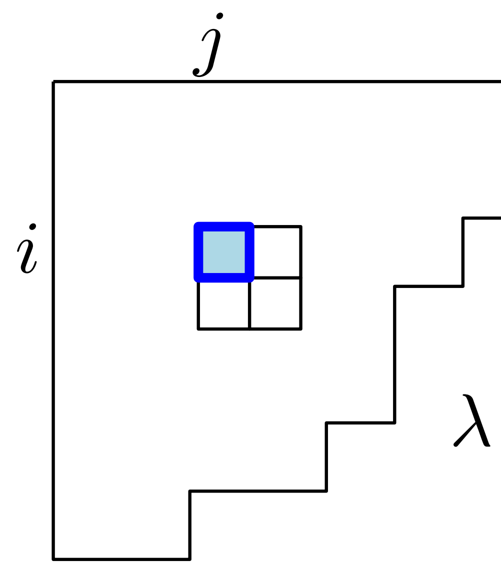

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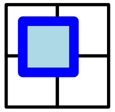

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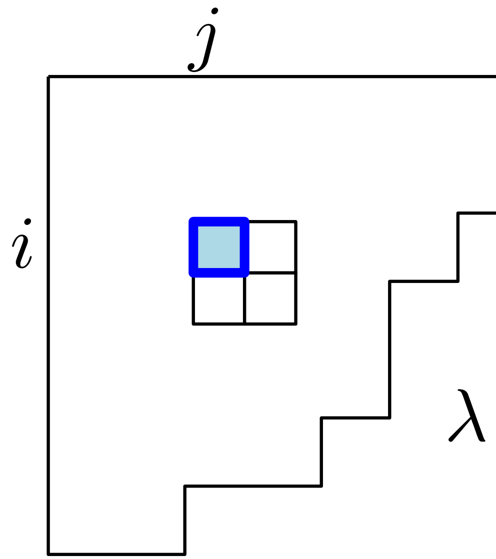

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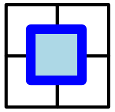

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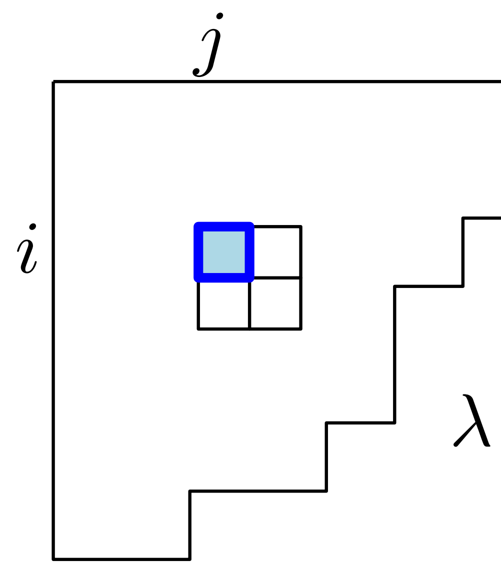

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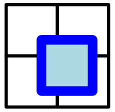

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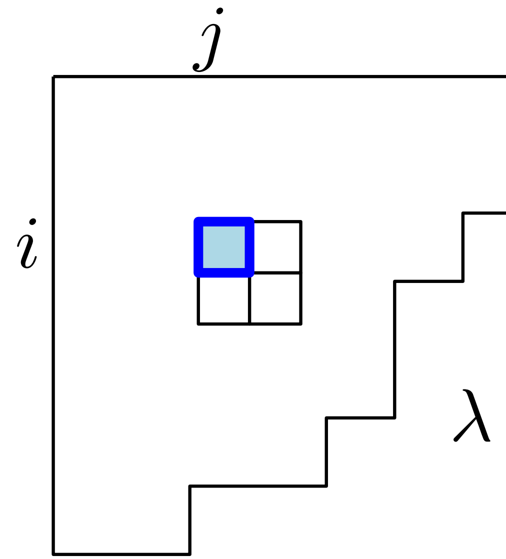

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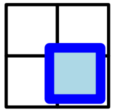

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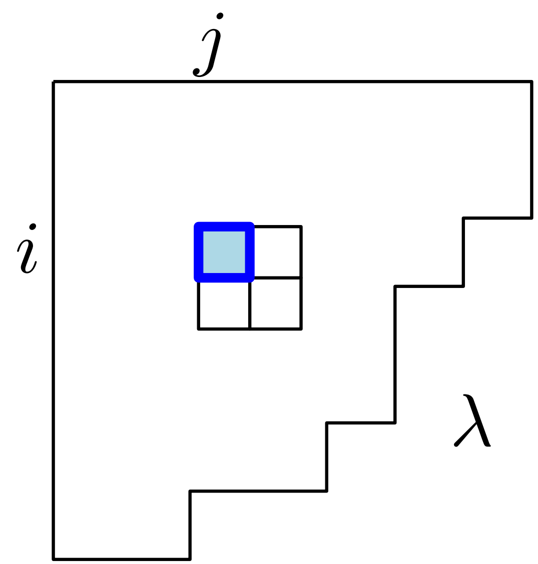

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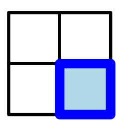

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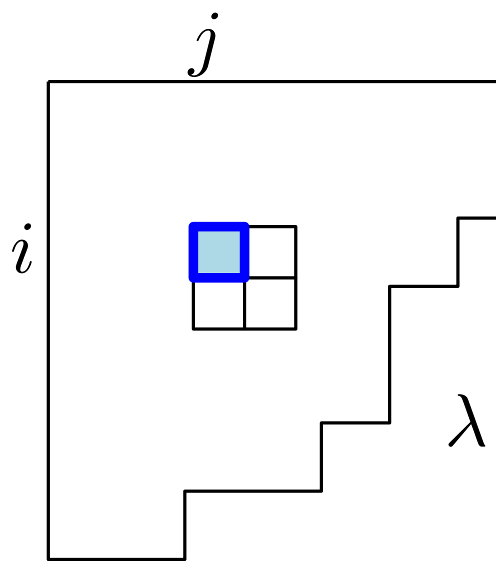

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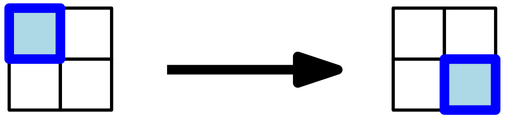

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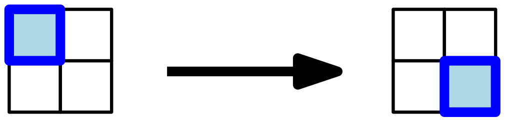

