# On the quantitative isoperimetric inequality in the plane

Chiara Bianchini

**Dipartimento di Matematica e Informatica U. Dini,**
Università degli Studi di Firenze

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joint work with Gisella Croce and Antoine Henrot

# From the classical isoperimetric inequality to the quantitative isoperimetric inequality

**Planar isoperimetric inequality:** Let $\Omega \subset \mathbb{R}^2$, $B$ be a ball s.t. $|B| = |\Omega|$ $\rightsquigarrow P(\Omega) \geq P(B)$, and equality holds iff $\Omega$ is a ball.

We are interested in a quantitative version:
if $P(\Omega) \approx P(B)$, can we say that $\Omega$ is "almost" a ball?

$$\rightsquigarrow \text{Define: } \delta(\Omega) = \frac{P(\Omega)}{P(B)} - 1 \text{ the isoperimetric deficit of } \Omega.$$

$\rightsquigarrow$if $\delta(\Omega)$ is small, can we say that $\Omega$ is "near to be" a ball?
Can we find $C > 0$, $\alpha$ s.t. $\lambda(\Omega) \leq C\ P^{\alpha}(\Omega)$ where $\lambda$ measures the asymmetry of $\Omega$?

Which kind of distance suitably measures how close $\Omega$ is to a ball?

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# The Fraenkel Asymmetry

Notice: $\lambda(\cdot) = d_H(\cdot\,;B_x)$, the Hausdorff distance:
with general non-convex sets we cannot expect $\delta$ to control $d_H(\cdot\,;B_x)$

$\rightsquigarrow$we consider the Fraenkel asymmetry:

$$\lambda(\Omega) = \min_{x\in\mathbb{R}^2}\left\{\frac{|\Omega\Delta B_x|}{|B_x|}\;:\;|B_x| = |\Omega|\right\}$$

Notice: $\lambda(\Omega) = 0$ iff $\Omega = B_o$; $\quad \lambda(\cdot) \leq 2$

$\rightsquigarrow$Problem: how to find *an* optimal ball $B_y$?

We will investigate this later...

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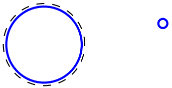

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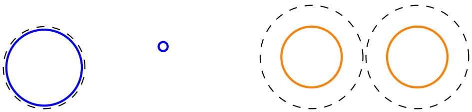

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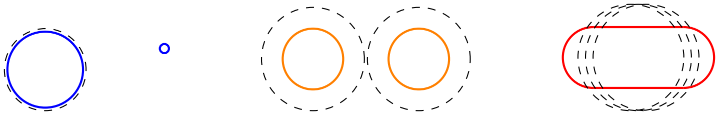

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**Theorem:**[N. Fusco, F. Maggi, A. Pratelli '08] There exists a constant $C_N$ s.t.

$$\lambda(\Omega) \leq \widetilde{C_N} \sqrt{\delta(\Omega)},$$

that is

$$inf_{\Omega \subset \mathbb{R}^N} \frac{\delta(\Omega)}{\lambda^2(\Omega)} \geq C_N.$$

Litterature: Bonnesen 1924 (planar case), Fuglede 1989 (nearly-spherical sets), Hall-Hayman-Weitsman 1991, Hall 1992 ($\alpha = 1/4$ axisymmetric sets), Fusco-Maggi-Pratelli 2008 (symmetrization techniques), Figalli-Maggi-Pratelli 2010 (mass transportation), Cicalese-Leonardi 2012 (selection principle), Fusco-Gelli-Pisante 2012 (Hausdorff distance)...

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# The best constant $C_2$ (i)

**Theorem:**[S. Campi '92],[A. Alvino, V. Ferone, C. Nitch '11] $[N=2]$ A particular stadium $D$ minimizes $\delta/\lambda^2$ among convex sets, that is

$$\inf_{\Omega \text{convex} \neq B} \frac{\delta(\Omega)}{\lambda^2(\Omega)} = \frac{\delta(D)}{\lambda^2(D)} \approx 0,406.$$

**Conjecture:**[M. Cicalese, G. Leonardi '12],[CB, G. Croce, A. Henrot '16] $[N=2]$ A particular peanut $D_0$ minimizes $\delta/\lambda^2$, that is

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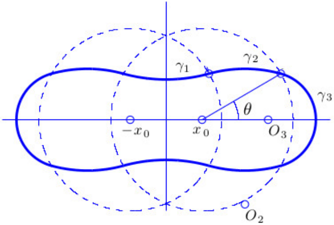

# The best constant $C_2$ (ii)

**Problem:** minimize the shape functional $\mathcal{F}(\cdot)$ among planar sets $\Omega \neq B$:

$$\mathcal{F}(\Omega) = \frac{\delta(\Omega)}{\lambda^2(\Omega)}.$$

**Theorem.** There exists a set $\Omega_0 \neq B$ s.t. $\min_{\Omega \subset \mathbb{R}^2} \mathcal{F}(\Omega) = \mathcal{F}(\Omega_0)$.

