SOME NEW CONJECTURES OF ROGERS-RAMANUJAN TYPE

Shashank Kanade

Joint work with Matthew C. Russell

PIMS Post-doctoral fellow
University of Alberta

ROGERS-RAMANUJAN IDENTITIES

Discovered by:

Discovered by:

L. J. Rogers (1894)

BACKGROUND

Discovered by:

L. J. Rogers (1894)

S. Ramanujan (1917)

Discovered by:

L. J. Rogers (1894)

S. Ramanujan (1917)

I. Schur (1917)

Discovered by:

L. J. Rogers (1894)

S. Ramanujan (1917)

I. Schur (1917)

Relevant to:

Discovered by:

L. J. Rogers (1894)

S. Ramanujan (1917)

I. Schur (1917)

Relevant to:

Number theory,

Discovered by:

L. J. Rogers (1894)

S. Ramanujan (1917)

I. Schur (1917)

Relevant to:

Number theory,

Representation theory of affine Lie algebras

Discovered by:

L. J. Rogers (1894)

S. Ramanujan (1917)

I. Schur (1917)

Relevant to:

Number theory,

Representation theory of affine Lie algebras

Representation theory of Virasoro Lie algebras

Discovered by:

L. J. Rogers (1894)

S. Ramanujan (1917)

I. Schur (1917)

Relevant to:

Number theory,

Representation theory of affine Lie algebras

Representation theory of Virasoro Lie algebras

Representation theory of vertex operator algebras

Discovered by:

L. J. Rogers (1894)

S. Ramanujan (1917)

I. Schur (1917)

Relevant to:

Number theory,

Representation theory of affine Lie algebras

Representation theory of Virasoro Lie algebras

Representation theory of vertex operator algebras

Statistical mechanics

Discovered by:

L. J. Rogers (1894)

S. Ramanujan (1917)

I. Schur (1917)

Relevant to:

Number theory,

Representation theory of affine Lie algebras

Representation theory of Virasoro Lie algebras

Representation theory of vertex operator algebras

Statistical mechanics

Conformal field theory

Discovered by:

L. J. Rogers (1894)

S. Ramanujan (1917)

I. Schur (1917)

Relevant to:

Number theory,

Representation theory of affine Lie algebras

Representation theory of Virasoro Lie algebras

Representation theory of vertex operator algebras

Statistical mechanics

Conformal field theory

Knot theory . . .

RR 1

Partitions of n whose adjacent parts differ by at least 2

RR 1

Partitions of n whose adjacent parts differ by at least 2 are equinumerous with

RR 1

Partitions of n whose adjacent parts differ by at least 2
are equinumerous with
partitions of n with each part $\equiv 1, 4 \pmod 5$

RR 1

Partitions of n whose adjacent parts differ by at least 2
are equinumerous with
partitions of n with each part $\equiv 1, 4 \pmod 5$

RR 2

Partitions of n whose adjacent parts differ by at least 2 and

RR 1

Partitions of n whose adjacent parts differ by at least 2
are equinumerous with
partitions of n with each part $\equiv 1, 4 \pmod 5$

RR 2

Partitions of n whose adjacent parts differ by at least 2 and
whose smallest part is at least 2

RR 1

Partitions of n whose adjacent parts differ by at least 2
are equinumerous with
partitions of n with each part $\equiv 1, 4 \pmod 5$

RR 2

Partitions of n whose adjacent parts differ by at least 2 and
whose smallest part is at least 2
are equinumerous with
partitions of n with each part $\equiv 2, 3 \pmod 5$

Rogers-Ramanujan 1

$$\begin{aligned} 9 &= 9 \\ &= 8+1 \\ &= 7+2 \\ &= 6+3 \\ &= 5+3+1 \end{aligned} \qquad \begin{aligned} 9 &= 9 \\ &= 6+1+1+1 \\ &= 4+4+1 \\ &= 4+1+1+1+1+1 \\ &= 1+1+1+1+1+1+1+1+1 \end{aligned}$$

