Bivariate Orthogonal Polynomials and Eigenvalues of Hankel Matrices

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1 Participants

The participants in this research team were:

1. Mourad E. H. Ismail, University of Central Florida, Orlando, Florida, USA.

2. Zeinab Mansour, King Saud University, Riyadh, Saudi Arabia

3. Ruiming Zhang, Northwest A&F University Yangling, Shaanxi P. R. China.

Both Ismail and Zhang are males and are Canadian citizens. Mansour is a female and is a young researcher.

We all stayed at Corbett Hall most of the time and met periodically for discussion.

2 Background

: The original proposal was to study bivariate orthogonal polynomials and eigenvalues of Hankel matrices. Before we came to Banff we made some progress on both topics. It became clear very quickly that we need to concentrate on one of the two topics so we decided to concentrate on the bivariate special functions and related topics. Although there have many many systems of orthogonal polynomials in two or more variables, the class of polynomial we are considering is only few years old, [5], [6], [7], [8], [9], and one of the early examples goes back to Ito, [10]. Contrast this with other types as in [2]. Ito's example is now central to a whole class of polynomial.

Our investigations also led us to consider several special functions in two real variables. A related problem which arose is a q-analogue of the Lidstone series and polynomial expansions [3]. This will lead to a conceptual approach to identities in the the theory of basic hypergeomtric functions and will lead to new identities. This corresponds to expansion of functions around two points. Ismail and his collaborators extensively studied the q-Taylor series and had a lot of success. There is an ongoing book project between Ismail and Dennis Stanton to develop the whole theory of q-special functions using q-Taylor series and Askey-WIlson operators. As such one would expect the q-Lidstone theory to be as rich and the hope is that the few stumbling blocks in the Ismail-Stanton approach to q-series will be resolved using q-Lidstone series. .

3 Progress Report

: The participants were engaged in the following projects.

1. Ismail and Zhang completed a 45 page survey article on bivariate orthogonal polynomials. This work started earlier but was completed while the authors were in residence at BIRS. It acknowledges the support of BIRS. This covers the recent progress in the subject and is expected to influence the development of this evolving subject. Ismail is involved in the Digital Library Project by NIST (National Institute of Science and Technology), which is part of the department of Energy. The NIST library will only list formulas so the paper will be a companion to the orthogonal polynomials in several variables material in the NIST project.

2. Ismail and Zhang spent the week before coming to BIRS visiting the university of Calgary, supported by their own research grants, and discussed the topic of inequalities and complete monotonicity of special functions with Peter Zvengrowski. This initiated another research project on special functions of two variables. They started a paper on the subject and typed about 15 pages of it. This work is still going strong and we expect the final product, which will acknowledge support from BIRS, to be about 30 pages long. In a way this is a contribution to the theory of monotonicity of the real, imaginary or modulus of the gamma function initiated by Ahern and Rudin in [1]. A note worthy reference is [11].

3. Ismail and Mansour started the q-Lidstone theory. During the stay they formulated how the expansion will look like and obtained preliminary results. The key step of expanding the Cauchy kernel in a certain Lidstone series has been achieved. Many technical details are left but we are positive that they will be carried out successfully. The finished product will acknowledge support from BIRS.

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