FACTORIAL CHARACTERS AND TOKUYAMA'S IDENTITY FOR CLASSICAL GROUPS

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ONE IDEA—TWO WAYS

1	1	1	$2'$	$3'$	3	4	4	4
	2	2	2	$3'$	$4'$	$5'$	5	5
		3	$4'$	4	4	5	6	
			4	$5'$	5	$6'$		
				5	$6'$	6		
					6			

9 8 6 4 3 1
9 8 5 3 1
9 5 4 1
6 4 1
4 3
3

SCHUR FUNCTIONS

Ex: $n = 3,\ x = (x_1, x_2, x_3),\ \lambda = (2, 1)$

The corresponding semistandard Young tableaux, T, and their weights, $\mathrm{wgt}(T)$, are given by:

1 1	1 2	1 1	1 2	2 2	1 3	1 3	2 3
2	2	3	3	3	2	3	3
$x_1^2x_2$	$x_1x_2^2$	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_2x_3$	$x_1x_3^2$	$x_2x_3^2$

$$s_{(2,1)}(x) = x_1^2x_2 + x_2^2x_1 + x_1^2x_3 + x_3^2x_1 + x_2^2x_3 + x_3^2x_2 + 2\,x_1x_2x_3$$

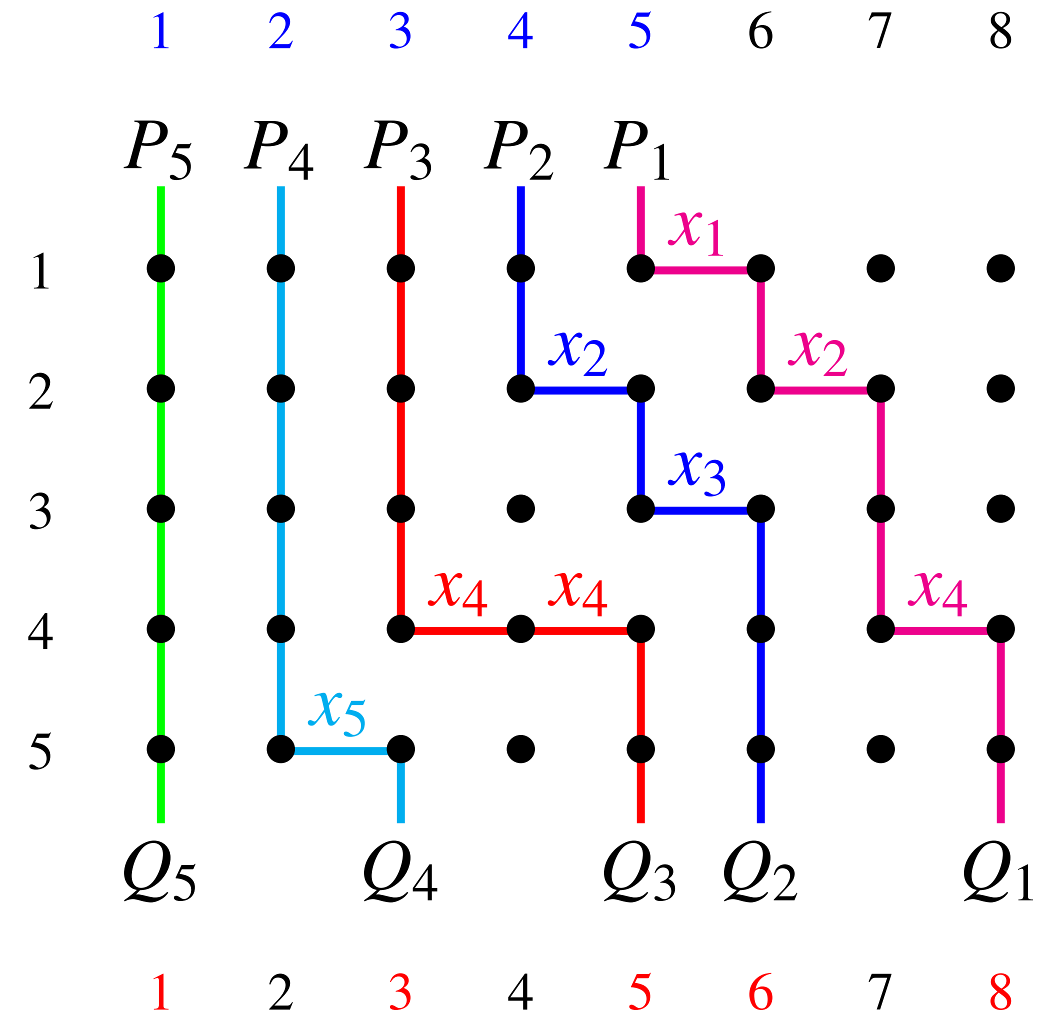
SCHUR FUNCTIONS AND LATTICE PATHS

Ex: $n = 5,\ \lambda = (3, 2, 2, 1)$.

- Starting points $P_i = (0, n - i + 1)$ for $i = 1, 2, \ldots, 5$.
- End points $Q_j = (n + 1, n - j + 1 + \lambda_j)$ for $j = 1, 2, \ldots, 5$.

