Korn type inequalities in Orlicz spaces

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2. D.Breit, A.C., Negative Orlicz-Sobolev norms and strongly nonlinear systems in fluid dynamics, J. Diff. Equat. 2015
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Namely,

$$\mathcal{E}u = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T),$$

where $(\nabla \mathbf{u})^T$ is the transpose of $\nabla \mathbf{u}$.

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When $p = 1$, instead of $E^1(\Omega, \mathbb{R}^n)$, the space

$$BD(\Omega, \mathbb{R}^n) = \{\mathbf{u} : \mathcal{E}\mathbf{u} \text{ is a Radon measure with finite total variation } in\Omega\}$$

is also of use in applications.

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Modern proofs, for general p, are due to Gobert, Nečas, Reshetnyak, Mosolov-Mjasnikov, Temam, Fuchs.

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The Korn inequality in $E^p_0(\Omega, \mathbb{R}^n)$ ensures that, in the case of functions vanishing on the boundary, such a matrix vanishes.

If Ω is connected, the kernel of the operator $\mathcal{E}$ is

$$\mathcal{R} = \{\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n : \mathbf{v}(x) = \mathbf{b} + \mathbf{Q}x \quad \text{for some } \mathbf{b} \in \mathbb{R}^n \text{ and } \mathbf{Q} \in \mathbb{R}^{n\times n} \text{ s.t. } \mathbf{Q} = -\mathbf{Q}^T\}.$$

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Namely, it can be rewritten as

$$\inf_{\mathbf{v}\in\mathcal{R}} \int_\Omega |\nabla \mathbf{u} - \nabla \mathbf{v}|^p \, dx \le C \int_\Omega |\mathcal{E}\mathbf{u}|^p \, dx \quad \forall\, \mathbf{u} \in E^p(\Omega, \mathbb{R}^n).$$

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A function A enjoying these properties is called a Young function in the literature.
The spaces obtained by replacing the power t^p in the definition of L^p with a Young function $A(t)$ are called Orlicz spaces.

In fact, the inequality

$$\int_\Omega A\big(|\nabla \mathbf{u}|\big)\, dx \le \int_\Omega A\big(C|\mathcal{E}\mathbf{u}|\big)\, dx$$

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A parallel result holds for functions with arbitrary boundary values.

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Pb.: Orlicz version of the Korn inequality, without Δ_2 and ∇_2 conditions, but possibly slightly different Young functions on the two sides.

Namely, inequalities of the form:

$$\int_\Omega B\big(|\nabla \mathbf{u}|\big)\,dx \le \int_\Omega A\big(C|\mathcal{E}\mathbf{u}|\big)\,dx$$

for $u = 0$ on $\partial\Omega$, where A and B are Young functions satisfying suitable "balance" conditions.

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Similarly,

$$\inf_{\mathbf{v}\in\mathcal{R}} \int_{\Omega} B(|\nabla \mathbf{u} - \nabla \mathbf{v}|)\,dx \le \int_{\Omega} A(C|\mathcal{E}\mathbf{u}|)\,dx\,,$$

for arbitrary u, where $\mathcal{R}$ is the kernel of the operator $\mathcal{E}$.

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$$\mathcal{E}^D\mathbf{u} = \mathcal{E}\mathbf{u} - \frac{\mathrm{tr}(\mathcal{E}\mathbf{u})}{n} I,$$

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L^p inequalities between $\mathcal{E}^D\mathbf{u}$ and $\nabla\mathbf{u}$ are known for $p \in (1, \infty)$.

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If Ω is bounded in $\mathbb{R}^n$, and $1 < p < \infty$, then

$$\int_\Omega |\nabla \mathbf{u}|^p \, dx \le C \int_\Omega |\mathcal{E}^D \mathbf{u}|^p \, dx$$

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In particular, it agrees with the whole space of holomorphic functions when $n = 2$. The inequalities in question require a distinct approach for $n = 2$ and for $n \ge 3$.