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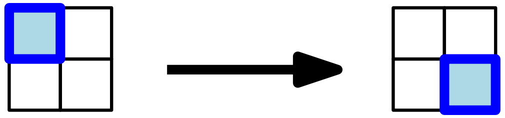

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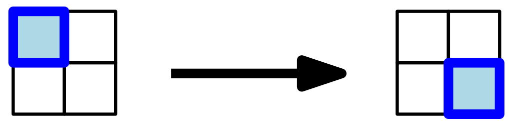

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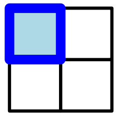
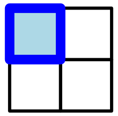
$\mathcal{E}($  $) = \{$  $\}$

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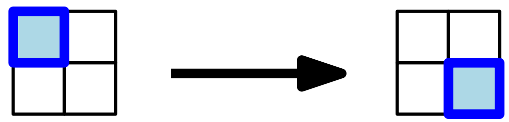

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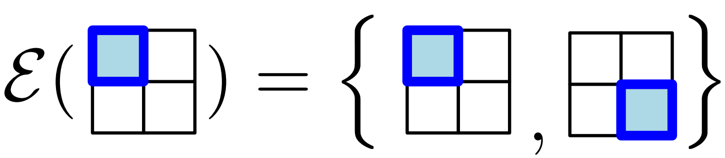

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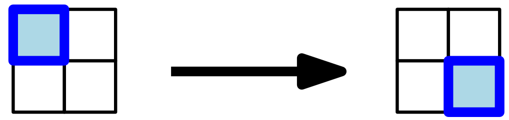

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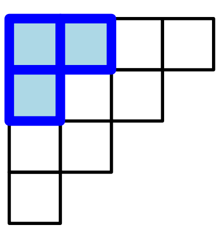
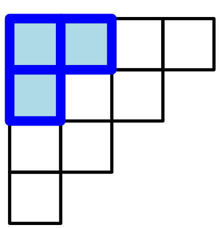
$\mathcal{E}($  $)$:



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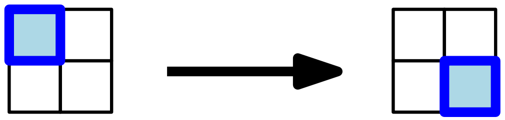

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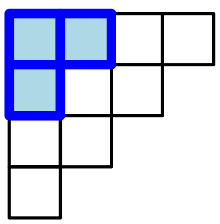
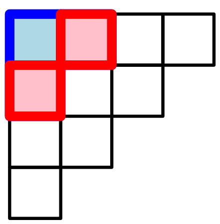
$\mathcal{E}($  $)$:



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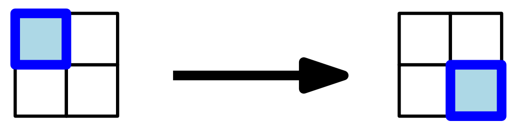

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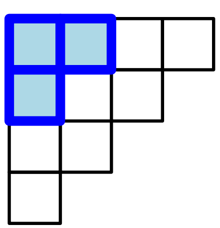
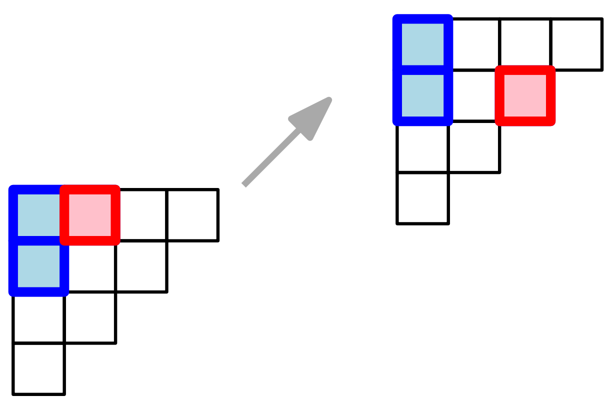
$\mathcal{E}($  $)$:



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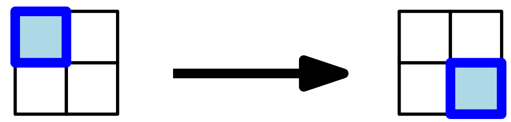

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Excited diagrams $\mathcal{E}(\lambda/\mu)$: diagrams obtained from μ by applying iteratively excited moves on excited cells.

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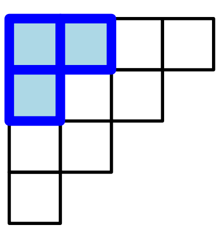
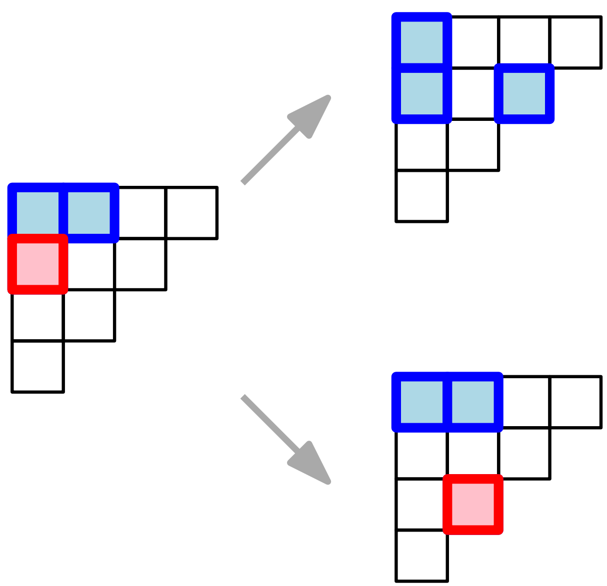
$\mathcal{E}($  $)$:



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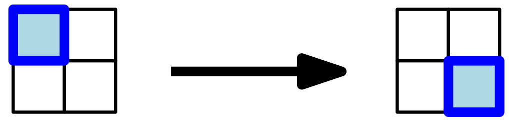