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

$$\mathrm{wgt}(T) = x_1\, x_2^2\, x_3\, x_4^3\, x_5$$



FACTORIAL SCHUR FUNCTIONS AND WEIGHTED TABLEAUX

1	$\overline{1}$	2	$\overline{4}$
$\overline{3}$	4	4	
4	$\overline{4}$	$\overline{4}$	

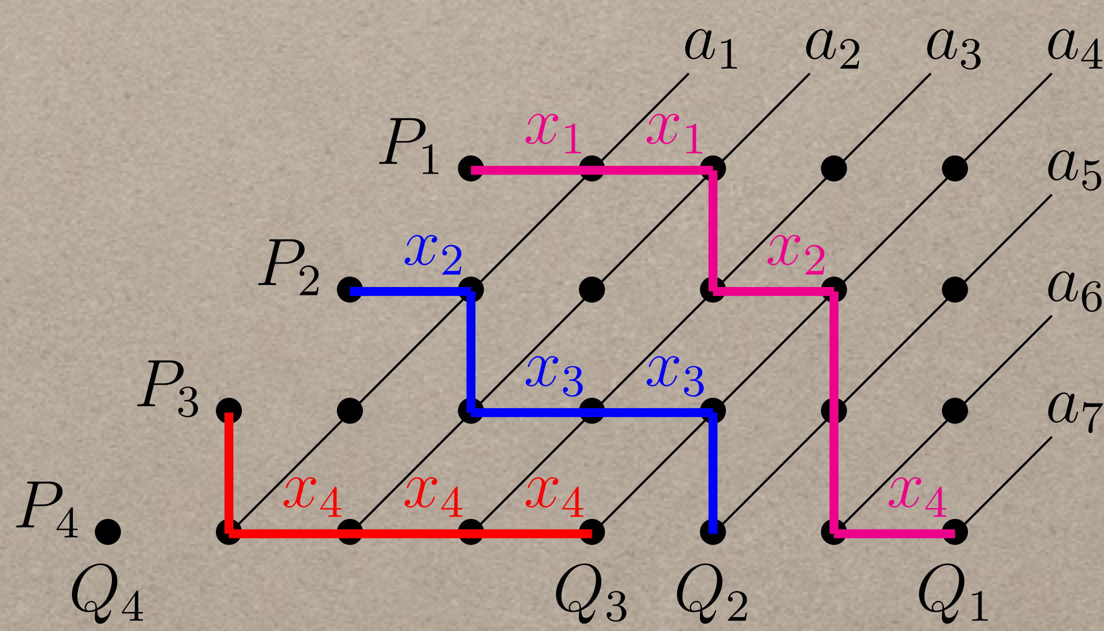
$\Rightarrow$

x_1	$\overline{x}_1$	$x_2 + a_1$	$x_{\overline{4}} + a_7$
$x_{\overline{3}} + a_1$	$x_4 + a_3$	$x_4 + a_4$	
$x_4 + a_1$	$x_{\overline{4}} + a_3$	$x_{\overline{4}} + a_4$	

- Entries weakly increase in rows.
- Entries strictly increase in columns.
- Weight each entry k in position i, j by $x_k + a_{k+j-i}$.

SCHUR FUNCTIONS AND LATTICE PATHS

$LP(T) =$



$T =$

1	1	2	4
2	3	3	
4	4	4	

$\mathrm{wgt}(T) =$

x_1+a_1	x_1+a_2	x_2+a_4	x_4+a_7
x_2+a_1	x_3+a_3	x_3+a_4	
x_4+a_2	x_4+a_3	x_4+a_4	

ORIGIN OF FACTORIAL SCHUR FUNCTIONS

- Biedenharn and Louck (1989) introduced the notion of the factorial Schur function.
- Chen and Louck (1993) further studied them.
- Macdonald (1992) and Goulden and Greene (1994) independently gave them a more general form, in the process making connections to supersymmetric functions.
- Macdonald also gave an alternate definition as a ratio of alternants. It is this definition that we now explore….

RATIO OF ALTERNANTS

Macdonald defined

$$s_\lambda(\mathbf{x}\,|\,\mathbf{a}) = \frac{\left|\,(x_i\,|\,\mathbf{a})^{\lambda_j+n-j}\,\right|}{\left|\,(x_i\,|\,\mathbf{a})^{n-j}\,\right|}$$

where

1

$$(x\,|\,\mathbf{a})^m = \begin{cases} (x+a_1)(x+a_2)\cdots(x+a_m) & \text{if } m>0; \\ 1 & \text{if } m=0 \end{cases}$$

If $a_i = -i+1$ this reduces to the falling factorial:

$(x)_i = x(x-1)\ldots(x-k+1)$.

FACTORIAL CHARACTERS FOR CLASSICAL GROUPS

$$s_\lambda(\mathbf{x}\,|\,\mathbf{a}) = \frac{\left|\,(x_i\,|\,\mathbf{a})^{\lambda_j+n-j}\,\right|}{\left|\,(x_i\,|\,\mathbf{a})^{n-j}\,\right|}\,;$$

$$so_\lambda(\mathbf{x},\overline{\mathbf{x}},1\,|\,\mathbf{a}) = \frac{\left|\,x_i^{1/2}(x_i\,|\,\mathbf{a})^{\lambda_j+n-j} - x_i^{-1/2}(x_i^{-1}\,|\,\mathbf{a})^{\lambda_j+n-j}\,\right|}{\left|\,x_i^{1/2}(x_i\,|\,\mathbf{a})^{n-j} - x_i^{-1/2}(x_i^{-1}\,|\,\mathbf{a})^{n-j}\,\right|}\,;$$

$$sp_\lambda(\mathbf{x},\overline{\mathbf{x}}\,|\,\mathbf{a}) = \frac{\left|\,x_i(x_i\,|\,\mathbf{a})^{\lambda_j+n-j} - x_i^{-1}(x_i^{-1}\,|\,\mathbf{a})^{\lambda_j+n-j}\,\right|}{\left|\,x_i(x_i\,|\,\mathbf{a})^{n-j} - x_i^{-1}(x_i^{-1}\,|\,\mathbf{a})^{n-j}\,\right|}\,;$$

$$o_\lambda(\mathbf{x},\overline{\mathbf{x}}\,|\,\mathbf{a}) = \frac{\eta\left|\,(x_i\,|\,\mathbf{a})^{\lambda_j+n-j} + (x_i^{-1}\,|\,\mathbf{a})^{\lambda_j+n-j}\,\right|}{\frac{1}{2}\left|\,(x_i\,|\,\mathbf{a})^{n-j} + (x_i^{-1}\,|\,\mathbf{a})^{n-j}\,\right|} \quad \textit{with} \quad \eta = \begin{cases} \frac{1}{2} & \textit{if } \lambda_n = 0; \\ 1 & \lambda_n > 0. \end{cases}$$

We propose these definitions as the most natural extension of the classical characters to the factorial case.

