Combinatorics Meets Ergodic Theory

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1 Overview of the Field

A classic combinatorial result is Szemerédi's Theorem: a set of integers with positive upper density contains arithmetic progressions of arbitrary length. This theorem has turned out to be a sort of Rosetta stone, connecting seemingly unrelated areas of mathematics by linking together ergodic theory, additive combinatorics, computer science, and harmonic analysis. The workshop focused on the ergodic theory and additive combinatorics link, and in particular on the role of the Gowers uniformity norms from the combinatorial side of Szemerédi's Theorem, and the Host-Kra structure theory from the ergodic-theoretic proof of Furstenberg's Multiple Recurrence Theorem. A prominent feature of many of the talks was the fruitful interactions between the two fields, and the conference gave a broad view of the current state of these rapidly maturing fields. A productive open problems session made it clear that there are still numerous directions to be developed in this area.

The topics covered can be broadly grouped into a few areas, with substantial overlap among the topics.

1. Ergodic Theory and Combinatorics. While the connection between the two fields is by now classical, we learnt of numerous new connections, both using ergodic theory to prove new combinatorial results, and importing finitary combinatorial statements to prove ergodic theoretic results.

2. Ergodic Averages. There are many variations on the mean and pointwise ergodic theorems, as well as versions with more terms (the multiple ergodic averages). Over the past few years, new techniques have been developed to address both recurrence problems (which correspond to positivity of the averages) and convergence problems (which correspond to structural results). Talks covered the use of topological models, hard analysis methods, and structural results to prove both types of results.

3. Combinatorics and Number Theory. Over the past ten years, the area of additive combinatorics (also referred to as arithmetic combinatorics) has been substantially developed, drawing on techniques from both combinatorics and number theory to develop our understanding of combinatorial patterns.

4. Gowers Norms and Nilsequences. The theory of Gowers norms and their ergodic analog, the Host-Kra seminorms, featured prominently throughout the conference. Talks covered both applications of these norms to particular combinatorial or ergodic problems, and the further development of the structural theory of these objects.

2 Presentation Highlights

The workshop included 18 talks overviewing recent progress and challenges, and we roughly group them in the four areas listed above.

2.1 Ergodic Theory and Combinatorics

Alexander Fish (University of Sydney) presented his recent joint work with M. Björklund [6] in which they establish the following recurrence result: For every set E of positive density in traceless square matrices with integer values, there exists $k \geq 1$ such that the set of characteristic polynomials of matrices in $E - E$ contains all characteristic polynomials of traceless matrices divisible by k. A key ingredient in the proof is a measure rigidity result of Benoist-Quint for algebraic actions on homogeneous spaces.

Randall McCutcheon (University of Memphis) discussed recent joint work with J. Zhou [12] in which they answer a question of V. Bergelson by proving that there exist IP-rich sets that are not D-sets. The construction uses tree-structure characterizations of both classes.

Joel Moreira (Ohio State University) discussed recent joint work with V. Bergelson and J. Jhonson [3] in which they prove partition regularity results of certain polynomial configurations in the integers and joint work with V. Bergelson [5] in which they prove partition regularity of the configuration $\{x + y, xy\}$ in the set of rationals. The latter result is deduced from a recurrence property of certain solvable measure-preserving group actions.

Donald Robertson (Ohio State University) discussed recent joint work with V. Bergelson, C. Christopherson, D. Robertson, and P. Zorin-Kranich [2] in which they give a characterization of countable amenable groups for which a density version of Hindman's Theorem fails, that is, the group has a positive density subset containing no shifts of any (multiplicative) IP set. The characterization involves a variant of the notion of minimal almost periodicity and its proof uses a nontrivial multiple recurrence result.

Pavel Zorin-Kranich (University of Bonn) presented two related results. The first was joint work with Q. Chu [7] in which they obtain lower bounds for certain double correlation sequences of $\mathbb{Z}^2$-actions, and the second [15] in which he obtains lower bounds for certain multiple correlations defined by several commuting actions of finite quasirandom groups. The latter work strengthens a result of V. Bergelson and T. Tao.

