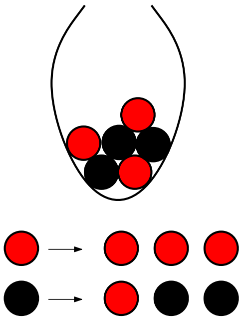
Measure-valued Pólya processes

– Cécile Mailler –
(Prob-L@B – University of Bath)

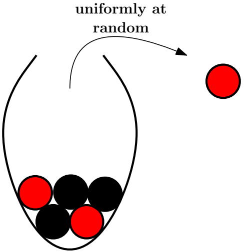
joint work with Jean François Marckert (Bordeaux)

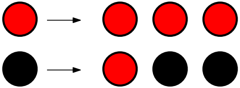
October 27, 2016

Pólya urns

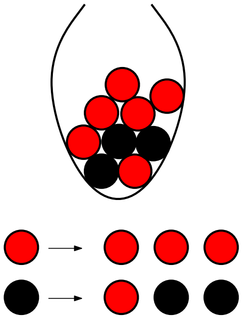


Pólya urns

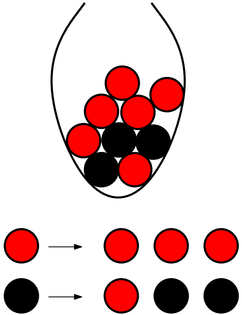




Pólya urns



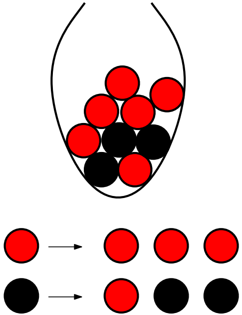
Pólya urns



Replacement matrix

$$R = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

Pólya urns



A discrete-time Markov process

$$U(n) = \begin{pmatrix} U_1(n) \\ \vdots \\ U_d(n) \end{pmatrix},$$

where $U_i(n)$ is the number of balls of colour i in the urn at time n.

Replacement matrix

$$R = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

Two parameters:

- the initial composition vector $U(0)$
- the replacement matrix $R = (R_{i,j})_{1 \le i,j \le d}$

Asymptotic composition

Standard assumptions:

Tenability

- $\|U(0)\|_1 > 0$ and
- $R_{i,j} \geq 0$ for all $1 \leq i,j \leq d$.

Irreducibility

For all $1 \leq i,j \leq d$, there exists n such that $R^n_{i,j} > 0$.

Under these assumptions, Perron–Frobenius applies and the spectral radius λ of R is a simple eigenvalue of R.

Theorem [see Athreya & Ney]

If $(U(n))_{n\geq 0}$ is tenable and irreducible, then

$$U(n)/n \to v_1, \text{ a.s.}$$

where v_1 is a (nicely chosen) positive eigenvector of R associated to λ.

A vast literature

Problems

- second/third order convergence theorems
- non-irreducible cases
- random replacement matrix
- multi-drawing
- etc.

Methods

- combinatorics
- probability
- stochastic approximation
- analytic combinatorics

Authors

Athreya & Karlin

Janson

Gouet

Flajolet, Dumas & Puyhaubert

Kuba, Mahmoud, Morcrette, Sulzbach

Knape & Neininger

Chauvin, Pouyanne, Sahnoun

etc.

Infinitely many colours?

Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
ELSEVIER Stochastic Processes and their Applications 110 (2004) 177–245
stochastic processes and their applications
www.elsevier.com/locate/spa

Functional limit theorems for multitype branching processes and generalized Pólya urns

Svante Janson*

Department of Mathematics, Uppsala University, PO Box 480, Uppsala S-751 06, Sweden

Received 20 February 2003; received in revised form 11 November 2003; accepted 1 December 2003

These examples suggest the possibility of (and desire for) an extension of the results in this paper to infinite sets of types (with suitable assumptions). Our matrix A would then be replaced by an operator acting in a suitable space, such as $\ell^1(\mathbb{N})$ or $L^2(\mathbb{T})$. It is far from clear how such an extension should be formulated, and we have not pursued this.

Infinitely many colours?

Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE @ DIRECT•
ELSEVIER Stochastic Processes and their Applications 110 (2004) 177–245
stochastic processes and their applications
www.elsevier.com/locate/spa

Functional limit theorems for multitype branching processes and generalized Pólya urns

Svante Janson*
Department of Mathematics, Uppsala University, PO Box 480, Uppsala S-751 06, Sweden

Received 20 February 2003; received in revised form 11 November 2003; accepted 1 December 2003

These examples suggest the possibility of (and desire for) an extension of the results in this paper to infinite sets of types (with suitable assumptions). Our matrix A would then be replaced by an operator acting in a suitable space, such as $\ell^1(\mathbb{N})$ or $L^2(\mathbb{T})$. It is far from clear how such an extension should be formulated, and we have not pursued this.

Bandyopadhyay & Thacker ['13, '14, '16] have set the first stone: we generalise and strengthen their results.

Measure-valued Pólya processes

We define $(\mathcal{M}_n)_{n\geq 0}$ a sequence of random positive measures on a Polish space $\mathcal{P}$.

Two parameters:

- the **initial composition** $\mathcal{M}_0$ (a positive measure on $\mathcal{P}$);
- the **replacement measures** $(\mathcal{R}_x)_{x\in\mathcal{P}}$ (a set of positive measures on $\mathcal{P}$).

dictionary

- $\mathcal{P}$ is the set of colours;
- $\mathcal{M}_n$ is the composition of the urn at time n;
- For any borelian set $\mathcal{B}$ of $\mathcal{P}$, $\mathcal{M}_n(\mathcal{B})$ is *the mass of balls whose colour is in $\mathcal{B}$ in the urn at time n.*

Definition of the Markov process $(\mathcal{M}_n)_{n\geq 0}$

At time $n+1$, one picks a colour $\xi_{n+1} \in \mathcal{P}$ with respect to $\mathcal{M}_n/\mathcal{M}_n(\mathcal{P})$; and set $\mathcal{M}_{n+1} = \mathcal{M}_n + \mathcal{R}_{\xi_{n+1}}$.

Definition of the Markov process $(\mathcal{M}_n)_{n\geq 0}$

At time $n+1$, one picks a colour $\xi_{n+1} \in \mathcal{P}$ with respect to $\mathcal{M}_n/\mathcal{M}_n(\mathcal{P})$; and set $\mathcal{M}_{n+1} = \mathcal{M}_n + \mathcal{R}_{\xi_{n+1}}$.

The original d-colour Pólya urn process can be seen as a MVPP:

$$\mathcal{M}_0 = \sum_{i=1}^{d} U_i(0)\delta_i \quad \text{and} \quad \mathcal{R}_i = \sum_{j=1}^{d} R_{i,j}\delta_j.$$

Definition of the Markov process $(\mathcal{M}_n)_{n\geq 0}$

At time $n+1$, one picks a colour $\xi_{n+1} \in \mathcal{P}$ with respect to $\mathcal{M}_n/\mathcal{M}_n(\mathcal{P})$; and set $\mathcal{M}_{n+1} = \mathcal{M}_n + \mathcal{R}_{\xi_{n+1}}$.

Remarks:

- We can now allow the colour set to be infinite, but also **uncountable**.
- The composition measure $\mathcal{M}_n$ can be atom free (meaning that all balls have infinitesimal weight).

Under what assumptions can we obtain a nice convergence result?

Convergence of measures: $\mu_n \to \mu$ weakly iff

$$\text{for all bounded continuous functions } f : \mathcal{P} \to \mathbb{R}, \int f\,\mathrm{d}\mu_n \to \int f\,\mathrm{d}\mu.$$

The companion Markov chain

Let $(\mathcal{M}_n)_{n\geq 0}$ be the MVPP of replacement measures $(\mathcal{R}_x)_{x\in\mathcal{P}}$.

The companion Markov chain

Let $(W_n)_{n\geq 0}$ be the Markov chain on $\mathcal{P}$ of initial distribution $\mathcal{M}_0/\mathcal{M}_0(\mathcal{P})$; and Kernel $K(x,\,\cdot\,) = \mathcal{R}_x$ (for all $x \in \mathcal{P}$).

In other words, conditionally on $W_n = x$, W_{n+1} has distribution $\mathcal{R}_x$.