- Now that we have these factorial characters for classical groups, let’s test their properties.
- But what properties….?
- Jacobi-Trudi! Tokuyama!
- What’s Jacobi-Trudi?

CLASSICAL JACOBI-TRUDI

$$s_\lambda(x) \quad = \quad \left| \, h_{\lambda_j - j + i}(x) \, \right|_{1 \le i,j \le n}$$

- Easily proved algebraically….
- Equally easily proved combinatorially…

COMPLETE SYMMETRIC FUNCTIONS

- Recall that the complete symmetric functions may be defined as:

Also the same as a Schur function for a single row of length m.

$$h_m(\mathbf{x}) = [t^m] \prod_{i=1}^{n} \frac{1}{1 - tx_i}\,;$$

$$h_m^{so}(\mathbf{x}, \overline{\mathbf{x}}, 1) = [t^m]\ (1+t) \prod_{i=1}^{n} \frac{1}{(1 - tx_i)(1 - tx_i^{-1})}\,;$$

$$h_m^{sp}(\mathbf{x}, \overline{\mathbf{x}}) = [t^m] \prod_{i=1}^{n} \frac{1}{(1 - tx_i)(1 - tx_i^{-1})}\,;$$

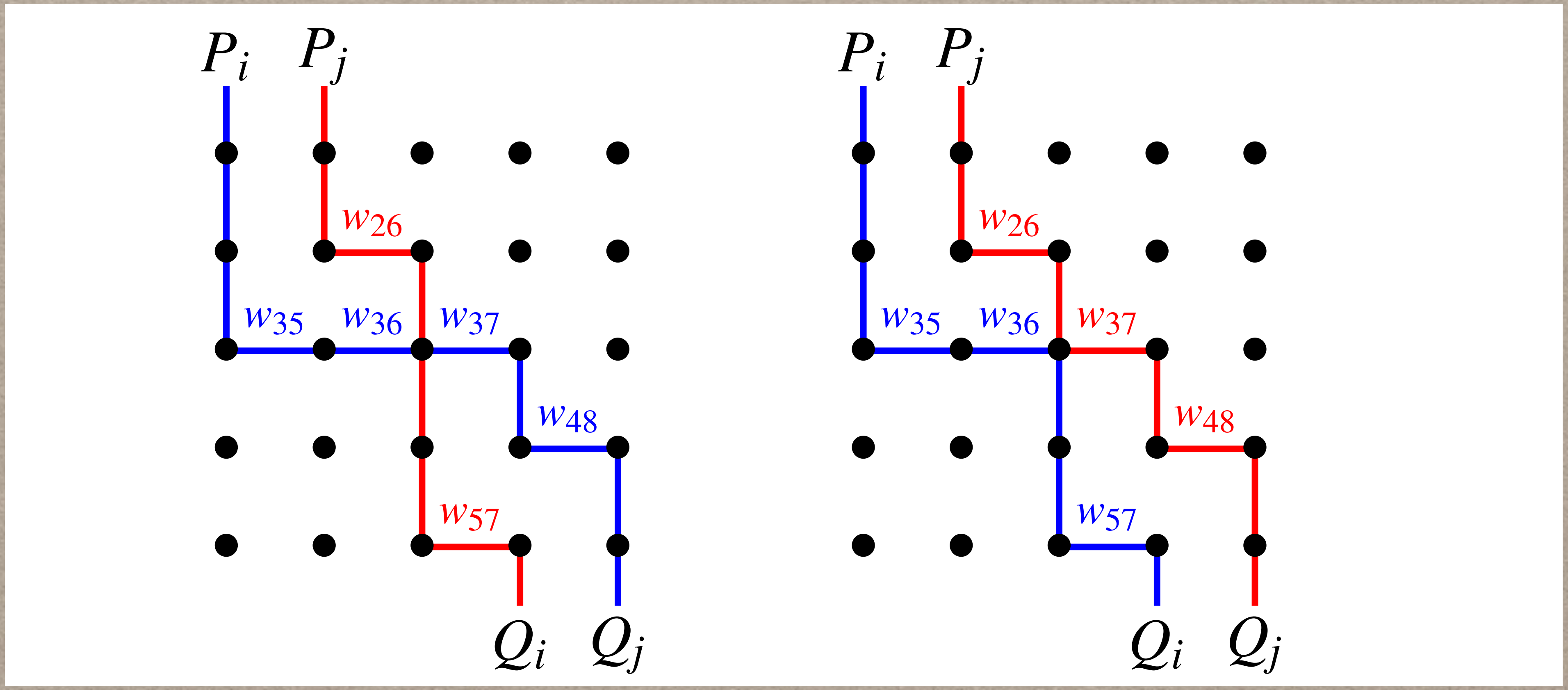
$$h_m^{o}(\mathbf{x}, \overline{\mathbf{x}}) = [t^m] \begin{cases} \left(\dfrac{1}{1 - tx_1} + \dfrac{1}{1 - tx_1^{-1}} - \delta_{m0}\right) & \textit{if } n = 1\,; \\ (1 - t^2) \displaystyle\prod_{i=1}^{n} \frac{1}{(1 - tx_i)(1 - tx_i^{-1})} & \textit{if } n > 1\,, \end{cases}$$

OUTLINE OF COMBINATORIAL PROOF OF JACOBI-TRUDI

$$s_\lambda(x) = \left| h_{\lambda_j - j + i}(x) \right|_{1 \le i,j \le n}$$

- Each row i in tableau ⇔ a lattice path in the plane from P_i to Q_i ⇔ a complete symmetric function.
- Each of these complete symmetric functions corresponds to a term on diagonal of J-T determinant.
- If we swap ending points, we can define an off-diagonal term in J-T determinant.