We focus on the case $n \geq 3$.

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$$\Sigma = \mathcal{D} \oplus \mathcal{R} \oplus \mathcal{S},$$

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If Ω has the cone property , then

$$\inf_{\mathbf{w}\in\Sigma} \int_\Omega |\nabla \mathbf{u} - \nabla \mathbf{w}|^p \, dx \leq C \int_\Omega |\mathcal{E}^D \mathbf{u}|^p \, dx.$$

Similarly to the Korn inequalities, trace-free Korn inequalities in Orlicz spaces, with t^p replaced with a Young function $A(t)$, hold if and only if $A \in \Delta_2 \cap \nabla_2$ [Bildhauer & Fuchs, 2011], [Breit & Schirra, 2012] (if), [Breit & Diening, 2012] (only if).

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with $u = 0$ on $\partial\Omega$, or for

$$\inf_{\mathbf{w}\in\Sigma} \int_\Omega B(|\nabla \mathbf{u} - \nabla \mathbf{w}|)\,dx \leq \int_\Omega A(C|\mathcal{E}^D \mathbf{u}|)\,dx\,,$$

for arbitrary u.

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Recall that the Orlicz space $L^A(\Omega, \mathbb{R}^n)$ built on the Young function A is endowed with the Luxemburg norm given by

$$\|\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \int_\Omega A\Big(\frac{|\mathbf{u}(x)|}{\lambda}\Big)\, dx \leq 1\right\}.$$

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In statements, $\widetilde{A}$ denotes the Young conjugate of A, defined as

$$\widetilde{A}(t) = \sup\{rt - A(r) : r \geq 0\} \qquad \text{for} \qquad t \geq 0 .$$

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In particular, we recover that the Korn inequalities hold with the same Young function A on both sides if and only if $A \in \Delta_2 \cap \nabla_2$.

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(iii) If Ω is regular, $\exists\, C > 0$ s.t.

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He showed that, if $1 < p < \infty$, then the $L^p(\Omega)$ norm of any function with zero mean-value over Ω is equivalent to the $W^{-1,p}(\Omega, \mathbb{R}^n)$ norm of its gradient.
Namely,

$$\frac{1}{C}\|u - u_\Omega\|_{L^p(\Omega)} \le \|\nabla u\|_{W^{-1,p}(\Omega,\mathbb{R}^n)} \le C\|u - u_\Omega\|_{L^p(\Omega)}.$$

The negative Orlicz-Sobolev norm can be defined accordingly as

$$\|\nabla u\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} = \sup_{\varphi \in C_0^\infty(\Omega,\mathbb{R}^n)} \frac{\int_\Omega u \operatorname{div} \varphi \, dx}{\|\nabla \varphi\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}.$$

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However, it can be restored if and only if A is replaced on the right-hand side by another Young function B related to A as in the Korn inequality [Breit & C., 2015] (if) and [Breit, C. & Diening, preprint] (only if).

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if and only if $\exists\, C > 0$ and $t_0 \geq 0$ s.t.

$$t\int_{t_0}^{t} \frac{B(s)}{s^2}\, ds \leq A(ct), \quad \text{and} \quad t\int_{t_0}^{t} \frac{\widetilde{A}(s)}{s^2}\, ds \leq \widetilde{B}(ct) \qquad \forall\, t \geq t_0.$$

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Since

$$\frac{\partial^2 v_i}{\partial x_k \partial x_j} = \frac{\partial \mathcal{E}_{ij}\mathbf{v}}{\partial x_k} + \frac{\partial \mathcal{E}_{ik}\mathbf{v}}{\partial x_j} - \frac{\partial \mathcal{E}_{jk}\mathbf{v}}{\partial x_i},$$

