Random $\mathbb{Z}^d$ SFTs

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October 25, 2016

joint work with Ronnie Pavlov

The probability model

Recall:

- "dimension" d and "alphabet" $\mathcal{A}$ are fixed;
- for $n \in \mathbb{N}$ and $\alpha \in [0, 1]$,

$$\mathbb{P}_{n,\alpha} \text{ is a prob. meas. on the power set of } \mathcal{A}^{[1,n]^d};$$

- if $\mathcal{F} \subset \mathcal{A}^{[1,n]^d}$, then

$$\mathbb{P}_{n,\alpha}(\mathcal{F}) = \alpha^{|\mathcal{A}|^{n^d} - |\mathcal{F}|}(1 - \alpha)^{|\mathcal{F}|}.$$

- We study properties of random SFT $X(\mathcal{F})$ under $\mathbb{P}_{n,\alpha}$ as $n \to \infty$.

Finite orbits

- Let $\{\gamma_k\}_k$ be an enumeration of the finite orbits in $\mathcal{A}^{\mathbb{Z}^d}$.
- For $t \in [0, 1]$, define
$$g(t) = \prod_k (1 - t^{|\gamma_k|}).$$
- The coefficients of $g(t)^{-1}$ are related to counts of finite orbits (Artin-Mazur zeta function for $d = 1$).
- $g(t) > 0$ if and only if $t < |\mathcal{A}|^{-1}$.

Emptiness theorem

Theorem (M.-Pavlov)

For $\alpha \in [0,1]$,

$$\lim_n \mathbb{P}_{n,\alpha}\big(X(\mathcal{F}) = \varnothing\big) = g(\alpha).$$

Moreover, for $\alpha \neq |\mathcal{A}|^{-1}$, *there exists* $\rho > 0$ *such that for all large n,*

$$\Big|\mathbb{P}_{n,\alpha}\big(X(\mathcal{F}) = \varnothing\big) - g(\alpha)\Big| \leq \exp\Big(-\rho n^d\Big).$$

- Recall: $g(\alpha) > 0$ if and only if $\alpha < |\mathcal{A}|^{-1}$.
- Emptiness is undecidable, but its asymptotic probability is tractable.

Intuition behind $g(\alpha)$

- For a finite orbit γ and all large n, there are $|\gamma|$ patterns with shape $[1, n]^d$ contained in γ.
- Thus, $\mathbb{P}_{n,\alpha}\big(\gamma \subset X(\mathcal{F})\big) = \alpha^{|\gamma|}$, and $\mathbb{P}_{n,\alpha}\big(\mathcal{F} \text{ forbids } \gamma\big) = 1 - \alpha^{|\gamma|}$.
- So
$$g(\alpha) = \prod_k (1 - \alpha^{|\gamma_k|})$$
looks like the probability of independently forbidding all finite orbits.

The intuition is not the proof

For the proof, we find $K_1 = K_1(n) \to \infty$, $K_2 = K_2(n) \to \infty$, and $C_n \to 1$ such that

$$C_n \prod_{k \leq K_1} (1 - \alpha^{|\gamma_k|}) \leq \mathbb{P}_{n,\alpha}\big(X(\mathcal{F}) = \varnothing\big) \leq \prod_{k \leq K_2} (1 - \alpha^{|\gamma_k|}).$$

- Difficulties:
  - finite orbits with "large period" relative to n;
  - finite orbits are not all independent;
  - emptiness is not equivalent to "no finite orbits" for $d > 1$;
- Most difficult part of our proof: show that $C_n \to 1$.
- Main technical tool: tight control on the number of large patterns with exactly j subpatterns of shape $[1, n]^d$.

Probability of aperiodic SFTs

Theorem (M.-Pavlov)

For each n, let $\mathcal{G}_n$ be the event that $X(\mathcal{F})$ is non-empty but contains no finite orbits. Then for $\alpha \in [0, 1]$,

$$\lim_n \mathbb{P}_{n,\alpha}(\mathcal{G}_n) = 0.$$

Moreover, for $\alpha \neq |\mathcal{A}|^{-1}$, the convergence rate is at least $\exp(-\rho n^d)$.

"Worst-case" behavior for $d > 1$ has asymptotically zero probability.