LINDSTRÖM-GESSEL-VIENNOT INVOLUTION



COMPLETE FACTORIAL SYM FNS

- To this we add the dependence on the factorial parameters *a*:

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ *and* $\mathbf{a} = (a_1, a_2, \ldots)$.

$$h_m(\mathbf{x}\,|\,\mathbf{a}) = [t^m] \prod_{i=1}^{n} \frac{1}{1 - tx_i} \prod_{j=1}^{n+m-1} (1 + ta_j)\,;$$

$$h_m^{so}(\mathbf{x}, \overline{\mathbf{x}}, 1\,|\,\mathbf{a}) = [t^m]\,(1+t) \prod_{i=1}^{n} \frac{1}{(1 - tx_i)(1 - tx_i^{-1})} \prod_{j=1}^{n+m-1} (1 + ta_j)\,;$$

$$h_m^{sp}(\mathbf{x}, \overline{\mathbf{x}}\,|\,\mathbf{a}) = [t^m] \prod_{i=1}^{n} \frac{1}{(1 - tx_i)(1 - tx_i^{-1})} \prod_{j=1}^{n+m-1} (1 + ta_j)\,;$$

$$h_m^{o}(\mathbf{x}, \overline{\mathbf{x}}\,|\,\mathbf{a}) = [t^m] \begin{cases} \left(\dfrac{1}{1 - tx_1} + \dfrac{1}{1 - tx_1^{-1}} - \delta_{m0}\right) \displaystyle\prod_{j=1}^{m} (1 + ta_j) & \text{if } n = 1\,; \\ (1 - t^2) \displaystyle\prod_{i=1}^{n} \dfrac{1}{(1 - tx_i)(1 - tx_i^{-1})} \prod_{j=1}^{n+m-1} (1 + ta_j) & \text{if } n > 1\,. \end{cases}$$

FLAGGED J-T FOR CLASSICAL GROUPS (CHEN ET AL., OKADA, H. & KING)

For $\mathbf{x}^{(i)} = (x_i, x_{i+1}, \ldots, x_n)$ ***and*** $\overline{\mathbf{x}}^{(i)} = (\overline{x}_i, \overline{x}_{i+1}, \ldots, \overline{x}_n)$

$$s_\lambda(\mathbf{x}\,|\,\mathbf{a}) = \left| h_{\lambda_j - j + i}(\mathbf{x}^{(i)}\,|\,\mathbf{a}) \right| ;$$

$$so_\lambda(\mathbf{x}, \overline{\mathbf{x}}, 1\,|\,\mathbf{a}) = \left| h^{so}_{\lambda_j - j + i}(\mathbf{x}^{(i)}, \overline{\mathbf{x}}^{(i)}, 1\,|\,\mathbf{a}) \right| ;$$

$$sp_\lambda(\mathbf{x}, \overline{\mathbf{x}}\,|\,\mathbf{a}) = \left| h^{sp}_{\lambda_j - j + i}(\mathbf{x}^{(i)}, \overline{\mathbf{x}}^{(i)}\,|\,\mathbf{a}) \right| ;$$

$$o_\lambda(\mathbf{x}, \overline{\mathbf{x}}\,|\,\mathbf{a}) = \left| h^{o}_{\lambda_j - j + i}(\mathbf{x}^{(i)}, \overline{\mathbf{x}}^{(i)}\,|\,\mathbf{a}) \right| ,$$

- Independently obtained by Okada (personal communication).

FLAGGED JACOBI-TRUDI FOR CLASSICAL GROUPS

Some History….

- Flagged factorial Schur due to Chen et al. (2002).
- Non-factorial symplectic and odd orthogonal due to Chen et al. (2002).
- Non-factorial flagged symplectic, odd orthogonal, and even orthogonal due to Okada (preprint).

SYMPLECTIC: TABLEAUX

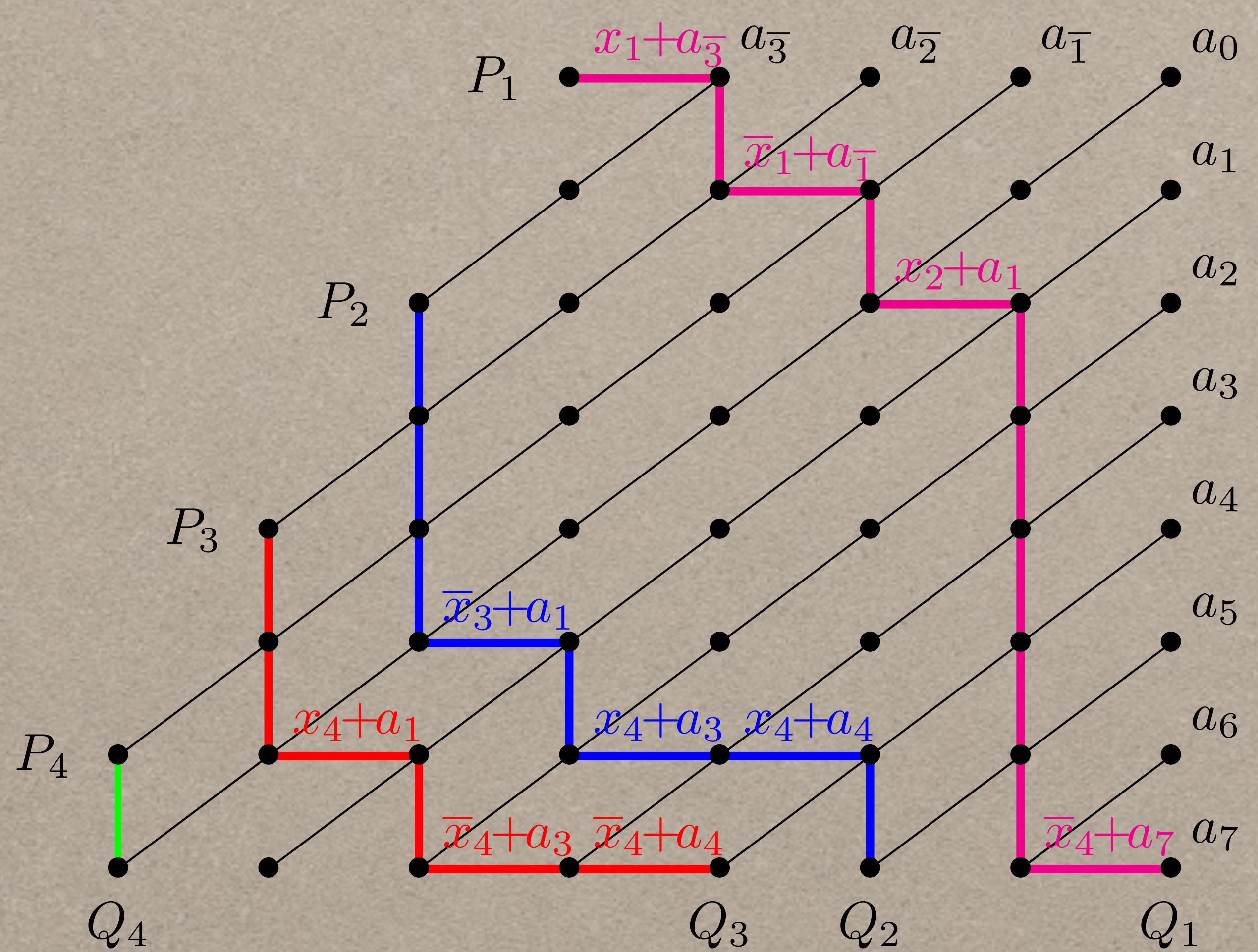
1	$\overline{1}$	2	$\overline{4}$
$\overline{3}$	4	4	
4	$\overline{4}$	$\overline{4}$	