- $\Omega_0$ is not convex;
- $\partial\Omega_0$ is $C^{1,1}$;
- $\Omega_0$ has at least two optimal balls for the Fraenkel asymmetry;
- $\Omega_0$ has at most six connected components.

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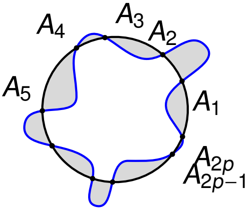
# Location of an optimal ball (for $\lambda(\Omega)$) (i)

In general, it is not easy to locate an optimal ball!
However, $B$ must satisfy some geometric conditions




**Theorem.**[BCH] Let $\Omega$ be a transversal set to an optimal ball $B$ $\rightsquigarrow$ the intersection points $A_i \equiv (x_i, y_i), i \in \{1, ..., 2p\}$ of $\partial\Omega \cap \partial B$ satisfy

$$x_1 + x_3 + ... + x_{2p-1} - (x_2 + x_4 + ... + x_{2p}) = 0,$$
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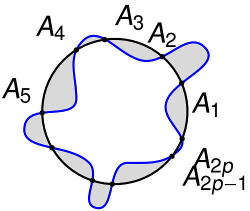


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# Location of an optimal ball (ii): symmetric case

**Proposition.**[BCH] Let $\Omega \subset \mathbb{R}^2$ be $\Pi$-axis symmetric, $\Omega$ is convex in the direction $\Pi^\perp \rightsquigarrow \exists$ an optimal ball centered on $\Pi$.

**Corollary.**[BCH] Assume $\Omega \subset \mathbb{R}^2$ has two (perpendicular) axis of symmetry crossing at $O$, $\Omega$ convex in both directions
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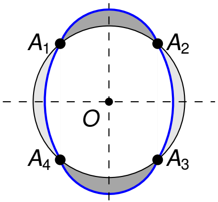

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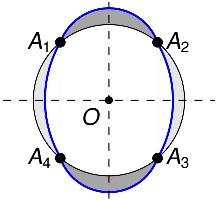
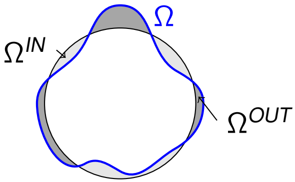


# Existence of a minimizer: a new proof of the quantitative isoperimetric inequality (i)

Let $\Omega_n$ be a minimizing sequence for $\min \mathcal{F}$.
$\rightsquigarrow$ Aim: $\Omega_n \to \Omega_0$, $\Omega_0 \neq B$.
[by contradiction!] We perform a rearrangement on $\Omega_n$:




Notice: $\rightsquigarrow \Omega^*$ is well defined if $\lambda(\Omega)$ is small!
$\rightsquigarrow$ the rearrangement (asymptotically) decreases $\mathcal{F}$: $\forall \alpha > 0$, $\exists \beta$ s.t. $\lambda(\Omega) < \beta$ implies $\mathcal{F}(\Omega^*) < \mathcal{F}(\Omega) + \alpha$.
$\rightsquigarrow$ $\liminf \mathcal{F}(\Omega_n^*) = \frac{\pi}{8(4-\pi)} \approx 0,457 > \mathcal{F}(D) = 0,406$.

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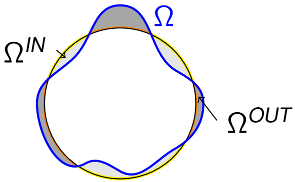


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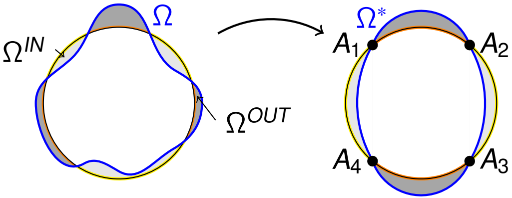


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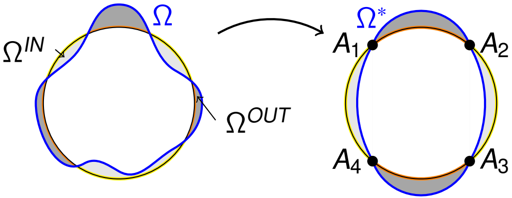