Rogers-Ramanujan 1

$$
\begin{aligned}
9 &= 9 \\
&= 8 + 1 \\
&= 7 + 2 \\
&= 6 + 3 \\
&= 5 + 3 + 1
\end{aligned}
\qquad
\begin{aligned}
9 &= 9 \\
&= 6 + 1 + 1 + 1 \\
&= 4 + 4 + 1 \\
&= 4 + 1 + 1 + 1 + 1 + 1 \\
&= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

Rogers-Ramanujan 2

$$
\begin{aligned}
9 &= 9 \\
&= 7 + 2 \\
&= 6 + 3
\end{aligned}
\qquad
\begin{aligned}
9 &= 7 + 2 \\
&= 3 + 3 + 3 \\
&= 3 + 2 + 2 + 2
\end{aligned}
$$

SOME NUMBER THEORETIC PROPERTIES

$$r(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}} \qquad (q = e^{2\pi i \tau})$$

Converges for $\tau \in \mathbb{H}$ ($|q| < 1$).

$$r(\tau) = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}} \qquad (q = e^{2\pi i\tau})$$

Converges for $\tau \in \mathbb{H}$ ($|q| < 1$).

Converges at q = roots of unity

$$r(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}} \qquad (q = e^{2\pi i \tau})$$

Converges for $\tau \in \mathbb{H}$ ($|q| < 1$).

Converges at q = roots of unity except if q is a mth primitive rooth with $5 \mid m$.

$$r(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}} \qquad (q = e^{2\pi i \tau})$$

Converges for $\tau \in \mathbb{H}$ ($|q| < 1$).

Converges at q = roots of unity except if q is a mth primitive rooth with $5 \mid m$.

Up to $q^{1/5}$, ratio of the two RR generating functions.

$$r(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}} \qquad (q = e^{2\pi i \tau})$$

Converges for $\tau \in \mathbb{H}$ ($|q| < 1$).

Converges at q = roots of unity except if q is a mth primitive rooth with $5 \mid m$.

Up to $q^{1/5}$, ratio of the two RR generating functions.

$r(\tau = 0) = \phi^{-1}$ (related to modular tensor categories)

Blow up the icosahedron to unit sphere.

EVALUATIONS

Blow up the icosahedron to unit sphere.

Now stereographically project.

Blow up the icosahedron to unit sphere.

Now stereographically project.

$SL_2(\mathbb{Z}) \cdot i$

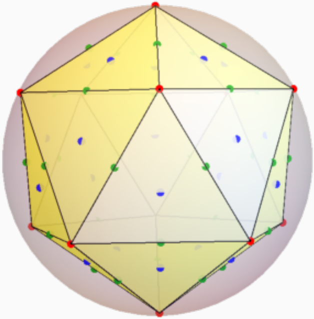
$\longmapsto$ Edge points

$SL_2(\mathbb{Z}) \cdot \rho$

$\longmapsto$ Face points

$SL_2(\mathbb{Z}) \cdot 0$

$\longmapsto$ Vertex points $\neq 0, \infty$



NEW CONJECTURES

$$n = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_j \qquad \lambda_i \geq \lambda_{i+1} \text{ (non-increasing order).}$$

$n = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_j$ $\qquad \lambda_i \geq \lambda_{i+1}$ (non-increasing order).

Difference at least 3 at distance 2:

$n = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_j$ $\lambda_i \geq \lambda_{i+1}$ (non-increasing order).

Difference at least 3 at distance 2:

$\lambda_i - \lambda_{i+2} \geq 3$.

$n = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_j \qquad \lambda_i \geq \lambda_{i+1}$ (non-increasing order).

Difference at least 3 at distance 2:

$\lambda_i - \lambda_{i+2} \geq 3$.

That is, if you jump over 2 plus signs, the parts fall by at least 3.

$n = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_j \qquad \lambda_i \geq \lambda_{i+1}$ (non-increasing order).

Difference at least 3 at distance 2:

$\lambda_i - \lambda_{i+2} \geq 3$.

That is, if you jump over 2 plus signs, the parts fall by at least 3.

(Rogers-Ramanujan has Difference at least 2 at distance 1.)

Condition ☺

Condition ☺

Difference at least 3 at distance 2 and

Condition ☺

Difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is divisible by 3.

`Condition ☺`
Difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is divisible by 3.