- Entries from alphabet

$$1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}$$

- Entries weakly increase across rows.
- Entries strictly increase down columns.
- No entry i or $\overline{i}$ appears below row i.

SYMPLECTIC: LATTICE PATHS



1	$\overline{1}$	2	$\overline{4}$
$\overline{3}$	4	4	
4	$\overline{4}$	$\overline{4}$	

x_1	$\overline{x}_1$	$x_2 + a_1$	$x_{\overline{4}} + a_7$
$x_{\overline{3}} + a_1$	$x_4 + a_3$	$x_4 + a_4$	
$x_4 + a_1$	$x_{\overline{4}} + a_3$	$x_{\overline{4}} + a_4$	

- the factorial contribution is simply to label the steps with an *x* + *a* weight instead of an *x* weight.

ODD ORTHOGONAL: TABLEAUX

1	$\overline{1}$	2	$\overline{4}$
3	4	0	
4	$\overline{4}$	0	

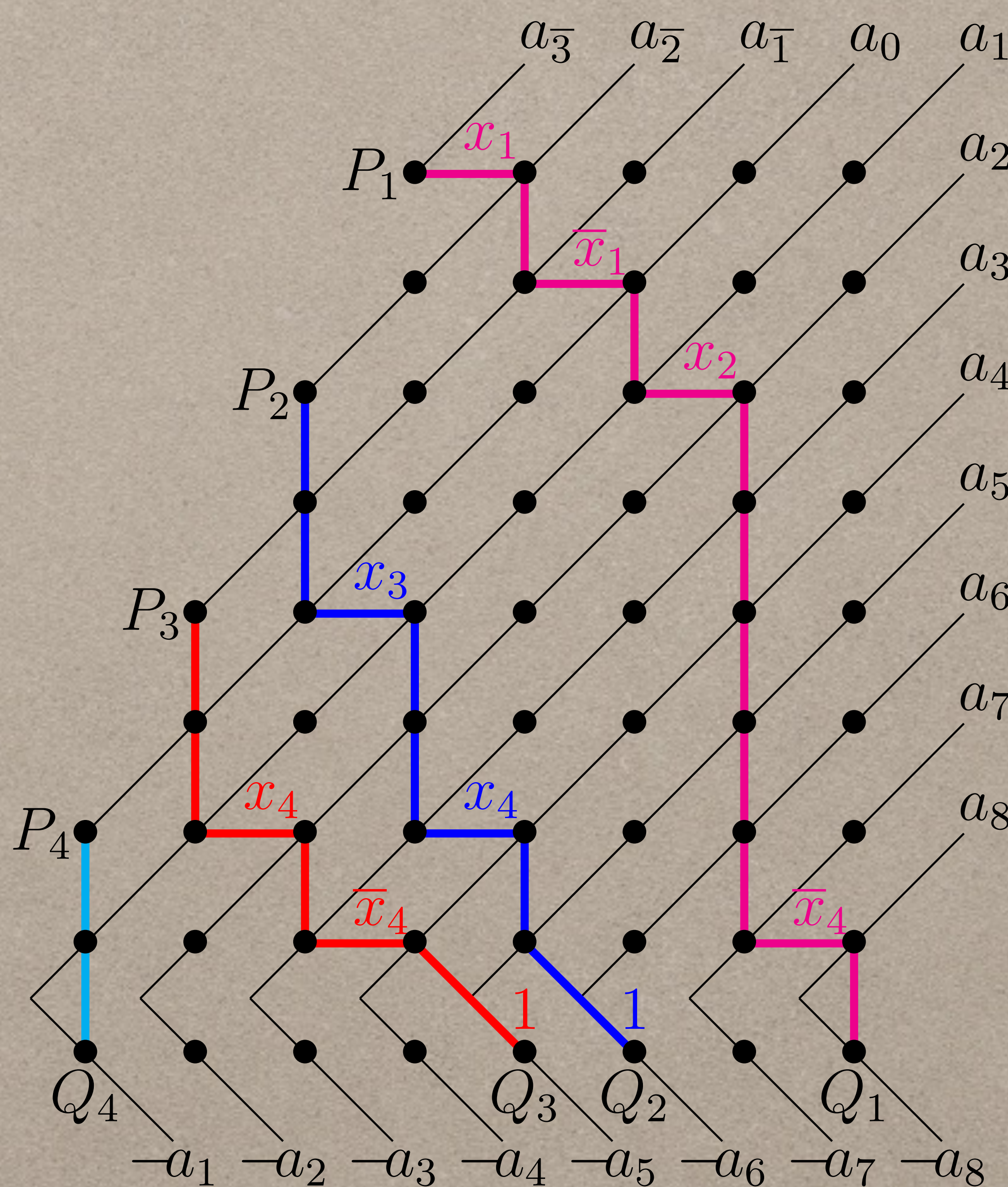
- Entries from alphabet

$$\{1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n} < 0\}$$

- Entries weakly increase across rows.
- Entries weakly increase down columns.
- No entry i or $\overline{i}$ appears below row i.
- No two non-zero entries in a column are equal.
- In any row, 0 appears at most once.