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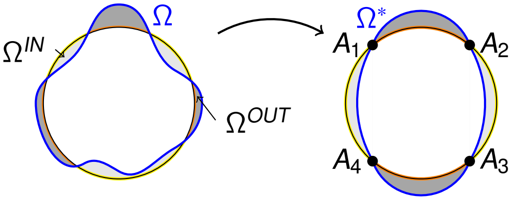


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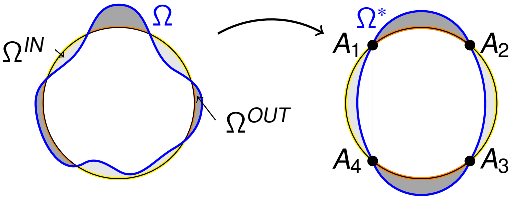


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# A minimizing sequence does not converge to a ball

Aim: let $\Omega_\varepsilon$ be sequence s.t. $\Omega_\varepsilon = \pi$ and $|\Omega_\varepsilon \Delta B| = 4\varepsilon/\pi$, then $\liminf \mathcal{F}(\Omega_\varepsilon^*) \geq \dfrac{\pi}{8(4-\pi)}$.

4 different cases: $i = 1, 2$

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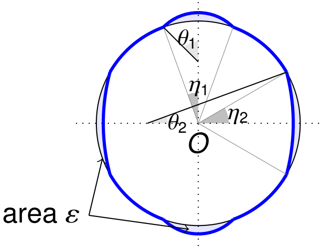


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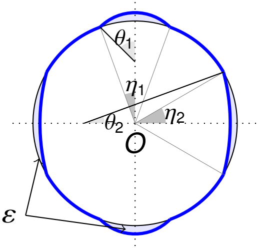


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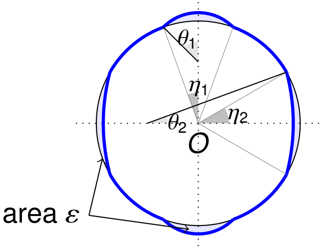


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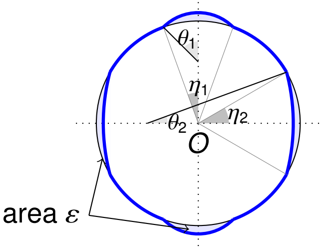


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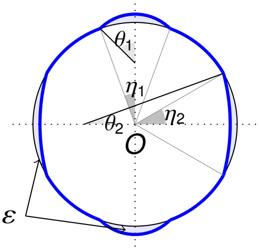


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We have seen: a minimizing sequence cannot converge to a ball.

But: **does a minimizing sequence converge?** YES! indeed...

$\lambda(\Omega_n) \leq 2$ , $\delta(\Omega_n)/\lambda^2(\Omega_n) \to M \leq \mathcal{F}(D) = 0.41 \rightsquigarrow P(\Omega_n) \leq 16.6$.

- [BCH] $P(\Omega) < 20 \implies \exists \widetilde{\Omega}$ composed by at most 7 connected component s.t. $\mathcal{F}(\widetilde{\Omega}) \leq \mathcal{F}(\Omega)$.

$\Omega$ has at most 4 components $\not\subset B_1$ $\rightsquigarrow$ we can replace all other by balls $\rightsquigarrow$ minimization problem involving the radii: the minimizer is achieved by 2 or 3 balls.

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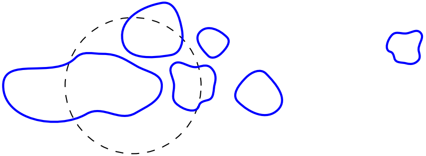

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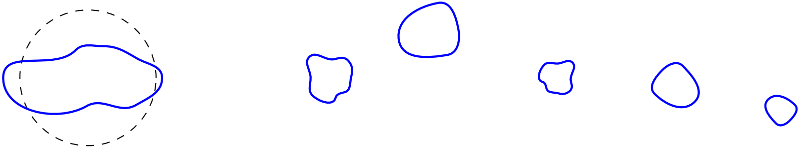

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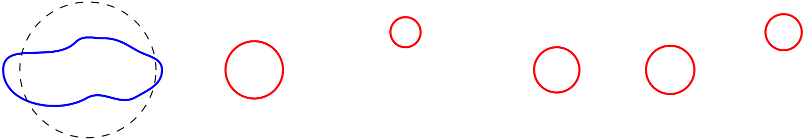