I_1 Partitions of n satisfying **`Condition ☺`**

Condition ☻
Difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is divisible by 3.

I_1 Partitions of n satisfying **Condition ☻**
are equinumerous with

Condition ☺

Difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is divisible by 3.

I_1 Partitions of n satisfying **Condition ☺**
are equinumerous with
partitions of n with each part $\equiv \pm 1, \pm 3 \pmod 9$

`Condition ☺`
Difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is divisible by 3.

- I_1 Partitions of n satisfying **`Condition ☺`**
  are equinumerous with
  partitions of n with each part $\equiv \pm 1, \pm 3 \pmod 9$
- I_2 Partitions of n satisfying **`Condition ☺`**

`Condition` ☺
Difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is divisible by 3.

- I_1 Partitions of n satisfying **`Condition` ☺**
are equinumerous with
partitions of n with each part $\equiv \pm 1, \pm 3 \pmod 9$
- I_2 Partitions of n satisfying **`Condition` ☺**
with smallest part at least 2 are equinumerous with

Condition ☺

Difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is divisible by 3.

I_1 Partitions of n satisfying **Condition ☺** are equinumerous with partitions of n with each part $\equiv \pm 1, \pm 3 \pmod 9$

I_2 Partitions of n satisfying **Condition ☺** with smallest part at least 2 are equinumerous with partitions of n with each part $\equiv \pm 2, \pm 3 \pmod 9$

Condition ☺
Difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is divisible by 3.

- I_1 Partitions of n satisfying **Condition ☺**
are equinumerous with
partitions of n with each part $\equiv \pm 1, \pm 3 \pmod 9$
- I_2 Partitions of n satisfying **Condition ☺**
with smallest part at least 2 are equinumerous with
partitions of n with each part $\equiv \pm 2, \pm 3 \pmod 9$
- I_3 Partitions of n satisfying **Condition ☺**

Condition ☺
Difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is divisible by 3.

- I_1 Partitions of n satisfying **Condition ☺**
  are equinumerous with
  partitions of n with each part $\equiv \pm 1, \pm 3 \pmod 9$
- I_2 Partitions of n satisfying **Condition ☺**
  with smallest part at least 2 are equinumerous with
  partitions of n with each part $\equiv \pm 2, \pm 3 \pmod 9$
- I_3 Partitions of n satisfying **Condition ☺**
  with smallest part at least 3 are equinumerous with

`Condition ☺`
Difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is divisible by 3.

- I_1 Partitions of n satisfying **`Condition ☺`**
  are equinumerous with
  partitions of n with each part $\equiv \pm 1, \pm 3 \pmod 9$
- I_2 Partitions of n satisfying **`Condition ☺`**
  with smallest part at least 2 are equinumerous with
  partitions of n with each part $\equiv \pm 2, \pm 3 \pmod 9$
- I_3 Partitions of n satisfying **`Condition ☺`**
  with smallest part at least 3 are equinumerous with
  partitions of n with each part $\equiv \pm 3, \pm 4 \pmod 9$

Condition ☺
Difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is divisible by 3.

I_1 Partitions of n satisfying **Condition ☺** are equinumerous with partitions of n with each part $\equiv \pm 1, \pm 3 \pmod 9$

I_2 Partitions of n satisfying **Condition ☺** with smallest part at least 2 are equinumerous with partitions of n with each part $\equiv \pm 2, \pm 3 \pmod 9$

I_3 Partitions of n satisfying **Condition ☺** with smallest part at least 3 are equinumerous with partitions of n with each part $\equiv \pm 3, \pm 4 \pmod 9$

I_4 Partitions of n satisfying difference at least 3 at distance 2 and

I_4 Partitions of n satisfying difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is $\equiv 2 \pmod 3$ and

I_4 Partitions of n satisfying difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is $\equiv 2 \pmod 3$ and
smallest part at least 2

I_4 Partitions of n satisfying difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is $\equiv 2 \pmod 3$ and
smallest part at least 2
are equinumerous with

I_4 Partitions of n satisfying difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is $\equiv 2 \pmod 3$ and
smallest part at least 2
are equinumerous with
partitions of n with each part $\equiv 2, 3, 5, 8 \pmod 9$