ODD ORTHOGONAL: LATTICE PATHS

$LP(T) =$



$T =$

1	$\overline{1}$	2	$\overline{4}$
3	4	0	
4	$\overline{4}$	0	

$\text{wgt}(T) =$

x_1	$\overline{x}_1$	x_2+a_2	$x_{\overline{4}}+a_8$
x_3+a_1	x_4+a_4	$1-a_6$	
x_4+a_2	$x_{\overline{4}}+a_4$	$1-a_5$	

THE TOKUYAMA STORY

$$Q_\lambda(\mathbf{x};\mathbf{y}\,|\,\mathbf{a}) = \prod_{1\le i\le j\le n} (x_i + y_j)\; s_\mu(\mathbf{x}\,|\,\mathbf{a})$$

$\lambda=\mu+\delta$

Symmetric Q function in two variables

Vandermonde

Schur function

Tokuyama,1988	$\mathbf{x}$	$\mathbf{y} = t\mathbf{x}$	$\mathbf{a} = 0$
Okada, 1990	$\mathbf{x}$	$\mathbf{y} = t\mathbf{x}$	$\mathbf{a} = 0$
Hamel and King, 2007	$\mathbf{x}$	$\mathbf{y}$	$\mathbf{a} = 0$
Brubaker, Bump, Friedberg, 2011	$\mathbf{x}$	$\mathbf{y}$	$\mathbf{a} = 0$
Ikeda, Mihalcea, Naruse, 2011	$\mathbf{x}$	$\mathbf{y} = \mathbf{x}$	$\mathbf{a}$
Bump, McNamara, Nakasuji, 2011	$\mathbf{x}$	$\mathbf{y} = t\mathbf{x}$	$\mathbf{a}$
Hamel and King, 2015	$\mathbf{x}$	$\mathbf{y}$	$\mathbf{a}$

THE TOKUYAMA STORY

$$Q_\lambda(\mathbf{x};\mathbf{y}\,|\,\mathbf{a}) = \prod_{1\le i\le j\le n} (x_i + y_j)\, s_\mu(\mathbf{x}\,|\,\mathbf{a})$$

- The Tokuyama story is all about shape. Combinatorially, the left hand side is a shifted shape; the right hand side is a special shifted shape (a staircase) along with a standard tableau shape.

$PST =$

1	1	1	2′	2	2	3	3	5
	2	2	3′	3	4′	5′	5	6′
		3	3	4′	4	5′	6	
			4	5′	5	5		
				5	6′	6		
					6			

$\longleftrightarrow$ $PD =$

1	2′	1	4′	5′	6′
	2	3′	2	5′	2
		3	4′	3	3
			4	5′	6′
				5	5
					6

, $T =$

1	2	3
3	5	5
4	6	
5		
6		

CLASSICAL CHARACTERS AND TOKUYAMA

- The key to Tokuyama is being able to split this shifted tableau into the staircase piece (with primed and unprimed entries) and the standard tableau piece (with unprimed entries only).
- the issue with the classical Q functions is: what are the shifted primed tableaux?
- and what is the appropriate factorial weighting?

SYMPLECTIC: PRIMED SHIFTED TABLEAUX

1	$\overline{1}$	$2'$	$\overline{2}'$	3	3	$4'$	$\overline{4}'$	$\overline{5}$
	$2'$	2	$\overline{2}$	$\overline{3}'$	4	4	$\overline{4}'$	
		$\overline{3}$	4	$\overline{4}'$	$\overline{4}$	$\overline{4}$	$\overline{4}$	
			$\overline{4}$	$\overline{4}$				
				5				

- Alphabet: $1' < 1 < \overline{1}' < \overline{1} < 2' < 2 < \overline{2}' < \overline{2} < \cdots < n' < n < \overline{n}' < \overline{n}$
- Entries weakly increase in rows and columns.
- Entries strictly increase along diagonals.
- Unprimed entries occur at most one in each column.
- Primed entries occur at most once in each row.

SYMPLECTIC: WEIGHTED TABLEAUX

1	$\bar{1}$	$2'$	$\bar{2}'$	3	3	$4'$	$\bar{4}'$	$\bar{5}$
	$2'$	2	$\bar{2}$	$\bar{3}'$	4	4	$\bar{4}'$	
		$\bar{3}'$	4	$\bar{4}'$	$\bar{4}$	$\bar{4}$	$\bar{4}$	
			$\bar{4}$	$\bar{4}$				
				5				

Weights:

