# Restricted Stirling numbers

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$$\begin{array}{rl} \text{alternating} & \text{if } r \text{ is even} \\ \textit{not} \text{ alternating} & \text{if } r \text{ is odd} \end{array}$$

**Question**: Is there a simple reason for this?

# Matryoshka doll numbers

$\left\{ {n \atop k} \right\} = \#(\text{partitions of } [n] \text{ into } k \text{ non-empty blocks})$

$$\left[ \left\{ {n \atop k} \right\} \right]_{n,k \geq 1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 7 & 6 & 1 & 0 & \cdots \\ 1 & 15 & 25 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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$\left[ {n \atop k} \right] = \#$(partitions of $[n]$ into $k$ non-empty *cycles*)

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Inverse entries (times $(-1)^{n-k}$) are *Bessel numbers*

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**Question**: For $r \geq 3$ is there an interpretation (up to sign) of $(n, k)$ entry of

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- $R = \mathbb{N}$: ordinary Matryoshka doll/Stirling numbers of second kind
- $R = \{1, \ldots, r\}$: Choi, Smith 2005; Choi, Long, Ng, Smith 2006
- $R = \{r, r+1, r+2, \ldots\}$: Comtet 1974

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**Question**: For $R$ with $1 \in R$ is there an interpretation (up to sign) of $(n, k)$ entry of

$$\left[\left\{ n \atop k \right\}_{R}\right]^{-1}_{n,k\geq 1}?$$

# A general setting

$\mathbf{a} = (a_1, a_2, a_3, \ldots)$

$$a_{n,k} = \sum \left\{ a_{|P_1|} \cdots a_{|P_k|} : \text{partitions } (P_1, \ldots, P_k) \text{ of } [n] \right\}$$

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- All $a_i = 1$: Stirling numbers of second kind
- $a_i = (i-1)!$: Stirling numbers of first kind
- $a_i = i!$: Lah numbers
- $a_i = i^{i-2}$: Count of labelled forests on $n$ vertices with $k$ components
- $\mathbf{a} = (1, 1, 1, 1, 0, 0, 0, \ldots)$:
  - $a_{5,1} = \left\{ {5 \atop 1} \right\}_{[4]} = 0 \neq \left\{ {5 \atop 1} \right\} \ (= 1)$
  - $a_{5,2} = \left\{ {5 \atop 2} \right\}_{[4]} = 15 = \left\{ {5 \atop 2} \right\}$

# The exponential formula

**a** determines $[a_{n,k}]_{n,k\geq 1}$ very cleanly:

$$\text{if } f(x) \text{ is egf of } \mathbf{a} \text{ then } \frac{f^k(x)}{k!} \text{ is egf of } (a_{n,k})_{n\geq 1}$$

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Example: $\mathbf{a} = (1, 1, 1, \dots)$

- $f(x) = e^x - 1$
- $\frac{f^2(x)}{2!} = \frac{e^{2x}-2e^x+1}{2} = \sum_{n\geq 1} \frac{2^{n-1}-1}{n!}x^n$
- $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$

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**a** determines $B$ very cleanly:

- $B$ is generated from $B$'s first column exactly as $A$ is generated from **a**
- if $g(x)$ is the egf of first column of $B$, then $g(x)$ is the compositional inverse (reversion) of $f(x)$

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Example: $a = (1, 1, 1, \ldots)$

- $f(x) = e^x - 1$
- $g(x) = \ln(1 + x)$
- $g(x)$ is egf of $(1, -1, 2, -6, 24, \ldots)$
- first column of $B$ is $(1, -1, 2, -6, 24, \ldots)$
- (signed) Stirling numbers of first kind generated by $(1, -1, 2, -6, 24, \ldots)$ exactly as Stirling numbers of second kind generated by $(1, 1, 1, \ldots)$

# Schröder trees (phylogenetic trees)

Rooted, $n$ labelled leaves, all non-leaves have at least two children

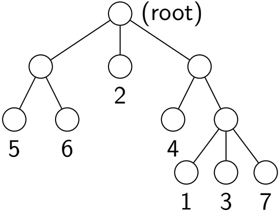


Figure: A Schroder tree

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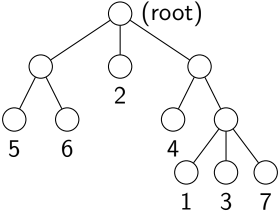


Figure: A Schroder tree with weight $(-1)^4 a_2^2 a_3^2$

Given $\mathbf{a} = (1, a_2, a_3, \ldots)$, weight $w(T)$ of Schröder tree $T$ is

$$w(T) = (-1)^{\#(\text{non-leaves})} \prod_{\text{non-leaves } v} a_{\#(\text{children of } v)}$$

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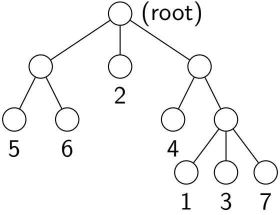


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If $\mathbf{a}$ is 0-1, supported on $R$, then $|w(T)| = \mathbf{1}_{\{\text{all down-degrees of } T \text{ in } R\}}$

# Schröder trees and the reversion

- $\mathbf{a} = (1, a_2, a_3, \ldots)$, $f(x)$ is egf of $\mathbf{a}$
- $g(x)$ the reversion of $f(x)$, egf of $\mathbf{b} = (1, b_2, b_3, \ldots)$