2.2 Ergodic Averages

Vitaly Bergelson (Ohio State University) presented mean convergence results related to multiple-parameter cubic averages, far reaching generalizations of a recurrence theorem of Khintchine, and also discussed related combinatorial implications and open problems. His talk was based partly on the article [1] and his recent joint work with A. Leibman [4].

Sebastian Donoso (University of Chile) presented joint work with W. Sun [9], proving a pointwise convergence result for cubic multiple ergodic averages involving two commuting transformations. The proof depends upon a construction of a suitable uniquely ergodic topological model of an extension of the system, combining variations on the machinery of magic extensions introduced by B. Host and the use of topological models for pointwise averages of W. Huang, S. Shao, and X. Ye.

Since Bourgain's work in the late eighties, random sequences have been used as a model for pointwise ergodic theorems. A few years ago P. LaVictoire proved an L^1 pointwise ergodic theorem with iterates given by random sequences of sub-quadratic growth. Ben Krause (University of California at Los Angeles) presented recent joint work with P. Zorin-Kranich [11] in which they prove a Wiener-Wintner variant of this result with weights of the form $e^{2\pi i n^{1+c}}$ where $c \in (0, 1)$.

2.3 Combinatorics and Number Theory

A crucial part of the recent spectacular progress made in the cap-set problem by Bateman and Katz was a new result relating smooth methods to measure the additive properties of a set (e.g. the additive energy) with structural methods (e.g. exploiting density within a subgroup). In his talk, Thomas Bloom (University of Bristol) presented a variant of this result and speculated regarding potential uses related to higher-order additive structures.

Issac Goldbring (University of Illinois at Chicago) presented a new quantitative version of a theorem of M. Beiglbock, V. Bergelson, and A. Fish, which states that if A and B are positive Banach density subsets of an arbitrary countable amenable group, then the set $A + B$ is piecewise syndetic. This new proof [8] is joint work with M. Di Nasso, R. Jin, S. Leth, M. Lupini, and K. Mahlburg and is purely combinatorial (modulo use of non-standard analysis techniques).

Tom Sanders (Oxford University) explained a new approach for obtaining quantitative versions of Freiman's theorem for Abelian groups of bounded exponent. His approach uses covering arguments and improves upon existing bounds for certain groups.

Anush Tserunyan (University of Illinois at Urbana-Champaign) presented her recent work [14] where she proves a van der Corput lemma for a large class of filters, generalizing many instances for which the van der Corput lemma was previously known. A key ingredient in the proof is a Ramsey theorem for graphs.

Trevor Wooley (University of Bristol) explained a technique for proving solubility of systems of homogeneous translation invariant equations on arbitrary positive density subsets of the integers. Mean value estimates of moments of weighted exponential sums play a key role in the arguments. These estimates are obtained via the "efficient congruencing" method, a technique recently used to make substantial progress to the main conjecture in Vinogradov's mean value theorem.

2.4 Gowers Norms and Nilsequences

Yonatan Gutman (Instytut Matematyczny Polskiej Akademii Nauk) discussed work in progress in which he uses an axiomatic characterization of dynamical nilspaces in order to give an alternative proof (and a strengthening in the context of $\mathbb{Z}^d$-actions) of the ergodic structure theorem of B. Host and B. Kra.

Freddie Manners (University of Oxford) discussed work in progress with Y. Gutman and P. Varjú in which they obtain equivalent characterisations of the axiomatic notions of "dimension-$(k+1)$ cubespaces" introduced by B. Host and B. Kra and of the closely related notion of "k-step nilspaces" that was subsequently introduced by A. Camarena and B. Szegedy.

Lilian Matthiesen (Leibniz Universität Hannover) presented her recent work [13] on the correlation of a certain class of not necessarily bounded multiplicative functions with polynomial nilsequences, as well as potential applications to evaluating asymptotic formulas for averages of multiple correlations of such multiplicative functions.