Entropy theorem

Theorem (M.-Pavlov)

Let $\alpha \in [0,1]$ and $\epsilon > 0$. Then there exists $\rho > 0$ such that for all large enough n,

$$\mathbb{P}_{n,\alpha}\left(\left| h(X(\mathcal{F})) - \log^{+}(\alpha|\mathcal{A}|) \right| \geq \epsilon \right) \leq \exp\left(-\rho n^{d}\right).$$

Entropy converges in probability to a point mass at $\log^{+}(\alpha|\mathcal{A}|)$.

Intuition about $\log^+(\alpha|\mathcal{A}|)$

Imagine trying to extend a pattern by filling one additional site.

- How many choices to fill the site? $|\mathcal{A}|$.
- Each choice is allowed with probability α.
- Looks like a branching process with "offspring distribution" given by Bin($|\mathcal{A}|, \alpha$), which has expectation $\alpha|\mathcal{A}|$.

This approximation holds until one of the "children" has appeared before (loss of independence).

Basic outline of the proof

For $k > n$, let

$$\varphi_{n,k} = \sum_{u \in L_k} \xi_u \quad \text{(number of allowed words on } [1,k]^d\text{)}$$
$$\psi_{n,k} = \sum_{u \in P_{n,k}} \xi_u \quad \text{(number of allowed “periodic" words on } [1,k]^d\text{).}$$

Here ξ_u is the characteristic function of the event that u contains no words from $\mathcal{F}$.

- Upper bound: $h(X(\mathcal{F})) \leq \frac{1}{k^d} \log \varphi_{n,k}$.
- Lower bound: $\frac{1}{k^d} \log \psi_{n,k} - r_{n,k} \leq h(X(\mathcal{F}))$.
- Choose $k = k(n)$ to control estimates on $\varphi_{n,k}$, $\psi_{n,k}$ and $r_{n,k}$.

Ideas in the proof: upper bound

- For $u \in L_k$, let $W_n(u) = \{u|_{p+[1,n]^d} : p + [1,n]^d \subset [1,k]^d\}$.
- Then the probability that u contains no pattern from $\mathcal{F}$ is $\alpha^{|W_n(u)|}$.
- Hence

$$\mathbb{E}_{n,\alpha}[\varphi_{n,k}] = \sum_{u \in L_k} \mathbb{E}_{n,\alpha}[\xi_u] = \sum_{u \in L_k} \alpha^{|W_n(u)|} = \sum_{j=1}^{(k-n+1)^d} \alpha^j N_{n,k}^j.$$

- Show that variance of $\varphi_{n,k}$ is small compared to its expectation squared, and use Chebychev's inequality.

Ideas in the proof: lower bound

- Let $p_{n,k}$ be the number of periodic frames of length k and thickness n.
- Then there exists at least one periodic frame such that the number of ways it may be legally filled is at least

$$\frac{1}{p_{n,k}} \sum_{u \in P_{n,k}} \xi_u = \frac{1}{p_{n,k}} \psi_{n,k}.$$

- Hence $h(X(\mathcal{F})) \geq \frac{1}{k^d} \log \psi_{n,k} - \frac{\log(p_{n,k})}{k^d}$.
- Use second moment method on $\psi_{n,k}$.

Periodic entropy

Let X be a SFT, and let X_{per} be the set of points with finite orbit in X. Define

$$h_{per}(X) = \lim_k \frac{1}{k^d} \log \#\{x|_{[1,k]^d} : x \in X_{per}\}.$$

Theorem (M.-Pavlov)

Let $\alpha \in [0,1]$ and $\epsilon > 0$. Then there exist $\rho > 0$ such that for all large enough n,

$$\mathbb{P}_{n,\alpha}\left(\left|h_{per}(X(\mathcal{F})) - \log^+(\alpha|\mathcal{A}|)\right| \geq \epsilon\right) \leq \exp\left(-\rho n^d\right).$$

Conclusions

- From the point of view of emptiness, periodic points, and entropy, it appears that “typical” $\mathbb{Z}^d$ SFTs behave as well as $\mathbb{Z}$ SFTs.
- It seems that even the Swamp of Undecidability is “typically" a pleasant place.

Acknowledgments

- KM acknowledges the support of NSF DMS 1613261.
- RP acknowledges the support of NSF DMS 1500685.