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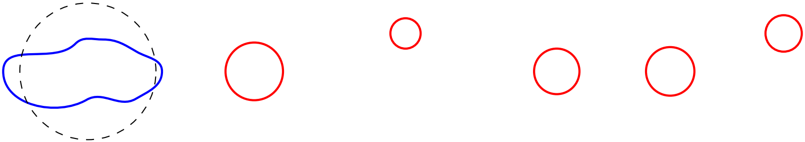

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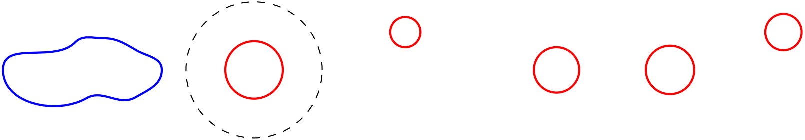

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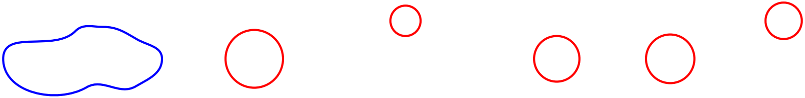

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**Thm.**[BCH] $\Omega_0$ has at most 6 connected components.

Indeed: look at the previous proof for the optimal domain $\Omega_0$:
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- hence $\Omega_0$ has at most 6 connected components. □

# Number of optimal balls

**Thm.**[BCH] $\Omega_0$ has at least 2 optimal balls for $\lambda(\cdot)$

Indeed: [by contradiction!] assume there is only one optimal ball.

⇝non-connected case: $\Omega_0 = E \cup B_r$.

⇝connected case: $\partial\Omega_0 = \cup C_i$: N copies of arcs of circle.

Considering all possible values for the parameters $\alpha, \theta, N$ we show that we always get a contradiction with one of the following facts:

- $\mathcal{F}(\Omega_0) < 0.4055$
- the first order optimality condition: $\frac{1}{R_1} + \frac{1}{R_2} = \frac{8\delta}{\lambda}$
- the second order optimality condition.

□

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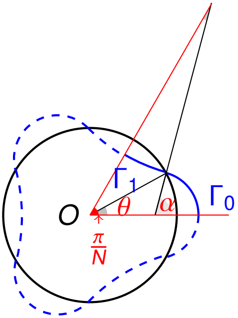


[case $N = 3$]

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# Open problems to determine $\Omega_0$ and hence $C_2 = \mathcal{F}(\Omega_0)$

Conjecture:

- $\Omega_0$ is connected;
- $\Omega_0$ has two orthogonal axis of symmetry;
- $\Omega_0$ has exactly 2 optimal balls.

⇝ $\partial\Omega_0$ can be parametrized by 8 arcs of circles:

⇝ the candidates are peanut shaped! (or masks)

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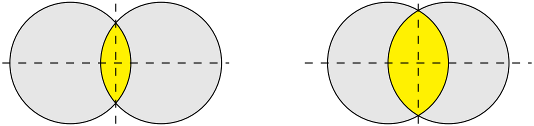

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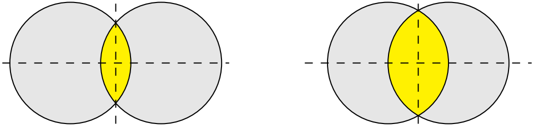

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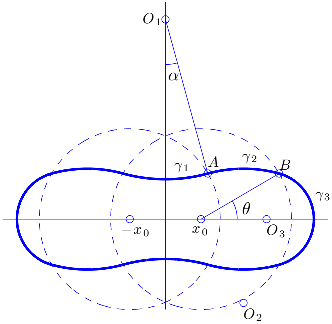
# Conjecture on the optimal domain $\Omega_0$

By solving the two-dimensional minimization problem, we get:

**Conjecture:** $\Omega_0$ is a "peanut" with $\alpha = 0.2686247$, $\theta = 0.5285017$, $x_0 = 0.3940769$. The value of $\mathcal{F}$ for the set $\Omega_0$ is

$$\mathcal{F}(\Omega_0) = C_2 = 0.39314,$$

so that $\widetilde{C_2} = 2.543625$.



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# Upcoming events:

- Workshop on **Partial Differential Equations and related topics**, Alghero (Italy), Septembre 2016.
  `www.dma.unina.it/ferone/alghero2016/index.html`

- CIME summer school on
  **Geometry of PDE's and related problems**
  Courses by: X. Cabré, A. Henrot, D. Peralta-Salas, W. Reichel, H. Shahgholian. Cetraro (Italy), June 2017.