Wenbo Sun (Northwestern University) discussed joint work with S. Donoso [9] in which they characterize a certain topological cube structure for minimal $\mathbb{Z}^2$-dynamical systems and then use this structural result to compute the automorphism group of specific topological dynamical systems; the example of the Robinson tiling system received particular attention.

Terence Tao (University of California at Los Angeles) presented work in progress with T. Ziegler in which they show that a function $f(m, n)$ which is Gowers anti-uniform in each variable of order d_1 and d_2 respectively, is Gowers anti-uniform of order $d_1 + d_2 - 1$ jointly in m, n (variants of this result give information on the characteristic factors of certain multidimensional ergodic averages). This turns out to be a crucial tool in obtaining asymptotics for the number of polynomial patterns in the primes, a result previously known only for linear patterns.

3 Open Problems

On Tuesday evening, Ben Green moderated an open problems session for all participants. The format, announced early on Monday, was unusual: anyone who wanted to pose a problem had to first explain the problem to another participant, and then that participant presented the problem to the audience. This set off numerous discussions over coffee and meals, with problems sketched in the air, dissected, and then ultimately presented to the whole conference. The session lasted more than two hours and generated numerous follow up discussions throughout the rest of the week.

1. (Bloom presenting a question of Carnovale) Let $f : G \to [-1, 1]$ be a function on some finite Abelian group G. Consider the functionals

$$\|f\|_{U^{s,k,p}} := \left(\mathbb{E}_{u_1,\ldots,u_s} \left(\mathbb{E}_x \left| (\triangle_{u_1,\ldots,u_s} f(x))^{*k} \right|^p \right) \right)^{\frac{1}{pk2^s}},$$

where $\triangle$ denotes the discrete derivative and $*k$ the k-fold self-convolution. These include the additive energies E_k and T_k, the Gowers norms U^s, and L^p norms of a function f.

There are many structural results in additive combinatorics analyzing extremizers of these functionals for special subsets of the parameters. Examples include the Balog-Szemerédi-Gowers theorem considered as a statement about the structure of sets having extremal E_2 in terms of E_1, work of Shkredov comparing $\|\cdot\|_{U^{0,1,3}}$ to E_2 as well as various other cases, the Bateman-Katz structure theorem comparing $E_2(A)$ to $T_4 := \|\cdot\|_{U^{0,4,2}}$, and the famous Gowers inverse theorem comparing U^s to L^∞.

Question: Are these norms? What is the theory of these norms and what can be said about their extremizers and inverse theory in general? Are there arithmetic problems which make use of information contained in these objects when they do not reduce to either the U^s norm, L^p norm, or additive energy?

2. (Zorin-Kranich presenting a question of Gutman) Let α be irrational and suppose that $\gamma \in (0, 1)$. Consider the following continuous time flow $(T^t_\alpha)_{t\in\mathbf{R}}$ on the space $X := \{(x, y) \in (0, 1]^2 : y \leq x^{-\gamma}\}$. Move vertically upwards at unit speed until we hit a point $(x, x^{-\gamma})$ on the graph, at which point we move to the point $(x + \alpha, 0)$, then move vertically upwards at unit speed, and so on.

It is known that such systems are mixing for all α, that is to say $\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$ whenever A, B are measurable. For some smaller set of α they are known to be mixing of order 2: $\lim_{n,m\to\infty} \mu(A\cap T^{-n}B\cap T^{-n-m}C) = \mu(A)\mu(B)\mu(C)$ whenever A, B, C are all measurable.

Question: Is such a flow in fact 2-mixing for all irrational α?

3. (Tao presenting a problem of Krause) Consider the discrete spheres $S^m = \{\vec{n} \subset \mathbb{Z}^d : n_1^2 + \cdots + n_d^2 = m\}$ of radius $\sqrt{m}$, where we restrict the radii to belong to some subset $E \subset \mathbb{N}$.