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for $\mathbf{v} : \Omega \to \mathbb{R}^n$, by the negative norm inequality applied to $\nabla \mathbf{u}$ we have

$$\begin{aligned}\|\nabla \mathbf{u} - (\nabla \mathbf{u})_\Omega\|_{L^B(\Omega,\mathbb{R}^{n\times n})} &\le C\|\nabla^2 \mathbf{u}\|_{W^{-1,A}(\Omega,\mathbb{R}^{n\times n})} \\ &\le C^{'}\|\nabla(\mathcal{E}\mathbf{u})\|_{W^{-1,A}(\Omega,\mathbb{R}^{n\times n})} \\ &\le C^{''}\|\mathcal{E}\mathbf{u} - (\mathcal{E}\mathbf{u})_\Omega\|_{L^A(\Omega,\mathbb{R}^{n\times n})}\,.\end{aligned}$$

In particular, if $\mathbf{u} = 0$ on $\partial\Omega$, then $(\nabla \mathbf{u})_\Omega = (\mathcal{E}\mathbf{u})_\Omega = 0$, In particular, if $\mathbf{u} = 0$ on $\partial\Omega$, then $(\nabla \mathbf{u})_\Omega = (\mathcal{E}\mathbf{u})_\Omega = 0$, and the above inequality yields

$$\|\nabla \mathbf{u}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} ,$$

namely the Korn inequality in $E^A_0(\Omega, \mathbb{R}^n)$.

An application to nonlinear systems in fluid mechanics.

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A simplified mathematical model for the stationary flow a homogeneous incompressible fluid in a bounded domain $\Omega \subset \mathbb{R}^n$ has the form

$$\begin{cases} -\operatorname{div} \mathbf{S}(\mathcal{E}\mathbf{v}) + \nabla \pi = \varrho \operatorname{div} \mathbf{F} & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega. \end{cases}$$

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In general, π belongs to some larger Orlicz space $L^B(\Omega)$.

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Theorem 5: Orlicz estimates for π

Let Ω be a bounded domain with the cone property in $\mathbb{R}^n$, $n \geq 2$. Let A and B be Young functions s.t.

$$t\int_{t_0}^{t} \frac{B(s)}{s^2}\, ds \leq A(ct), \quad \text{and} \quad t\int_{t_0}^{t} \frac{\widetilde{A}(s)}{s^2}\, ds \leq \widetilde{B}(ct) \qquad \forall\, t \geq t_0.$$

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Assume that $\mathbf{H} \in L^A(\Omega, \mathbb{R}^{n\times n})$ and satisfies

$$\int_\Omega \mathbf{H} : \nabla\boldsymbol{\varphi}\,dx = 0 \quad \forall\, \boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(\Omega, \mathbb{R}^n).$$

Then $\exists!\ \pi \in L^B_\perp(\Omega)$ s.t.

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$$\int_\Omega B(|\pi|)\, dx \leq \int_\Omega A(C|\mathbf{H} - \mathbf{H}_\Omega|)\, dx.$$

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If Ω is starshaped with respect to a ball, and ω is a smooth, nonnegative function, compactly supported in such ball and with integral equal to 1, the Bogovskii operator $\mathcal{B}$ is defined as

$$\mathcal{B}f(x) = \int_\Omega f(y)\left(\frac{x-y}{|x-y|^n}\int_{|x-y|}^\infty \omega\Big(y+r\frac{x-y}{|x-y|}\Big)\zeta^{n-1}\,dr\right)dy \quad \text{for } x\in\Omega,$$

for $f \in C^\infty_{0,\perp}(\Omega)$.

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for $f \in C^\infty_{0,\perp}(\Omega)$.
This operator is often used to construct a solution to the divergence equation, coupled with zero boundary conditions, since

$$\mathrm{div}\mathcal{B}f = f.$$

The necessary and sufficient conditions on A and B for

$$\nabla\mathcal{B} : L^A(\Omega) \to L^B(\Omega)$$

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- Sequences of trial functions converging to laminates for the condition $t \int_{t_0}^{t} \frac{B(s)}{s^2}\, ds \leq A(ct)$.