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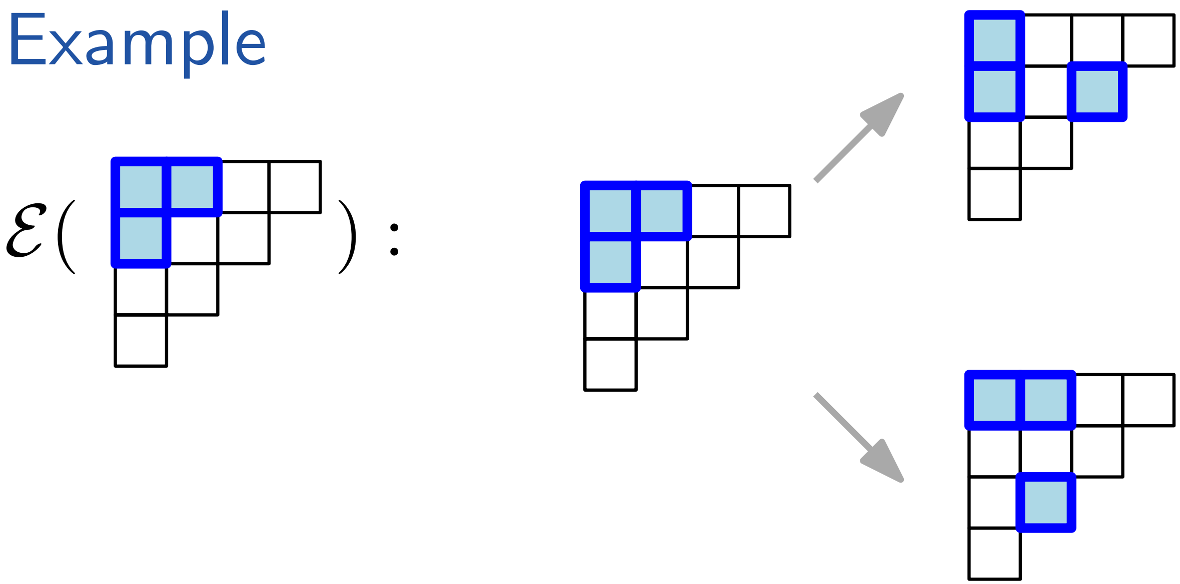

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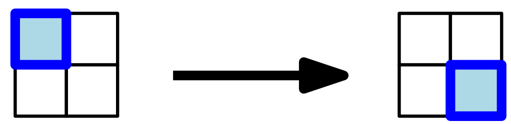

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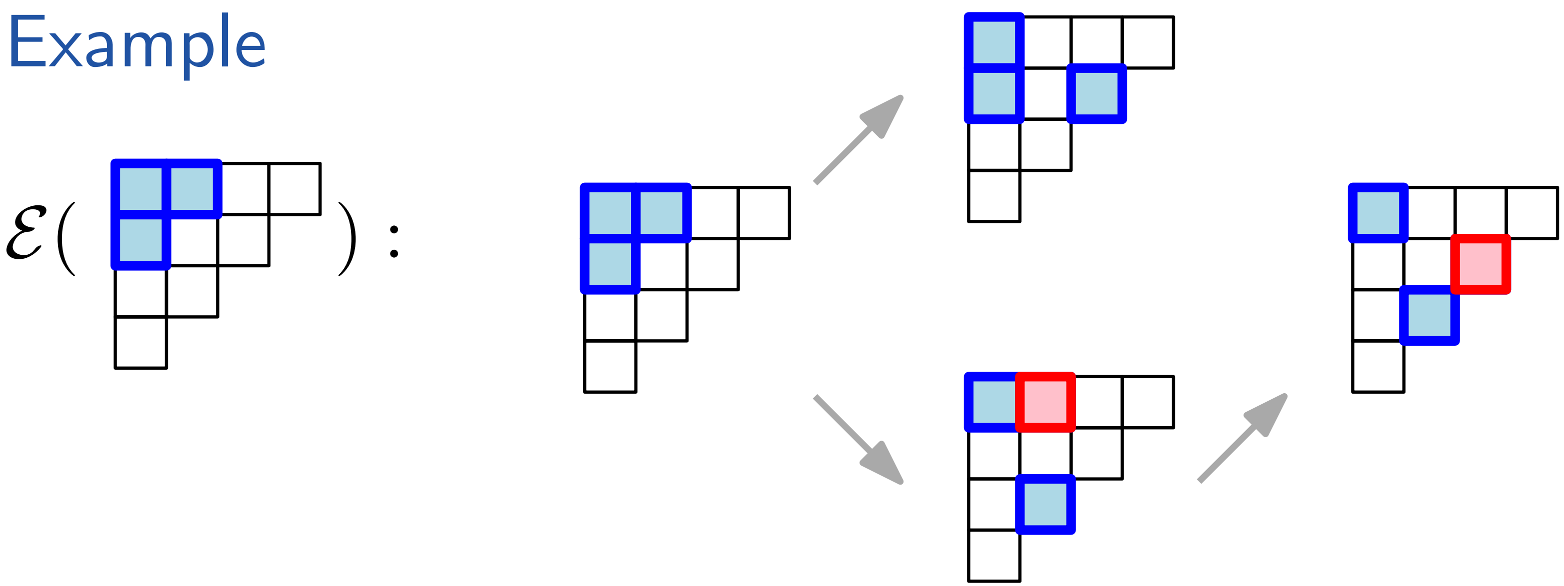

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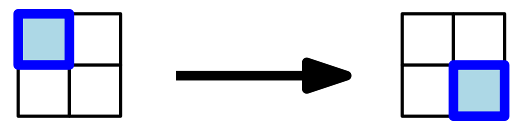

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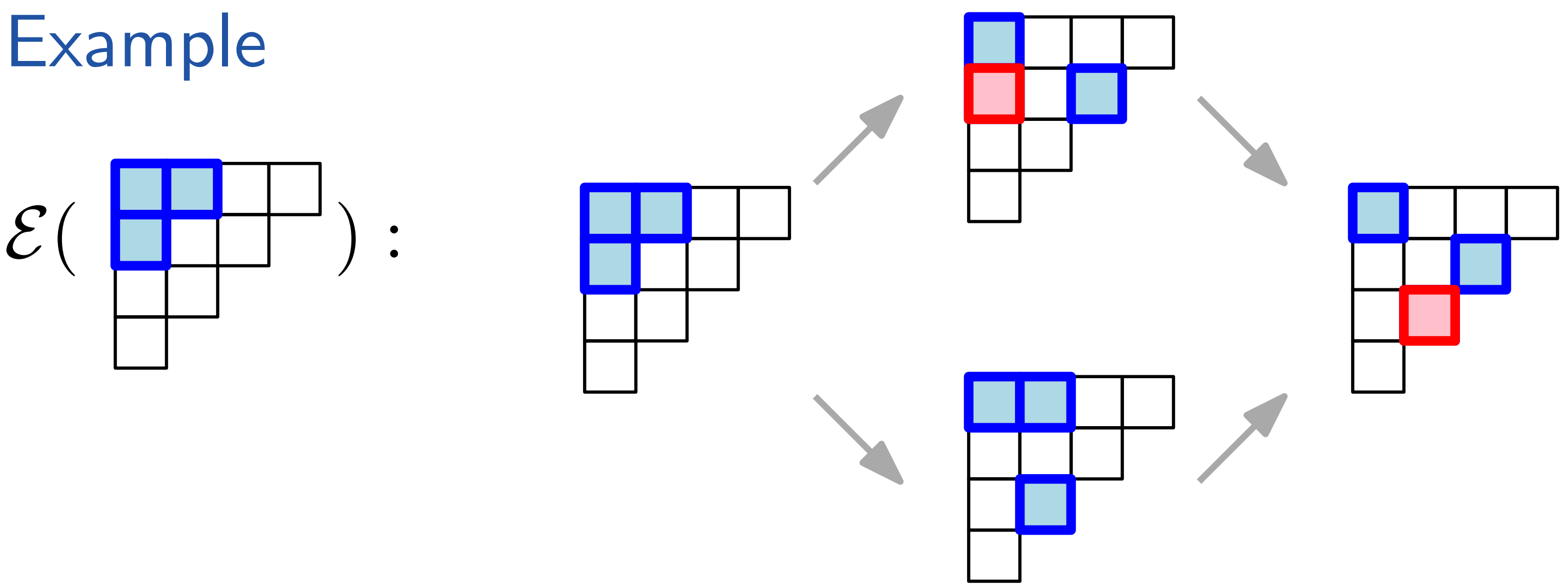