Definition:

A Markov chain $(Q_n)_{n\geq 0}$ on $\mathcal{P}$ is said (a_n, b_n)-**convergent** if

$$\frac{Q_n - b_n}{a_n} \Rightarrow \gamma.$$

We say that it is (a_n, b_n)-**ergodic** if the limit distribution γ does not depend on the initial distribution of Q_0.

Main result

Let $(\mathcal{M}_n)_{n\geq 0}$ be a MVPP of initial composition $\mathcal{M}_0$ and replacement measures $(\mathcal{R}_x)_{x\in\mathcal{P}}$. Denote by $(W_n)_{n\geq 0}$ its companion Markov chain.

Assumptions:

- $0 < \mathcal{M}_0(\mathcal{P}) < \infty$,
- $\mathcal{R}_x(\mathcal{P}) = 1$ for all $x \in \mathcal{P}$ (balance hypothesis),
- $(W_n)_{n\geq 0}$ is (a_n, b_n)-ergodic with limit distribution γ,
- for all $x \in \mathcal{P}$, for all $\varepsilon_n = o(\sqrt{n})$,

$$\lim_{n\to\infty} \frac{b_{n+x\sqrt{n}+\varepsilon_n} - b_n}{a_n} = f(x) \text{ and } \lim_{n\to\infty} \frac{a_{n+x\sqrt{n}+\varepsilon_n}}{a_n} = g(x),$$

where f and g are two measurable functions.

Main result

Theorem [MM++]

Under all these assumptions,

$$n^{-1}\mathcal{M}_n\big(a_{\log n}\ \cdot\ + b_{\log n}\big) \to \nu \quad \text{in probability},$$

where ν is the distribution of $\Gamma g(\Lambda) + f(\Lambda)$, where $\Gamma \sim \gamma$ and $\Lambda \sim \mathcal{N}(0,1)$ are independent.

NB: in a general Polish space, addition and division by a scalar might not be defined. We set $x - 0 = x$ and $x/1 = x$ for all $x \in \mathcal{P}$.

Examples:

- the d-colour case: our result implies that $U(n)/n \to v_1$ **in probability**;
- your favourite ergodic Markov chain will define a convergent MVPP.

The random walk case

If $\mathcal{R}_x$ is the distribution of $x + \Delta$ where Δ is a real random variable, of finite mean m, then $(W_n)_{n\geq 0}$ is a random walk with i.i.d. increments:

- if Δ has finite variance σ^2, then $(W_n)_{n\geq 0}$ is $(\sqrt{n}, mn)$-ergodic and converges to $\gamma = \mathcal{N}(0, \sigma^2)$.
  Thus, in probability
  $$n^{-1}\mathcal{M}_n(\sqrt{\log n}\,\cdot\, + m\log n) \to \mathcal{N}(0, \sigma^2 + m^2)$$
  (it also holds in higher dimensions)
- if $\mathbb{P}(\Delta \geq u) \sim u^{-\alpha}$ with $\alpha < 2$, then $(W_n)_{n\geq 0}$ is $(n^{1/\alpha}, b_n)$-ergodic with $b_n = 0$ if $\alpha < 1$ and mn otherwise, and its limit law, γ, is alpha-stable. In both cases, in probability
  $$n^{-1}\mathcal{M}_n((\log n)^{1/\alpha}\,\cdot\,) \to \gamma.$$

The random walk case

If $\mathcal{R}_x$ is the distribution of $x + \Delta$ where Δ is a real random variable, of finite mean m, then $(W_n)_{n\geq 0}$ is a random walk with i.i.d. increments:

- if Δ has finite variance σ^2, then $(W_n)_{n\geq 0}$ is $(\sqrt{n}, mn)$-ergodic and converges to $\gamma = \mathcal{N}(0, \sigma^2)$. Thus, if there exists $\delta > 0$ such that $\mathbb{E}e^{\delta\Delta} < \infty$, then, almost surely

$$n^{-1}\mathcal{M}_n(\sqrt{\log n}\,\cdot\, + m\log n) \to \mathcal{N}(0, \sigma^2 + m^2)$$

  (it also holds in higher dimensions)