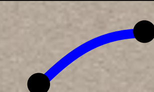
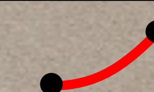
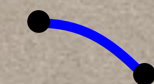
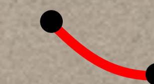
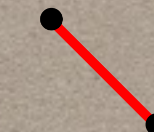
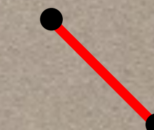
P_{ii}	$\mathrm{wgt}(P_{ii})$	P_{ij} $(i<j)$	$\mathrm{wgt}(P_{ij})$ $(i<j)$
k	x_k	k	$x_k + a_{j-i}$
k'	y_k	k'	$y_k - a_{j-i}$
$\bar{k}$	x_k^{-1}	$\bar{k}$	$x_k^{-1} + a_{j-i}$
$\bar{k}'$	y_k^{-1}	$\bar{k}'$	$y_k^{-1} - a_{j-i}$

x_1	$\bar{x}_1+a_1$	y_2+a_2	$\bar{y}_2+a_3$	x_3+a_4	x_3+a_5	y_4+a_6	$\bar{y}_4+a_7$	$\bar{x}_5+a_8$
	y_2	x_2+a_1	$\bar{x}_2+a_2$	$\bar{y}_3+a_3$	x_4+a_4	x_4+a_5	$\bar{y}_4+a_6$	
		$\bar{y}_3$	x_4+a_1	$\bar{y}_4+a_2$	$\bar{x}_4+a_3$	$\bar{x}_4+a_4$	$\bar{x}_4+a_5$	
			$\bar{x}_4$	$\bar{x}_4+a_1$				
				x_5				

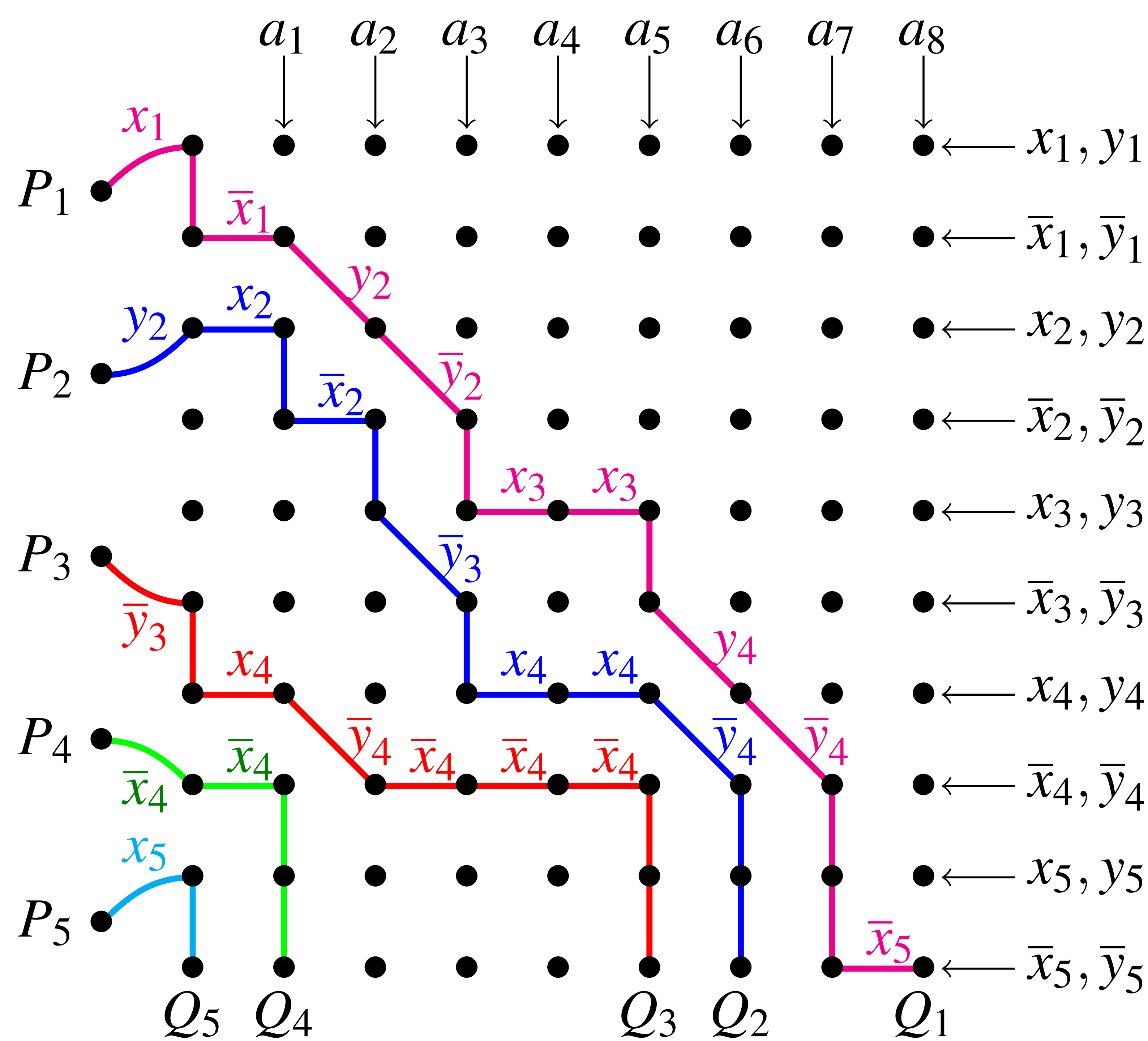
SYMPLECTIC: WEIGHTS AND PATHS

P_{ii}	$\mathrm{wgt}(P_{ii})$	$P_{ij}\ \ (i<j)$	$\mathrm{wgt}(P_{ij})\ \ (i<j)$
k	x_k	k	$x_k + a_{j-i}$
k'	y_k	k'	$y_k - a_{j-i}$
$\overline{k}$	x_k^{-1}	$\overline{k}$	$x_k^{-1} + a_{j-i}$
$\overline{k'}$	y_k^{-1}	$\overline{k}'$	$y_k^{-1} - a_{j-i}$

1	$\overline{1}$	$2'$	$\overline{2}'$	3	3	$4'$	$\overline{4}'$	$\overline{5}$
	$2'$	2	$\overline{2}$	$\overline{3}'$	4	4	$\overline{4}'$	
		$\overline{3}$	4	$\overline{4}'$	$\overline{4}$	$\overline{4}$	$\overline{4}$	
			$\overline{4}$	$\overline{4}$				
				5				