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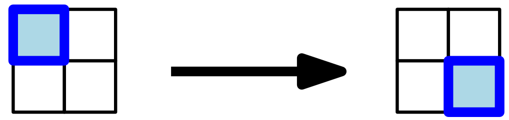

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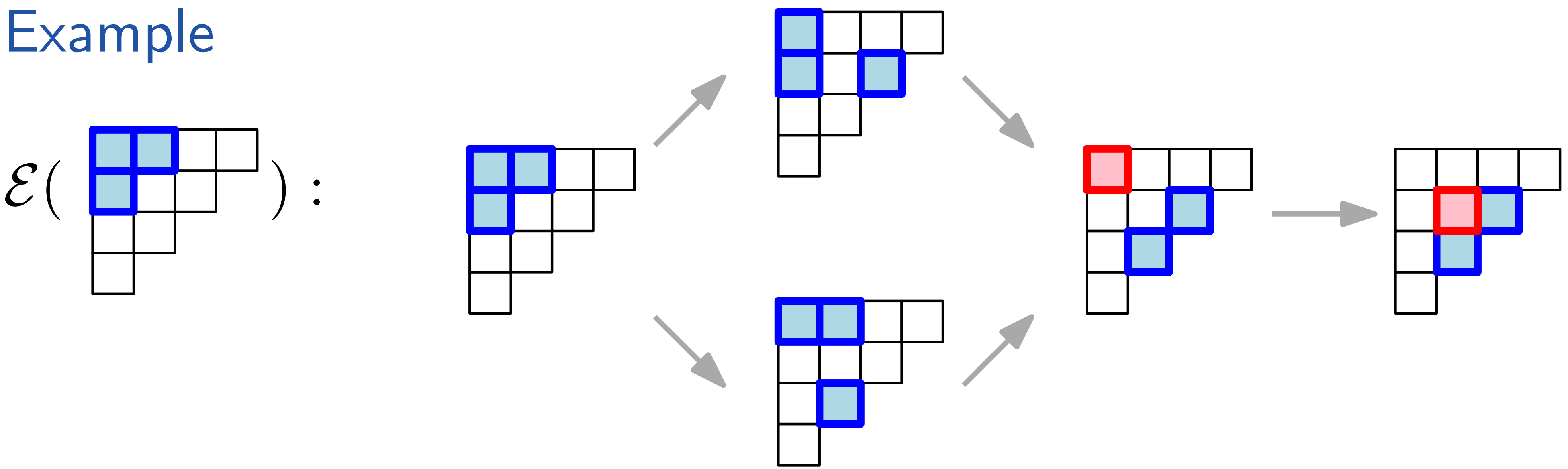

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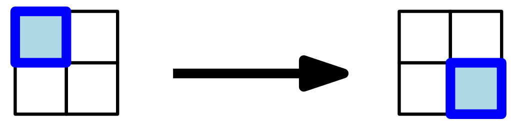

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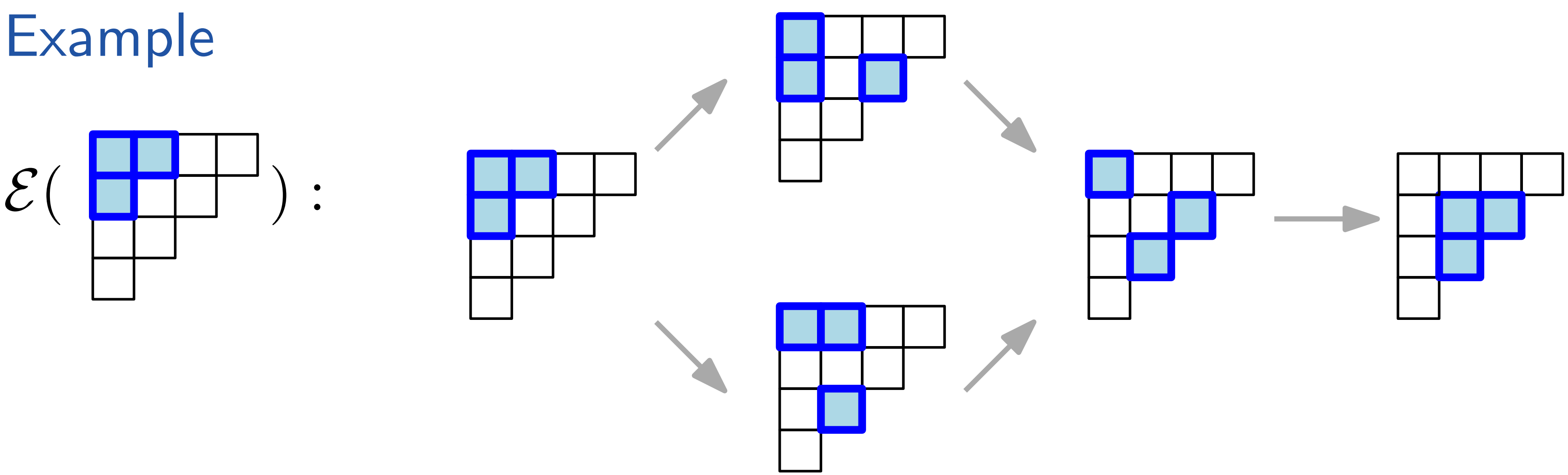