- if $\mathbb{P}(\Delta \geq u) \sim u^{-\alpha}$ with $\alpha < 2$, then $(W_n)_{n\geq 0}$ is $(n^{1/\alpha}, b_n)$-ergodic with $b_n = 0$ if $\alpha < 1$ and mn otherwise, and its limit law, γ, is alpha-stable. In both cases, in probability

$$n^{-1}\mathcal{M}_n((\log n)^{1/\alpha}\,\cdot\,) \to \gamma.$$

A corollary on the profile of the random recursive tree

The sequence $(\mathrm{RRT}_n)_{n\geq 0}$ is the Markov chain defined as

- $\mathrm{RRT}_0 = \{\varnothing\}$;
- To get RRT_{n+1}, take RRT_n, pick one of its nodes uniformly at random and attach a new child to this node.

RRT_n is the n-node random recursive tree;

If $\mathcal{R}_x$ is the distribution of $x + \Delta$ with Δ having exponential moments, then $n^{-1}\mathcal{M}_n(\sqrt{\log n}\,\cdot\, + m\log n) \to \mathcal{N}(0, \sigma^2 + m^2)$ a.s.

Corollary [MM++]: (take $\Delta = 1$ a.s. in the theorem)

Let $\mathrm{Prof}_n = n^{-1}\sum_{u\in\mathrm{RRT}_n}\delta_{|u|}$, where $|u|$ is the height of u (distance to the root). We have

$$\mathrm{Prof}_n(\sqrt{\log n}\,\cdot\, + \log n) \to \mathcal{N}(0,1) \quad \text{a.s.}$$

[BST – Chauvin, Drmota, Jabbour-Hattab '01] [PORT – Katona '05]

Main results - summary

Theorem [MM++]

Under our set of assumptions,

$$n^{-1}\mathcal{M}_n\big(a_{\log n}\,\cdot\,+b_{\log n}\big)\to\nu\qquad\text{in probability,}$$

where ν is the distribution of $\Gamma g(\Lambda)+f(\Lambda)$, where $\Gamma\sim\gamma$ and $\Lambda\sim\mathcal{N}(0,1)$ are independent.

Theorem [MM++]

If $\mathcal{R}_x$ is the distribution of $x+\Delta$, where Δ is a real random variable with mean m and variance σ^2. If there exists $\delta>0$ such that $\mathbb{E}e^{\delta\Delta}<\infty$, then, almost surely,

$$n^{-1}\mathcal{M}_n(\sqrt{\log n}\,\cdot\,+m\log n)\to\mathcal{N}(0,\sigma^2+m^2).$$

Main results - summary

Theorem [MM++]

Under our set of assumptions,

$$n^{-1}\mathcal{M}_n\big(a_{\log n}\,\cdot\,+b_{\log n}\big) \to \nu \quad \text{in probability},$$

where ν is the distribution of $\Gamma g(\Lambda) + f(\Lambda)$, where $\Gamma \sim \gamma$ and $\Lambda \sim \mathcal{N}(0,1)$ are independent.

Theorem [MM++]

If $\mathcal{R}_x$ is the distribution of $x + \Delta$, where Δ is a real random variable with mean m and variance σ^2. If there exists $\delta > 0$ such that $\mathbb{E}e^{\delta\Delta} < \infty$, then, almost surely,

$$n^{-1}\mathcal{M}_n(\sqrt{\log n}\,\cdot\,+m\log n) \to \mathcal{N}(0,\sigma^2+m^2).$$

Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$,
where ξ_i is the colour drawn at time i.

Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$,
where ξ_i is the colour drawn at time i.

Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

Let us couple the MVPP with a branching Markov chain on the random recursive tree:

Main idea:

Pick a colour ξ_{i+1} according to $\mathcal{M}_i$ is equivalent to

- pick an integer u uniformly at random in $\{0, \dots, i\}$;
- if $u = 0$, pick ξ_{i+1} according to $\mathcal{M}_0$;
- otherwise, pick ξ_{i+1} according to $\mathcal{R}_{\xi_u}$.

Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$,
where ξ_i is the colour drawn at time i.

> Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

Let us couple the MVPP with a branching Markov chain on the random recursive tree:

Main idea:

Pick a colour ξ_{i+1} according to $\mathcal{M}_i$ is equivalent to

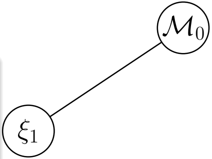
- pick an integer u uniformly at random in $\{0, \ldots, i\}$;
- if $u = 0$, pick ξ_{i+1} according to $\mathcal{M}_0$;
- otherwise, pick ξ_{i+1} according to $\mathcal{R}_{\xi_u}$.

Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$,
where ξ_i is the colour drawn at time i.

Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

Let us couple the MVPP with a branching Markov chain on the random recursive tree:



Main idea:

Pick a colour ξ_{i+1} according to $\mathcal{M}_i$ is equivalent to

- pick an integer u uniformly at random in $\{0, \dots, i\}$;
- if $u = 0$, pick ξ_{i+1} according to $\mathcal{M}_0$;
- otherwise, pick ξ_{i+1} according to $\mathcal{R}_{\xi_u}$.

Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$,
where ξ_i is the colour drawn at time i.

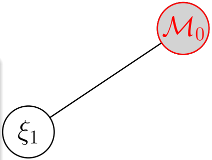
Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

Let us couple the MVPP with a branching Markov chain on the random recursive tree:

Main idea:

Pick a colour ξ_{i+1} according to $\mathcal{M}_i$ is equivalent to

- pick an integer u uniformly at random in $\{0, \dots, i\}$;
- if $u = 0$, pick ξ_{i+1} according to $\mathcal{M}_0$;
- otherwise, pick ξ_{i+1} according to $\mathcal{R}_{\xi_u}$.

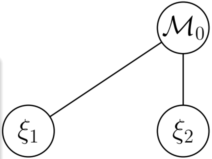


Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$, where ξ_i is the colour drawn at time i.

Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

Let us couple the MVPP with a branching Markov chain on the random recursive tree:



Main idea:

Pick a colour ξ_{i+1} according to $\mathcal{M}_i$ is equivalent to

- pick an integer u uniformly at random in $\{0, \ldots, i\}$;
- if $u = 0$, pick ξ_{i+1} according to $\mathcal{M}_0$;
- otherwise, pick ξ_{i+1} according to $\mathcal{R}_{\xi_u}$.

Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$,
where ξ_i is the colour drawn at time i.

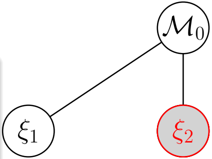
Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

Let us couple the MVPP with a branching Markov chain on the random recursive tree:

Main idea:

Pick a colour ξ_{i+1} according to $\mathcal{M}_i$ is equivalent to

- pick an integer u uniformly at random in $\{0, \ldots, i\}$;
- if $u = 0$, pick ξ_{i+1} according to $\mathcal{M}_0$;
- otherwise, pick ξ_{i+1} according to $\mathcal{R}_{\xi_u}$.



Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$,
where ξ_i is the colour drawn at time i.

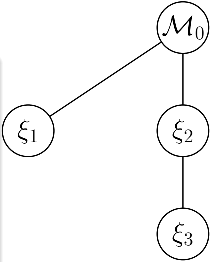
Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

Let us couple the MVPP with a branching Markov chain on the random recursive tree:

Main idea:

Pick a colour ξ_{i+1} according to $\mathcal{M}_i$ is equivalent to

- pick an integer u uniformly at random in $\{0, \dots, i\}$;
- if $u = 0$, pick ξ_{i+1} according to $\mathcal{M}_0$;
- otherwise, pick ξ_{i+1} according to $\mathcal{R}_{\xi_u}$.



Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$,
where ξ_i is the colour drawn at time i.

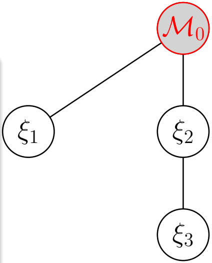
Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

Let us couple the MVPP with a branching Markov chain on the random recursive tree:

Main idea:

Pick a colour ξ_{i+1} according to $\mathcal{M}_i$ is equivalent to

- pick an integer u uniformly at random in $\{0, \dots, i\}$;
- if $u = 0$, pick ξ_{i+1} according to $\mathcal{M}_0$;
- otherwise, pick ξ_{i+1} according to $\mathcal{R}_{\xi_u}$.



Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$,
where ξ_i is the colour drawn at time i.