I_4 Partitions of n satisfying difference at least 3 at distance 2 and
two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is $\equiv 2 \pmod 3$ and
smallest part at least 2
are equinumerous with
partitions of n with each part $\equiv 2, 3, 5, 8 \pmod 9$

Condition 😈$_t$

Condition 😈$_t$

Difference at least 3 at distance 3 and

Condition 😈$_t$
Difference at least 3 at distance 3 and

if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is $\equiv t \pmod 3$, and

Condition 😈$_t$
Difference at least 3 at distance 3 and

if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is $\equiv t \pmod 3$, and

smallest part at least t and

Condition 😈$_t$
Difference at least 3 at distance 3 and

if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is $\equiv t \pmod 3$, and

smallest part at least t and at most one occurance of t in the partition.

I_5 Partitions of n satisfying **Condition** 😈$_1$

Condition 😈$_t$
Difference at least 3 at distance 3 and

if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is $\equiv t \pmod 3$, and

smallest part at least t and at most one occurance of t in the partition.

I_5 Partitions of n satisfying **Condition** 😈$_1$ are equinumerous with

ASYMMETRIC CONJECTURES

Condition 😼$_t$
Difference at least 3 at distance 3 and

if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is $\equiv t \pmod 3$, and

smallest part at least t and at most one occurance of t in the partition.

I_5 Partitions of n satisfying **Condition** 😼$_1$ are equinumerous with partitions of n with each part $\equiv 1, 3, 4, 6, 7, 10, 11 \pmod{12}$.

ASYMMETRIC CONJECTURES

`Condition` 😼$_t$
Difference at least 3 at distance 3 and

if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is $\equiv t \pmod 3$, and

smallest part at least t and at most one occurance of t in the partition.

I_5 Partitions of n satisfying **`Condition`** 😼$_1$
are equinumerous with
partitions of n with each part $\equiv 1, 3, 4, 6, 7, 10, 11 \pmod{12}$.

I_6 Partitions of n satisfying **`Condition`** 😼$_2$

ASYMMETRIC CONJECTURES

`Condition` 😈$_t$
Difference at least 3 at distance 3 and

if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is $\equiv t \pmod 3$, and

smallest part at least t and at most one occurance of t in the partition.

- I_5 Partitions of n satisfying **`Condition`** 😈$_1$ are equinumerous with partitions of n with each part $\equiv 1, 3, 4, 6, 7, 10, 11 \pmod{12}$.
- I_6 Partitions of n satisfying **`Condition`** 😈$_2$ are equinumerous with

ASYMMETRIC CONJECTURES

`Condition` 😼$_t$
Difference at least 3 at distance 3 and

if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is $\equiv t \pmod 3$, and

smallest part at least t and at most one occurance of t in the partition.

- I_5 Partitions of n satisfying **`Condition`** 😼$_1$ are equinumerous with partitions of n with each part $\equiv 1, 3, 4, 6, 7, 10, 11 \pmod{12}$.
- I_6 Partitions of n satisfying **`Condition`** 😼$_2$ are equinumerous with partitions of n with each part $\equiv 2, 3, 5, 6, 7, 8, 11 \pmod{12}$.

Condition 😈$_t$
Difference at least 3 at distance 3 and

if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is $\equiv t \pmod 3$, and

smallest part at least t and at most one occurance of t in the partition.

I_5 Partitions of n satisfying **Condition** 😈$_1$
are equinumerous with
partitions of n with each part $\equiv 1, 3, 4, 6, 7, 10, 11 \pmod{12}$.

I_6 Partitions of n satisfying **Condition** 😈$_2$
are equinumerous with
partitions of n with each part $\equiv 2, 3, 5, 6, 7, 8, 11 \pmod{12}$.

- G. E. Andrews, *The theory of partitions*, Cambridge University Press, Cambridge, 1998.
- W. Duke, Continued fractions and modular functions. *Bull. Amer. Math. Soc. (N.S.)* **42** (2005), no. 2, 137–162.
- S. Kanade and M. C. Russell, `IdentityFinder` and some new identities of Rogers-Ramanujan type, *Exp. Math.* **24** (2015), no. 4, 419–423.

Thanks!