Given a $\mathbb{Z}^d$ action on a probability space (X, μ), one may consider the pointwise convergence of ergodic averages along spheres of restricted radii

$$\lim_{m\to\infty, m\in E} \frac{1}{|S^m|} \sum_{n\in S_m} T^n f$$

and the maximal function

$$\sup_{m\in E} \frac{1}{|S^m|} \sum_{n\in S_m} T^n |f|$$

for, say, $f \in L^2$.

Is the maximal function bounded? In $d = 5$ Magyar and Magyar-Stein-Wainger and Ionescu show the answer to be yes when $E = \mathbb{N}$. Stein has shown the failure of these questions for $E = \mathbb{N}$ when $d = 4$. Hughes has shown that for some sparse E the answer is yes.

Question: What about random E? (Defined via $P(n \in E) = \frac{1}{n}$).

4. (Maass presenting a question of Tao)

Theorem 3.1 (Tao-Ziegler)**.** *If the group G acts on the space X, and $H_1, H_2 < G$, then*

$$\cap_{i=1,2} L^\infty(Z^{d_i-1}_{H_i}) \subset L^\infty(Z^{d_1+d_2-1}_{H_1+H_2}).$$

For the definitions of these factors, see their paper.

Question: Given a sequence $(a_{n_1,n_2}) \in \ell^\infty(\mathbb{Z}^2)$ which restricted to lines in the n_i direction is a d_i-step nilsequence (with some uniformity in the other coordinate), the above theorem can be used to show that this sequence is a joint (d_1+d_2)-step nilsequence in both variables. However, this argument requires quite heavy machinery, for instance it uses the structural classification of the factors Z_H^d due to Host and Kra. Is there a direct proof of this statement? This could potentially be used to give an alternative approach to certain inverse theorems for the Gowers uniformity norms.

5. (Matthiesen presenting a question of Glasscock)

   Given a polynomial $p \in \mathbb{Z}[x]$ of degree ≥ 2, let $S = \{n : n = p(m), m \in \mathbb{Z}\}$. Prove that S does not contain arbitrarily long arithmetic progressions.

   Comment: For a discussion of this question, see

   `http://mathoverflow.net/questions/59471`

   In particular, it was observed there by Noam Elkies that, thanks to the work of Caporoso, Harris, and Mazur, this question would be resolved if one was willing to assume a suitable form of the Bombieri-Lang conjecture.

6. (Lyall presenting a question of Griesmer) Say that a set $R \subset \mathbb{Z}$ is a set of optimal recurrence (elsewhere known as "nice recurrence") if given any measure preserving system (X, μ, T), $A \subset X$ and $\varepsilon > 0$,there exists $n \in R$ such that

   $$\mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon\,.$$

   Example (Poincaré-Khintchine): Let S be any infinite set and $R := \{a - b : b < a \in S\}$. Then R is a set of optimal recurrence.

   Question: If S is infinite, must

   $$R = \{a - 2b + c : a, b, c \in S, a, b, c \text{ mutually distinct}\}$$

   be a set of optimal recurrence?

7. (Bergelson presenting a question of Kra)

   Theorem 3.2 (Bergelson-Host-Kra)**.** *If T is ergodic, then for every $\varepsilon > 0$*

   $$\{n : \mu(A \cap T^{-n}A \cdots \cap T^{-(k-1)n}A) \geq \mu^k(A) - \varepsilon\}$$

   is syndetic for $k = 2, 3, 4$, but for $k = 5$ the set may be empty.

   On the other hand, Qing Chu has shown

   Theorem 3.3 (Chu)**.** *If T, S are commuting transformations generating an ergodic action, then for every $\varepsilon > 0$*

   $$\{n : \mu(A \cap T^{-n}A \cap S^{-n}A) \geq \mu^4(A) - \varepsilon\}$$

   is syndetic.