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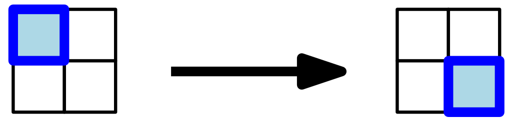

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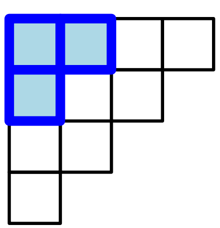
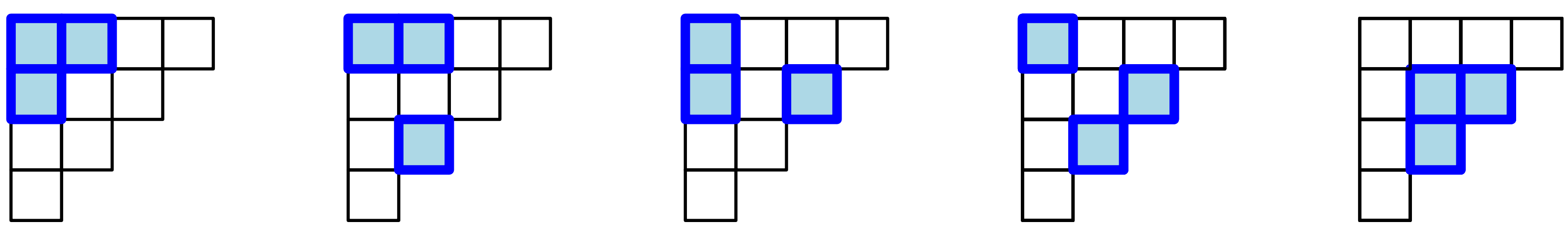
$\mathcal{E}\big($  $\big)$:



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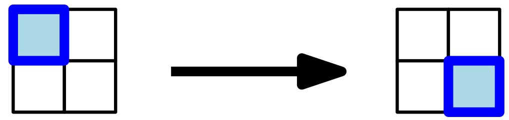

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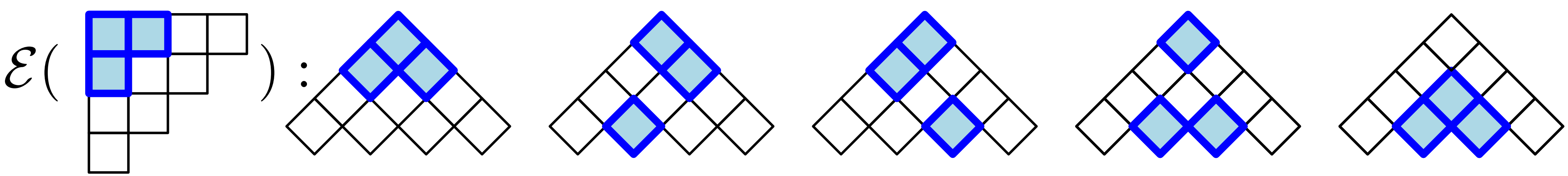

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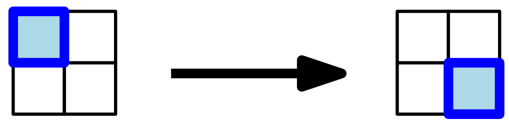

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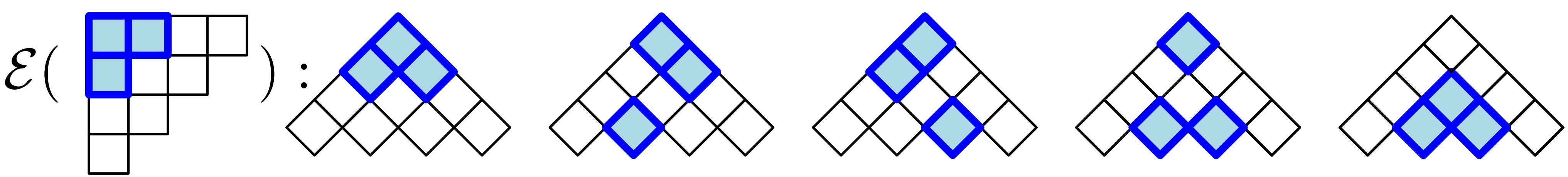

Proposition $|\mathcal{E}(z(n))| = \frac{1}{n+1}\binom{2n}{n}.$

Naruse's "hook-length" formula for $f^{\lambda/\mu}$

Theorem (Naruse 2014)

$$f^{\lambda/\mu} = n! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

where $\mathcal{E}(\lambda/\mu)$ is the set of **excited diagrams** of λ/μ.

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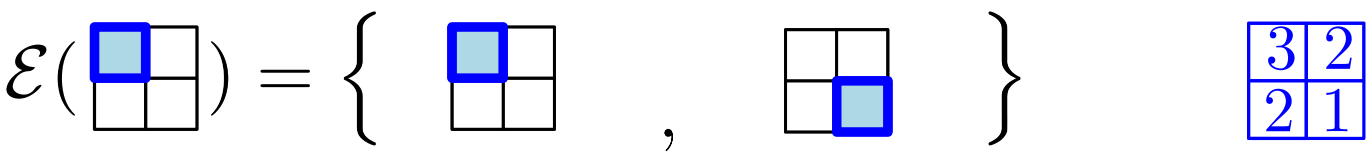

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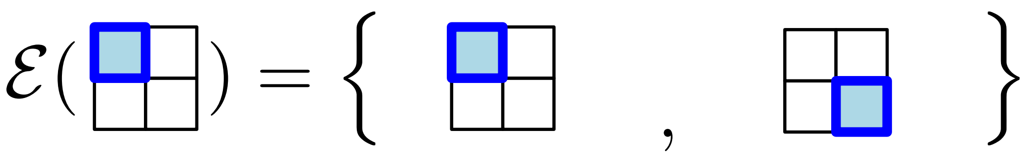

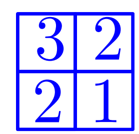

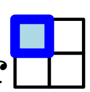

$$f^{\square} = 3! \cdot \left(\frac{1}{1 \cdot 2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 3} \right)$$

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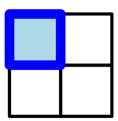
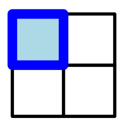
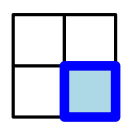
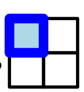
$\mathcal{E}($  $) = \{$  ,  $\}$