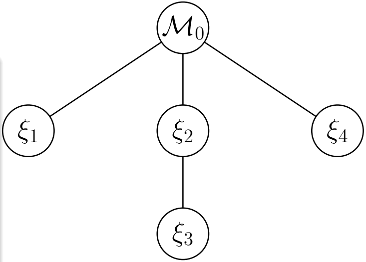
Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

Let us couple the MVPP with a branching Markov chain on the random recursive tree:

Main idea:

Pick a colour ξ_{i+1} according to $\mathcal{M}_i$ is equivalent to

- pick an integer u uniformly at random in $\{0, \dots, i\}$;
- if $u = 0$, pick ξ_{i+1} according to $\mathcal{M}_0$;
- otherwise, pick ξ_{i+1} according to $\mathcal{R}_{\xi_u}$.



Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$,
where ξ_i is the colour drawn at time i.

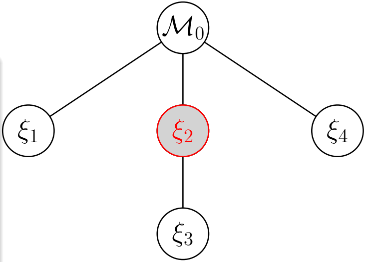
Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

Let us couple the MVPP with a branching Markov chain on the random recursive tree:

Main idea:

Pick a colour ξ_{i+1} according to $\mathcal{M}_i$ is equivalent to

- pick an integer u uniformly at random in $\{0, \dots, i\}$;
- if $u = 0$, pick ξ_{i+1} according to $\mathcal{M}_0$;
- otherwise, pick ξ_{i+1} according to $\mathcal{R}_{\xi_u}$.



Coupling with a branching Markov chain

Note that $\mathcal{M}_n = \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{R}_{\xi_i}$,
where ξ_i is the colour drawn at time i.

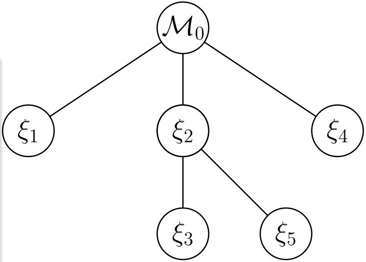
Assume for the proof that $\mathcal{M}_0(\mathcal{P}) = 1$.

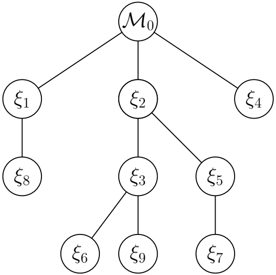
Let us couple the MVPP with a branching Markov chain on the random recursive tree:

Main idea:

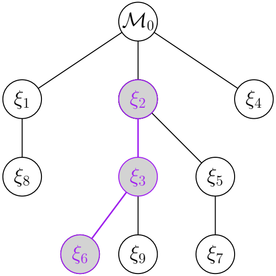
Pick a colour ξ_{i+1} according to $\mathcal{M}_i$ is equivalent to

- pick an integer u uniformly at random in $\{0, \dots, i\}$;
- if $u = 0$, pick ξ_{i+1} according to $\mathcal{M}_0$;
- otherwise, pick ξ_{i+1} according to $\mathcal{R}_{\xi_u}$.

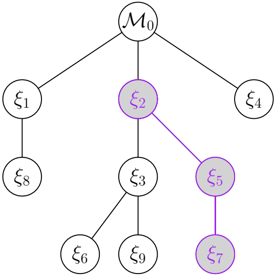




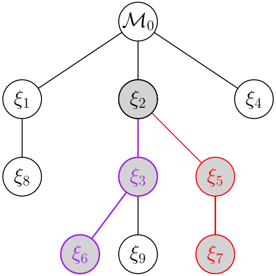
- the underlying tree is the random recursive tree;



- the underlying tree is the random recursive tree;
- the labels are a branching Markov chain of the same Kernel as the companion Markov chain: i.e.
  - the sequence of labels along each branch has the same law as the companion Markov chain (kernel $K(x, \cdot) = \mathcal{R}_x$);



- the underlying tree is the random recursive tree;
- the labels are a branching Markov chain of the same Kernel as the companion Markov chain: i.e.
  - the sequence of labels along each branch has the same law as the companion Markov chain (kernel $K(x, \cdot) = \mathcal{R}_x$);



- the underlying tree is the random recursive tree;
- the labels are a branching Markov chain of the same Kernel as the companion Markov chain: i.e.
  - the sequence of labels along each branch has the same law as the companion Markov chain (kernel $K(x, \cdot) = \mathcal{R}_x$);
  - two distinct branches are independent.