   Question: What is the threshold $\eta_0 \in [0, 1]$ such that the following holds: For every $\varepsilon > 0$ the set

   $$\{n : \mu(A \cap T^{-n}A \cap S^{-n}A) \geq \mu^{3+\eta}(A) - \varepsilon\}$$

   is syndetic for all $\eta > \eta_0$ but not for $\eta < \eta_0$? It is known (by Chu and Zorin-Kranich) that η_0 is at least 0.19.

8. (Richter presenting a question of Bergelson) Let $E \subset \mathbb{N}$ denote the set of multiplicatively even numbers, that is

$$E = \{n \in \mathbb{N} : n = \prod_{i=1}^{k} p_i^{e_i} \text{ with } \sum_i e_i \in 2\,\mathbb{Z} \text{ and } p_i \text{ prime}\},$$

or in other words those n for which the Liouville function $\lambda(n)$ equals 1.

Question 1: Is every point in E a starter of an IP-set contained in E? More precisely, for all $x_1 \in E$, does there exist a sequence $(x_n)_{n\geq 2}$ such that the set $FS(x_i)_{i\in\mathbb{N}} = \left\{ \sum_{i\in F} x_i : F \subset \mathbb{N},\ 0 < |F| < \infty \right\}$ is contained in E?

Remark: The above question is equivalent to asking that $E \cap E - 1$ is IP-large, which is to say, it contains a set of the form $FS(y_i)_{i\geq 2}$ for some numbers $y_i \in \mathbb{N}$, $i \geq 2$. This is because if $FS(y_i)_{i\geq 2} \subset E \cap E - 1$, then for arbitrary $x_1 \in E$ the set $FS(x_i)_{i\in\mathbb{N}}$ with $x_i = x_1 y_i, i \geq 2$ is contained in E. Note, it follows from Hindman's theorem and from the multiplicativity of the Liouville function that E is IP-large. Also, Question 1 can be affirmatively answered if one assumes the Chowla conjecture on correlations of the Liouville function, since Chowla's conjecture implies that E is a thick set, meaning that it contains arbitrarily long intervals, and any point in a thick set is a starter of an IP-set contained in this thick set. However, one would like an unconditional proof.

Question 2: Is it true that for all $k \geq 1$ and $x \in E$ (by convention $0 \notin E$), the set E contains a kAP of the form $x, x+r, \ldots, x+kr$?

9. (Fish presenting a question of Zorin-Kranich) Recall the following return times theorems,

Theorem 3.4 (Bourgain)**.** *For every measure preserving system (X, μ, T) and $f \in L^\infty(X)$ there exists a universal set $X' \subset X$ with $\mu(X') = 1$ such that for every $x \in X'$, for any other measure preserving system (Y, ν, S) and $g \in L^\infty(Y)$, there exists $Y' \subset Y$ with $\nu(Y') = 1$ such that for every $y \in Y'$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g(S^n y) \text{ exists.}$$

This result has a short proof using the Bourgain–Furstenberg–Katznelson–Ornstein (BFKO) orthogonality criterion. The BFKO criterion and Bourgain's bilinear pointwise ergodic theorem can also be used to prove the following result.

Theorem 3.5 (Zorin-Kranich)**.** *For every measure preserving system (X, μ, T) and $f_1, f_2 \in L^\infty(X)$ there exists a universal set $X' \subset X$ with $\mu(X') = 1$ such that for every $x \in X'$, for any other measure preserving system (Y, ν, S) and $g \in L^\infty(Y)$, there exists $Y' \subset Y$ with $\nu(Y') = 1$ such that for every $y \in Y'$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) g(S^n y) \text{ exists.}$$

However, the BFKO criterion does not seem to be applicable in the following situation:

Conjecture 3.6. *For every measure preserving system (X, μ, T) and $f \in L^\infty(X)$ there exists a universal set $X' \subset X$ with $\mu(X') = 1$ such that for every $x \in X'$, for any other measure preserving system (Y, ν, S) and $g_1, g_2 \in L^\infty(Y)$, there exists $Y' \subset Y$ with $\nu(Y') = 1$ such that for every $y \in Y'$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g_1(S^n y) g_2(S^{2n} y) \text{ exists.}$$

Is there any substitute for the BFKO criterion that would settle this conjecture using Bourgain's bilinear pointwise ergodic theorem as a black box?