3	2
2	1

f  $= 3! \cdot \left(\frac{1}{1 \cdot 2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 3} \right) = 3! \left(\frac{1}{4} + \frac{1}{12} \right) = 2.$

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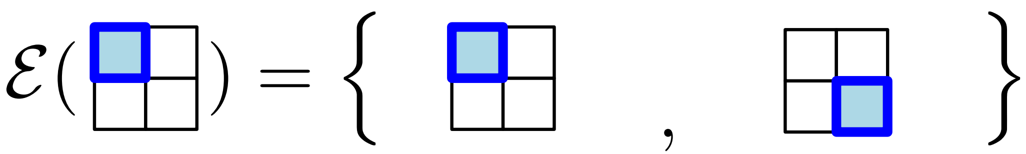

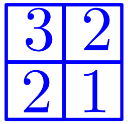

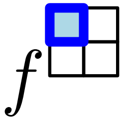

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- we have two q-analogues and a combinatorial proof (M, Pak, Panova, 2015,2016)

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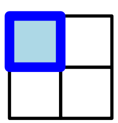
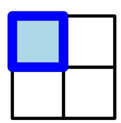
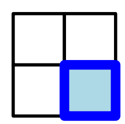
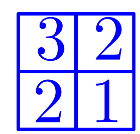
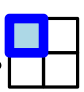
$\mathcal{E}($  $) = \{$  ,  $\}$



f  $= 3! \cdot \left(\frac{1}{1 \cdot 2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 3} \right) = 3! \left(\frac{1}{4} + \frac{1}{12} \right) = 2.$

- we have two q-analogues and a combinatorial proof (M, Pak, Panova, 2015,2016)
- Konvalinka (2016+) announced a probabilistic proof

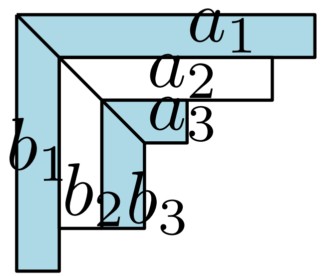
Outline

- $f^\lambda = \dfrac{n!}{\prod_{u \in \lambda} h(u)}$
- asymptotics

- $f^{\lambda/\mu} = n! \sum_{D \in \mathcal{E}(\lambda/\mu)} \cdots$
- asymptotics?

Some known bounds

- Thoma-Verskhi-Kerov limit: let $\lambda^n \to (\alpha \mid \beta)$ in Frobenius coordinates,



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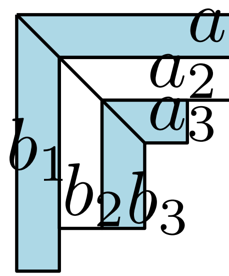


$$a_i/n \to \beta_1 \qquad b_i/n \to \beta_i$$

fix μ

Stanley 1993

$$f^{\lambda^n/\mu} = f^{\lambda^n} s_\alpha(\alpha/-\beta)(1+O(1/n))$$

Okounkov-Olshanski have explicit formulas for $f^{\lambda/\mu}/f^{\lambda}$.

Related work by Corteel-Goupil-Schaeffer 2004

Main bound from Naruse's formula

$$f^{\lambda/\mu} = n! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)},$$

Let the naive hook-length formula

$$F(\lambda/\mu) := \frac{n!}{\prod_{(i,j)\in\lambda/\mu} h(i,j)}$$

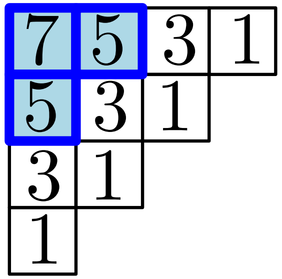
Corollary

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)| \cdot F(\lambda/\mu)$$

Proof

LB: μ is an excited diagram

UB: The diagram that contributes the most is $D = \mu$.



Bounds for number of excited diagrams

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)| \cdot F(\lambda/\mu)$$

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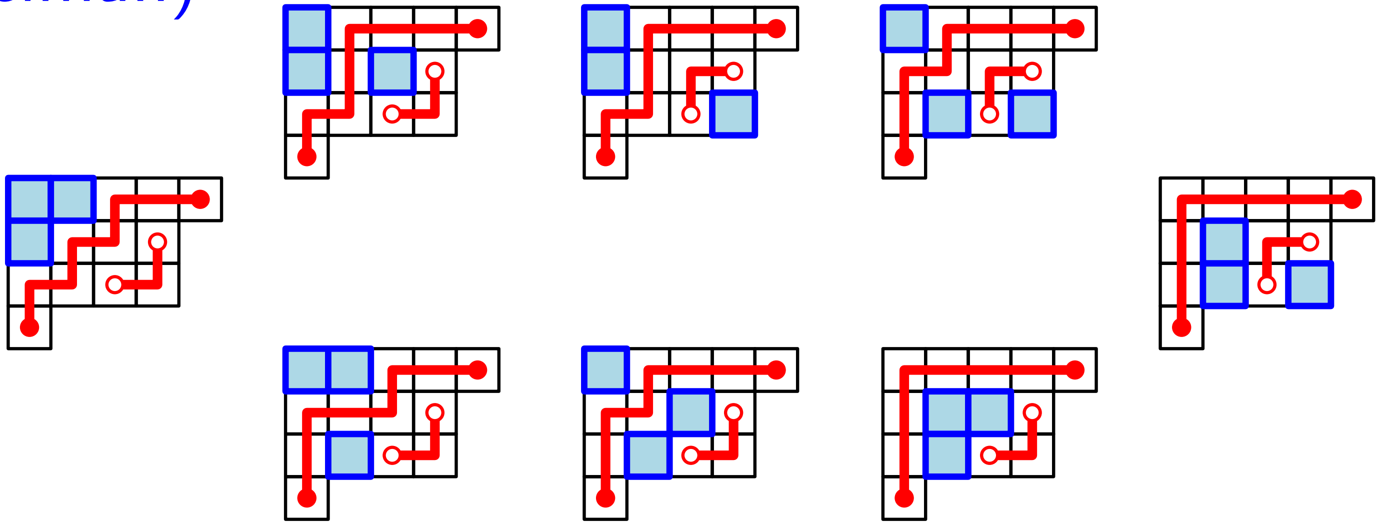
Proof:

Excited diagrams correspond to certain non-intersecting paths in λ (Kreiman)