(i,i)	i	x_i		(i,i)	i'	y_i	
(i,i)	$\overline{i}$	$\overline{x}_i$		(i,i)	$\overline{i}'$	$\overline{y}_i$	
$(i,j)\ \ i<j$	k	$x_k + a_{j-i}$		$(i,j)\ \ i<j$	k'	$y_k - a_{j-i}$	
$(i,j)\ \ i<j$	$\overline{k}$	$\overline{x}_k + a_{j-i}$		$(i,j)\ \ i<j$	$\overline{k}'$	$\overline{y}_k - a_{j-i}$	

1	$\bar{1}$	$2'$	$\bar{2}'$	3	3	$4'$	$\bar{4}'$	$\bar{5}$
	$2'$	2	$\bar{2}$	$\bar{3}'$	4	4	$\bar{4}'$	
		$\bar{3}'$	4	$\bar{4}'$	$\bar{4}$	$\bar{4}$	$\bar{4}$	
			$\bar{4}$	$\bar{4}$				
				5				

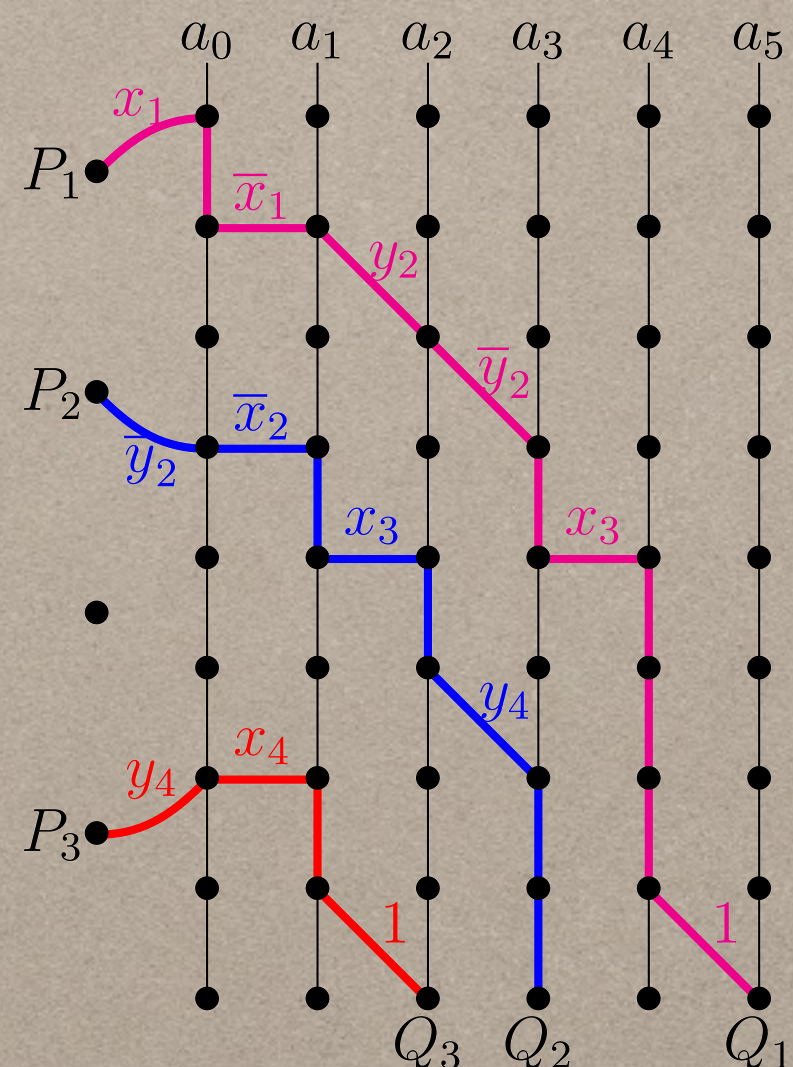


OUTLINE OF PROOF OF FACTORIAL TOKUYAMA

- Start from expression for $Q_\lambda(x;y|a)$ as a determinant.
- Extract factors (x_i+y_i).
- Subtract successive rows from one another to give factors of the form (x_i+y_{i+1}).
- Repeat the process to obtain factors of the form (x_i+y_{i+2}).
- Continue until all factors of the form (x_i+y_j) are extracted for $i \leq j$.
- Show that what remains is a factorial character $s_\mu(x|a)$.

ODD ORTHOGONAL: LATTICE PATHS

$LP(P) =$



$P =$

1	$\overline{1}$	$2'$	$\overline{2}'$	3	0
	$\overline{2}'$	$\overline{2}$	3	$4'$	
		$4'$	4	0	

$\mathrm{wgt}(P) =$

x_1	$\overline{x}_1+a_1$	y_2-a_2	$\overline{y}_2-a_3$	x_3+a_4	$1-a_5$
	$\overline{y}_2$	$\overline{x}_2+a_1$	x_3+a_2	y_4-a_3	
		y_4	x_4+a_1	$1-a_2$	

THE RESULT (H. & KING, 2016)

$$Q_\lambda(\mathbf{x};\mathbf{y}\,|\,\mathbf{a}) = \prod_{1\le i\le j\le n} (x_i + y_j)\; s_\mu(\mathbf{x}\,|\,\mathbf{a})\,;$$

$$Q^{sp}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}}\,|\,\mathbf{a}) = \prod_{1\le i\le j\le n} (x_i + y_j + \overline{x}_i + \overline{y}_j)\; sp_\mu(\mathbf{x},\overline{\mathbf{x}}\,|\,\mathbf{a})\,;$$

$$Q^{so}_\lambda(\mathbf{x},\overline{\mathbf{x}};\mathbf{y},\overline{\mathbf{y}},1\,|\,\mathbf{a}) = \prod_{1\le i\le j\le n} (x_i + y_j + \overline{x}_i + \overline{y}_j)\; so_\mu(\mathbf{x},\overline{\mathbf{x}},1\,|\,\mathbf{a})\,.$$

STILL TO DO....

- Finish the combinatorial proof of the even orthogonal Tokuyama....
- But, for the combinatorial proof, defining exactly the correct tableaux and lattice paths is tricky....

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