10. (Bloom presenting a question of Wooley) Given a set $A \subset [N]$ and $\varepsilon = \varepsilon(N) > 0$, find a density condition on A which ensures that $A + [-\varepsilon, \varepsilon]$ contains a configuration of the form

$$\{a, a+d, a+\sqrt{2}d\}.$$

One would hope to combine Roth type arguments with Davenport-Heilbronn arguments in order to achieve this. But then one may ask for the generalization to Szemerédi type problems, such as conditions to show there exist configurations of the form

$$\{a, a+\alpha_1 d, \ldots, a+\alpha_k d\}$$

for arbitrary real numbers $\alpha_1, \ldots, \alpha_k$.

11. (Tao presenting a question of Goldbring) Given an ultrafilter in $\beta\,\mathbb{N}$, recall that one may define a limit along this ultrafilter as

$$\lim_{n \to p} f(n) = x \text{ iff } \{n : |f(n) - x| < \varepsilon\} \in p \forall\, \varepsilon > 0.$$

Recall also that one may define an associative addition on $\beta\,\mathbb{N}$, and that an idempotent ultrafilter is defined to be one which satisfies $p+p = p$. As observed by Galvin-Glazer, the existence of an idempotent ultrafilter gives a short proof of Hindman's theorem that any finite coloring of $\mathbb{N}$ contains a monochromatic IP-set: If A is p-large, then $A-n$ is p-large for p-many n, and iterating yields Hindman's theorem.

One must use Zorn's Lemma twice in order to prove the existence of idempotent ultrafilters.

Goldbring and Andrews show that Hindman's theorem is equivalent to the existence of idempotent type, something weaker than existence of idempotent ultrafilters. Roughly speaking, a type is like an ultrafilter, but it only measures definable sets $\{x : \phi(x) = \text{true}\}$ in some language (e.g. arithmetic), rather than all sets.

Question 1: Can one replace every use of idempotent ultrafilters in ergodic theory by the use of idempotent types, and thus, by the above theorem, by the use of Hindman's theorem?

Question 2: The same question, but replace "idempotent" by "minimal idempotent" and Hindman's theorem by the statement that any finite coloring of $\mathbb{N}$ contains a monochromatic central set.

12. (A problem of Tao, added after the end of the problem session) It is a corollary of the work of Host and Kra that if (X, T) is an inverse limit of k-step nilsytems, and (Y, S) is a factor of (X, T), then (Y, S) is also an inverse limit of k-step nilsystems. Is there a direct proof of this fact which avoids the full structure theory of Host-Kra (or the related structure theory of Ziegler)? Solving this problem would enable one to deduce the ergodic structure theorem for the Gowers-Host-Kra seminorms from the combinatorial one for the Gowers norms on $\mathbb{Z}_N$. For more details see

`https://terrytao.wordpress.com/2015/07/23`

4 Outcome of the Meeting

BIRS provided an excellent atmosphere for research and collaboration. The latest developments in the field were covered in the talks, followed by intense discussions in the many spaces provided by the outstanding facilities. The common room, the coffee area, and the small collaboration rooms were filled throughout the breaks with follow-up discussions, and these discussions continued during walks around the Banff Centre and surrounding national park.

Further attention was brought to the conference when Terence Tao, well known throughout the mathematical community, posted several blog entries during the meeting about topics covered in the workshops. These blog entries, along with numerous comments from around the world, are available here:

`https://terrytao.wordpress.com/2015/07/20`
`https://terrytao.wordpress.com/2015/07/22`
`https://terrytao.wordpress.com/2015/07/23`
`https://terrytao.wordpress.com/2015/07/24`

During the workshop, talks were recorded using the BIRS video facilities, and several speakers received email from viewers not present at the conference with comments or questions.

References

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