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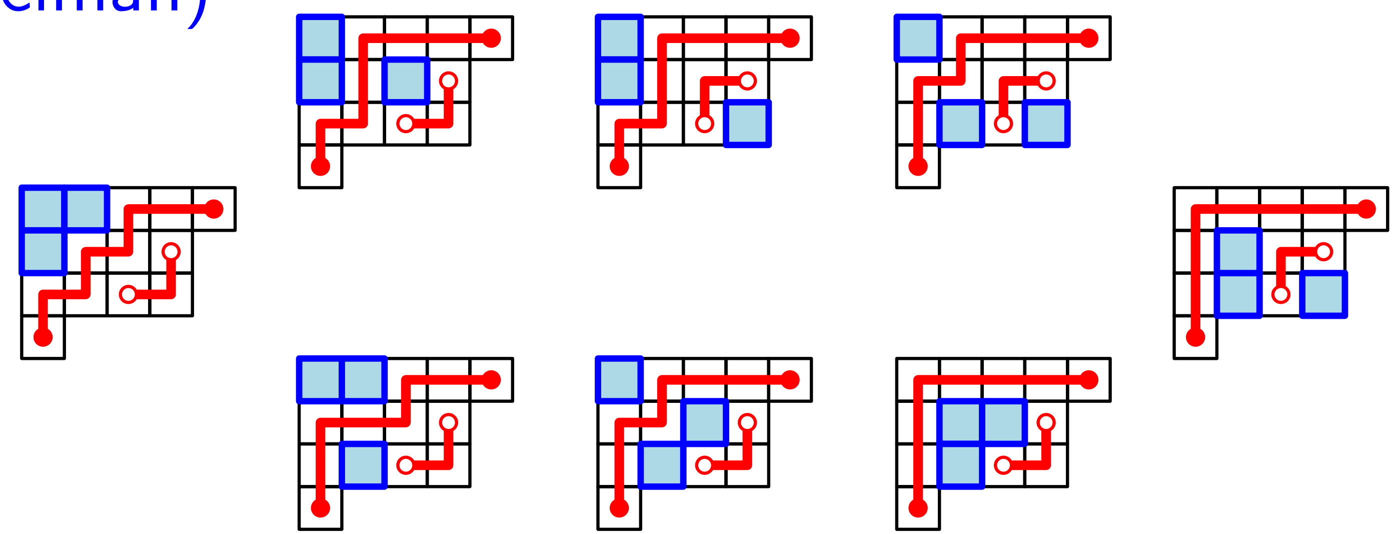
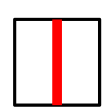
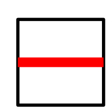


Each path determined by steps  or 

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$$|\mathcal{E}(\lambda/\mu)| \leq 2^n$$

$$|\mathcal{E}(\lambda/\mu)| \leq n^{2d^2}$$

where d size Durfee square of λ

in some special cases $F(\lambda/\mu)$ dwarfs $|\mathcal{E}(\lambda/\mu)|$

Comparing bounds

general poset bound:

$$r_1! \cdots r_\ell! \leq f^{\lambda/\mu} \leq \frac{n!}{c_1! \cdots c_m!}$$

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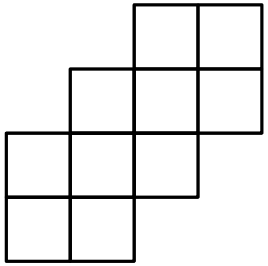

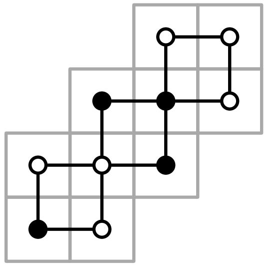

$$864 = 3!4!3! \leq f^{\lambda/\mu} \leq \frac{10!}{3!3!3!1!} = 16800$$

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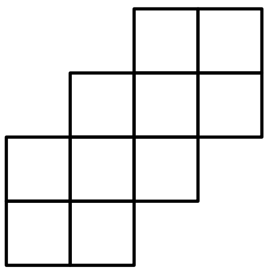
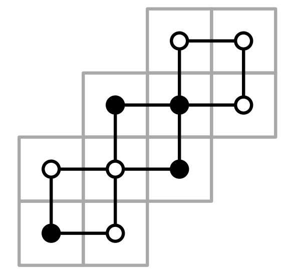
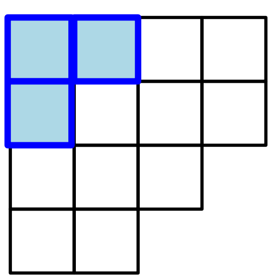
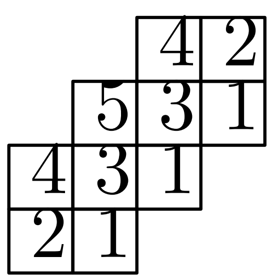




$$864 = 3!4!3! \le f^{\lambda/\mu} \le \frac{10!}{3!3!3!1!} = 16800$$

$$F(\lambda/\mu) \le f^{\lambda/\mu} \le |\mathcal{E}(\lambda/\mu)| \cdot F(\lambda/\mu)$$

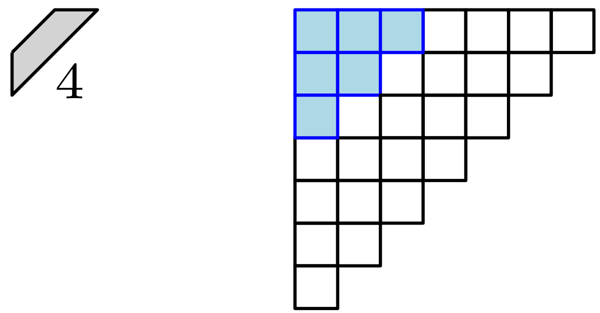




$$1260 = \frac{10!}{5 4^2 3^2 2^2} \le f^{\lambda/\mu} \le 5 \cdot 1260 = 6300$$

Main application:

Let $\diagup_k$ be shape $(2k-1, 2k-2, \ldots, 1)/(k-1, k-2, \ldots, 1)$



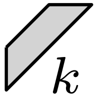
$n = k(3k-1)/2$

Theorem (M., Pak, Panova 16)

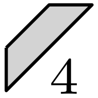
$$-0.3237 \leq \frac{1}{n}\left(\log f^{\diagup_k} - \frac{1}{2} n \log n\right) \leq -0.0621$$

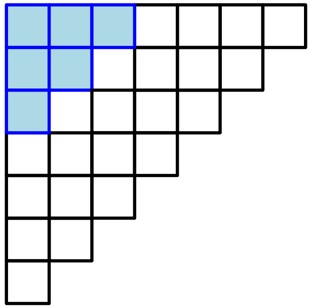
Compare with general bound for $e(\mathcal{P})$:

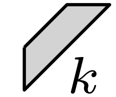
$$-0.7785 \leq \frac{1}{n}\left(\log f^{\diagup_k} - \frac{1}{2} n \log n\right) \leq 0.3694$$

Why bound from Naruse's formula is good for 

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)| \cdot F(\lambda/\mu)$$





Why bound from Naruse's formula is good for 

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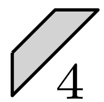
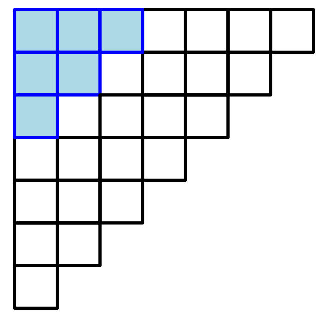
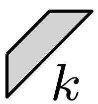
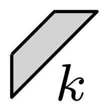
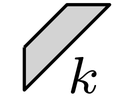
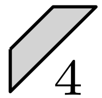


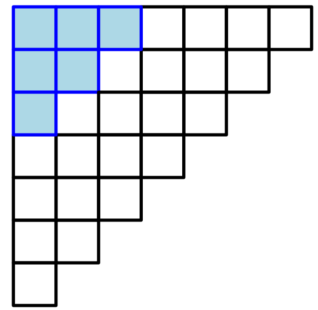


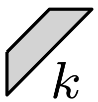
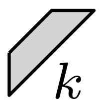
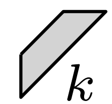
- $F($  $) = \dfrac{n!}{\prod_{u \in \text{ [shape] }} h(u)}$  ratio of easy hooks

Why bound from Naruse's formula is good for 

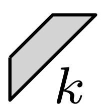
$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)| \cdot F(\lambda/\mu)$$

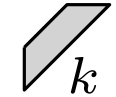




- $F($  $) = \dfrac{n!}{\prod_{u \in}$  $h(u)}$ ratio of easy hooks
- For this shape there is a product formula for $|\mathcal{E}($  $)|$

Lemma (Proctor 1990)

$|\mathcal{E}($  $)| = \displaystyle\prod_{1 \leq i < j \leq k} \frac{k+i+j-1}{i+j-1}$

Why bound from Naruse's formula is good for 

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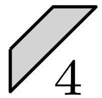
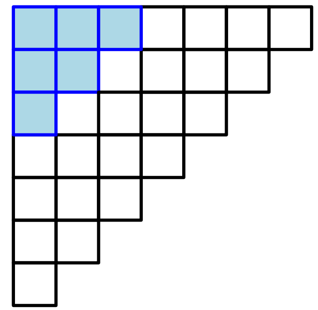
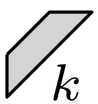
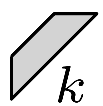
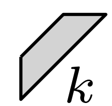
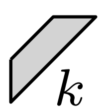




- $F($  $) = \dfrac{n!}{\prod_{u \in}$  $h(u)}$ ratio of easy hooks
- For this shape there is a product formula for $|\mathcal{E}($  $)|$

Lemma (Proctor 1990)

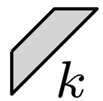
$|\mathcal{E}($  $)| = \displaystyle\prod_{1 \le i < j \le k} \frac{k+i+j-1}{i+j-1}$

- express bounds in terms of (double) factorials and use Stirling's formula

Summary

- bounds from Naruse's formula

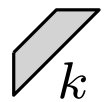
$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)| \cdot F(\lambda/\mu)$$

- thick zigzags: f  $_k \approx \sqrt{n!}$ get good bounds for second asymptotic term

Summary

- bounds from Naruse's formula

$$F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)| \cdot F(\lambda/\mu)$$

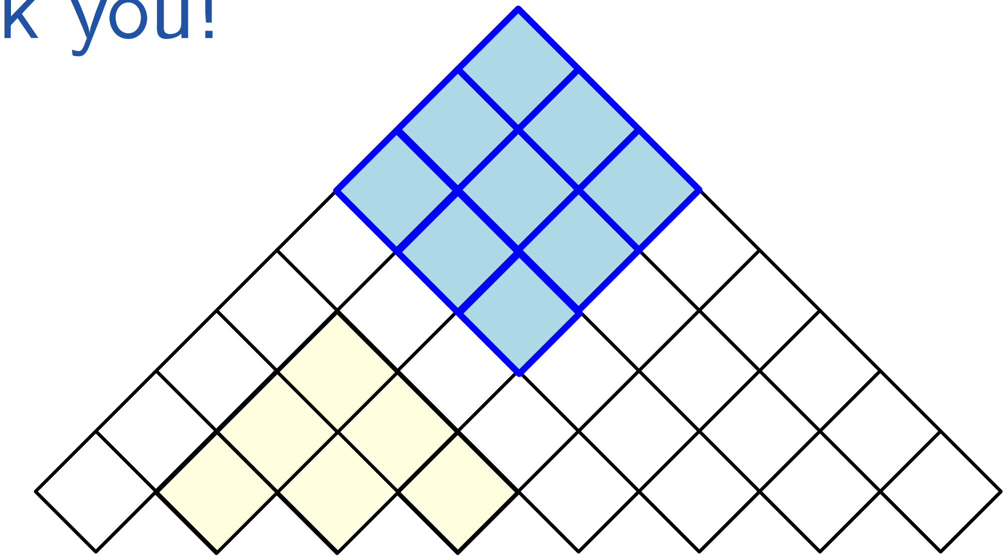
- thick zigzags: f 

$\approx \sqrt{n!}$ get good bounds for second asymptotic term

- other shapes where row/col lengths grow like $\sqrt{n}$ then $f^{\lambda/\mu} \approx \sqrt{n!}$

- λ, μ have Thoma-Vershik-Kerov limit, $f^{\lambda/\mu}$ has exponential growth

Thank you!



Some references

- **Increasing and decreasing subsequences and their variants**, R.P. Stanley, Proc. ICM, Vol I, 545-579
- **Schubert calculus and hook formula**, H. Naruse, slides Séminaire Lotharingien de Combinatoire 73, Strobl, Austria, 2014
- **Asymptotics for the number of standard Young tableaux of skew shape**, M., I. Pak, G. Panova, `arxiv:1610.07561`
- **Hook formulas for skew shapes I and II**, M., I. Pak, G. Panova, `arxiv:1512:08348`, `arxiv:1610.04744`

Theorem (Brightwell, Tetali 2003)

$$\frac{\log_2(e(B_n))}{2^n} = \log_2 \binom{n}{\lfloor n/2 \rfloor} - \frac{3}{2}\log_2(e) + o(1)$$