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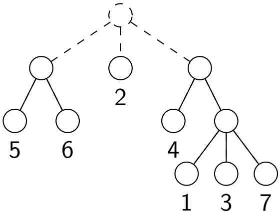
**Proof sketch**:

- Delete root to get collection of smaller Schröder trees, all down-degrees unchanged



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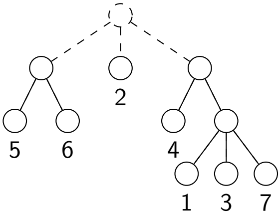


- Implies recurrence for $\sum \{w(T) : \text{Schröder trees } T \text{ on } [n]\}$ that coincides with recurrence for $b_n$

# Combinatorially interpreting inverses

**Corollary**: If **a** is 0-1 with support $R$, $1 \in R$,

$$b_n = \begin{array}{c}\text{number of even } T \text{ [\#(non-leaves) even], all down-degrees in } R \\ - \\ \text{number of odd } T \text{ [\#(non-leaves) odd], all down-degrees in } R\end{array}$$

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$$b_{n,k} = \begin{array}{c} \#(k\text{-comp } R\text{-Schröder forests on } [n], \#(\text{odd components}) \text{ even}) \\ - \\ \#(k\text{-comp } R\text{-Schröder forests on } [n], \#(\text{odd components}) \text{ odd}) \end{array}$$

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Gives combinatorial interpretation of $\left[\left\{ {n \atop k} \right\}_R\right]^{-1}$ for *every* $R$ with $1 \in R$

# Back to $r$-restricted Stirling numbers

$$\left[\left\{ {n \atop k} \right\}_{[3]}\right]_{n,k\geq 1}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & -3 & 1 & 0 & 0 & 0 & \cdots \\ -5 & 11 & -6 & 1 & 0 & 0 & \cdots \\ 10 & -45 & 35 & -10 & 1 & 0 & \cdots \\ \mathbf{35} & 175 & -210 & 85 & -15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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# Alternating first column

If $\mathbf{b} = (1, b_1, b_2, b_3, \ldots)$ is alternating,

$$\begin{aligned} b_{n,k} &:= \sum \left\{ b_{|P_1|} \cdots b_{|P_k|} : \text{partitions } (P_1, \ldots, P_k) \text{ of } [n] \right\} \\ &= (-1)^{n-k} \sum \left\{ \left| b_{|P_1|} \right| \cdots \left| b_{|P_k|} \right| : \text{partitions } (P_1, \ldots, P_k) \text{ of } [n] \right\} \end{aligned}$$

so $(-1)^{n-k} b_{n,k}$ positive for all $n, k$

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If $\mathbf{b} = (1, b_1, b_2, b_3, \ldots)$ is alternating for 0-1 $\mathbf{a}$ with support $R$, $1 \in R$, recalling

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**If** there is injection from

$$\{R\text{-Schröder trees on } [n], \#(\text{non-leaves}) \text{ has parity } x\}$$

to

$$\{\text{same}, \#(\text{non-leaves}) \text{ has parity } 1 - x\}$$

($x$ depending on parity of $n$), **then**

$$(-1)^{n-1} b_n = \#(\text{trees not in range of injection})$$

# Alternating first column

If $\mathbf{b} = (1, b_1, b_2, b_3, \ldots)$ is alternating for 0-1 $\mathbf{a}$ with support $R$, $1 \in R$, recalling

$$b_n = \begin{array}{c} \text{number of even } T \text{ [\#(non-leaves) even], all down-degrees in } R \\ - \\ \text{number of odd } T \text{ [\#(non-leaves) odd], all down-degrees in } R: \end{array}$$

**If** there is injection from

$$\{R\text{-Schröder trees on } [n], \#(\text{non-leaves}) \text{ has parity } x\}$$

to

$$\{\text{same}, \#(\text{non-leaves}) \text{ has parity } 1 - x\}$$

($x$ depending on parity of $n$), **then**

$$\begin{aligned} (-1)^{n-1} b_n &= \#(\text{trees not in range of injection}) \\ (-1)^{n-k} b_{n,k} &= \#(k\text{-component forests of these special trees}) \end{aligned}$$

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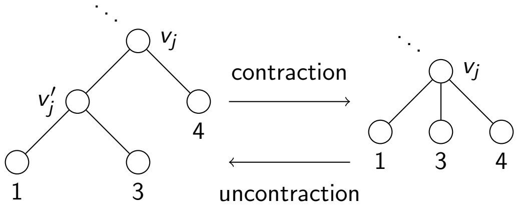
# Main idea for the injection

*Contraction* and *uncontraction*

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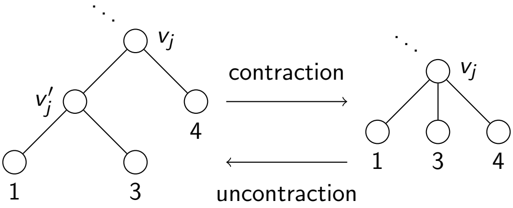


- involution operations that flips parity of number of non-leaves, changes down-degrees predictably
- key is to find "first" non-leaf where operation is possible

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- Can deal with homothetic copies of $R$ with no exposed odds

$$\text{e.g. } \{1,4,7,10\}, \quad \{1,7,9,11\}$$

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**Research question**

Characterize those $R \subseteq \mathbb{N}$ with $1 \in R$ such that

$$\text{series reversion of } \sum_{n \in R} \frac{x^n}{n!} \text{ is alternating}$$