Let $(\mathrm{RRT}_n)_{n\geq 0}$ be the random recursive tree sequence.
Let $(X(u), u \in \mathrm{RRT}_n)_{n\geq 0}$ be the branching Markov chain of initial distribution $\mathcal{M}_0$ and kernel $K(x, \cdot) = \mathcal{R}_x$.

Then, in distribution, $\mathcal{M}_n = \mathcal{M}_0 + \sum_{u \in \mathrm{RRT}_n \setminus \{\varnothing\}} \mathcal{R}_{X(u)}$.

Let $(\mathrm{RRT}_n)_{n\geq 0}$ be the random recursive tree sequence.
Let $(X(u), u \in \mathrm{RRT}_n)_{n\geq 0}$ be the branching Markov chain of initial distribution $\mathcal{M}_0$ and kernel $K(x, \cdot\,) = \mathcal{R}_x$.

Then, in distribution, $\mathcal{M}_n = \mathcal{M}_0 + \sum_{u\in\mathrm{RRT}_n\setminus\{\varnothing\}} \mathcal{R}_{X(u)}$.

In fact, $n^{-1}\mathcal{M}_n = \mathcal{R}_{X(U_n)}$, where U_n is a node chosen uniformly at random in RRT_n. (We set $\mathcal{R}_{X(\varnothing)} = \mathcal{M}_0$.)

It is enough to prove that $\mathcal{R}_{X(U_n)}(a_{\log n}\,\cdot\,+b_{\log n}) \Rightarrow \nu$,
which implies $\mathcal{R}_{X(U_n)}(a_{\log n}\,\cdot\,+b_{\log n}) \to \nu$ in probability

Recall that ν is the (deterministic) law of $\Gamma g(\Lambda) + f(\Lambda)$ where $\Lambda \sim \mathcal{N}(0,1)$ and $\Gamma \sim \gamma$ are independent.

Lemma:

Let $(\mu_n)_{n\geq 0}$ be a sequence of random probability measures.
For all n, take (A_n, B_n) i.i.d. of distribution μ_n.

If $(A_n, B_n) \Rightarrow (A, B)$,
where A and B are i.i.d. with **deterministic** distribution μ, then $\mu_n \Rightarrow \mu$.

Proposition [MM++]

Take U_n and V_n two independent uniform nodes of RRT_n, and denote by K_n the height of their deepest common ancestor, then, when $n \to \infty$,

$$\left(\frac{|U_n| - \log n}{\sqrt{\log n}}, \frac{|V_n| - \log n}{\sqrt{\log n}}, K_n\right) \Rightarrow (\Lambda_1, \Lambda_2, G),$$

where $\Lambda_1, \Lambda_2 \sim \mathcal{N}(0,1)$ and $G \sim \mathrm{Geom}(1/2)$ are independent.

[marginals Dobrow '96 and Kuba & Wagner '10]

Conclusion

We have:

- generalised Pólya urns to infinitely-many (and even **uncountably-many**) colours;
- proved first order convergence in probability;
- proved convergence almost sure in some particular case;
- proved almost sure convergence of the profile of the RRT;
- defined and studied Branching Markov Chains.

Open problems:

- fluctuations around the limit?
- non-balanced urns? (i.e. $\mathcal{R}_x(\mathcal{P}) \neq 1$)
- characterise a.s. convergence?

Conclusion

We have:

- generalised Pólya urns to infinitely-many (and even **uncountably-many**) colours;
- proved first order convergence in probability;
- proved convergence almost sure in some particular case;
- proved almost sure convergence of the profile of the RRT;
- defined and studied Branching Markov Chains.

Open problems:

- fluctuations around the limit?
- non-balanced urns? (i.e. $\mathcal{R}_x(\mathcal{P}) \neq 1$)
- characterise a.s. convergence?

Thanks!!