

Subconvexity bounds and simple zeros of modular L-functions

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1 Overview of the Field

This Research in Teams meeting focussed on questions concerning the behavior of L -functions in the critical strip, an old and fundamental problem in analytic number theory. Analytic number theory dates back to work of Riemann (1859) and Dirichlet (1837) who studied the behaviour of prime numbers via analytic methods using functions now known as L -functions. In particular, they defined what we know call the Riemann zeta-function, $\zeta(s)$, and Dirichlet L -functions, $L(s, \chi)$, which are certain meromorphic functions attached to group characters χ of $(\mathbb{Z}/q\mathbb{Z})^\times$. These are special cases of arithmetic L -functions which may be thought of as Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ where the sequence $\{a_n\}$ typically encodes arithmetic information. The Riemann zeta function and Dirichlet L -functions are the only examples of degree one L -functions. In the twentieth century, it was realized that there were many more examples of higher degree L -functions. Hecke and Maass discovered certain degree two L -functions which are attached to what are now known as holomorphic modular forms and to Maass forms. Godement and Jacquet constructed higher degree L -functions attached to irreducible cuspidal automorphic representation of $GL(n)$ over the rationals. From another point of view, Selberg [18] gave an axiomatic definition of L -functions. This set of L -functions became to be known as the Selberg class. Conjectures of Langlands suggest that Selberg class of L -functions coincides with the set of cuspidal automorphic L -functions.

L -functions are important since they encode information about many arithmetic problems. For instance, an exact formula for $\pi(x)$, the number of primes less than x , can be obtained by studying the Riemann zeta function and its zeros. The famous Prime Number Theorem, which asserts that $\pi(x) \sim \frac{x}{\log x}$ as $x \rightarrow \infty$, may be deduced from the non-trivial fact that $\zeta(s)$ does not vanish on the line $\Re(s) = 1$. Dirichlet's theorem on the infinitude of primes in arithmetic progressions follows from the non-trivial fact that Dirichlet L -functions do not vanish at the point $s = 1$. Higher degree L -functions are related to other problems in arithmetic. For instance, the Birch and Swinnerton-Dyer conjecture (one of Clay Mathematics Institute's million dollar Millennium Prize Problems) describes a relationship between the group of rational points on an elliptic curve over \mathbb{Q} to order of vanishing of a degree two L -function at a certain point in the complex plane.

Due to this connection between arithmetic and L -functions, researchers are interested in studying the analytic properties of L -functions. Our Research in Teams project involves studying the *size* of certain degree two L -functions and the properties of their *zeros*. The Lindelöf hypothesis asserts that any L -function $L(s)$ satisfies the bound $|L(\frac{1}{2} + it)| = O(|t|^\epsilon)$. The convexity bound for a degree d L -function, a consequence of the Phragmén–Lindelöf convexity principle, states that $|L(\frac{1}{2} + it)| = O(|t|^{\frac{d}{4} + \epsilon})$. A *subconvexity bound*

is an improvement of the convexity bound for a degree d L -function of the form $|L(\frac{1}{2} + it)| = O(|t|^a)$ where $a < \frac{d}{4}$. For any given L -function it is desirable to prove such subconvexity bounds as many arithmetic consequences can be derived from such bounds.

It is also important to have knowledge of the zeros of an L -function. It is widely believed that all non-trivial zeros of an L -function lie on the line $\Re(s) = 1/2$. This is known as the ‘‘Generalized Riemann Hypothesis,’’ and it is another of Clay Mathematics Institute’s million dollar Millennium Prize Problems. This is perhaps the most famous open problem in number theory. Another important conjecture is that all non-real, non-trivial zeros of an L -function are simple. As a first step toward establishing this conjecture, it is an open and difficult question to show that an L -function has (quantifiably) many simple zeros. Showing that certain degree two L -functions have many simple zeros is one of the key topics we focussed on in this Research in Teams project.

2 Recent Developments and Open Problems

In recent years, establishing a subconvexity result for an L -function has been a very popular research subject. In a series of influential articles Duke, Friedlander, and Iwaniec [6], [7], [8], [9] established some of the first results for degree two L -functions. Many of their arguments were generalized considerably in the following years (see, for instance, the article by Michel and Venkatesh [16] and the references contained within). Attached to a modular L -function are various parameters, such as the level and the weight. In some of the above articles, the authors were interested in subconvexity results with respect to the level or weight. We are interested in a subconvexity result with respect to the parameter t , the height of the L -function above the real axis. In this case, we would like to establish there exists $\delta \in (0, \frac{1}{2})$ such that for any $\varepsilon > 0$

$$|L_f(\frac{1}{2} + it)| \ll |t|^{\delta + \varepsilon} \quad (1)$$

where $L_f(s)$ is the L -function attached to a modular form f . In the special case where f is a modular form on the full modular group, Good [10] proved that $\delta = \frac{1}{3}$ is admissible. A different proof of Good’s result was later provided by Jutila [12]. Recently, Wu [20] established (1) with $\delta = \frac{1}{2} - \frac{25}{256}$ for all newforms f and this was slightly improved by Kuan [14] to $\delta = \frac{25}{178}$ in some (but not all) cases. Despite these advances, it remained an open problem to establish the analogue of Good’s subconvexity estimate for modular form L -functions of all weights and levels with the value $\delta = \frac{1}{3}$.

The question of showing that a fixed L -function has infinitely many simple zeros began with work of Levinson [15] for the Riemann zeta function. He showed that at least $1/3$ of the zeros of the Riemann zeta function are simple and lie on the critical line. His work was extended to Dirichlet L -functions by Bauer [1]. For primitive degree two L -functions this first result is due to Conrey and Ghosh [5], established in 1989. They considered Ramanujan’s delta function $\Delta(z)$, a level one, weight 12 modular form. They showed that the $L_\Delta(s)$, the L -function attached to Δ , has infinitely many simple zeros. More precisely, let $N_\Delta^s(T)$ denote the number of simple zeros of $L_\Delta(s)$ to height T in the critical strip. They showed the following:

$$\text{For any } \varepsilon > 0 \text{ there are arbitrarily large values of } T \text{ such that } N_\Delta^s(T) \geq T^{\frac{1}{6} - \varepsilon}. \quad (2)$$

For L -functions attached to both arbitrary modular forms or to Maass form f , we would like to show that

$$\text{for any } \varepsilon > 0 \text{ there are arbitrarily large values of } T \text{ such that } N_f^s(T) \geq T^{\frac{1}{6} - \varepsilon} \quad (3)$$

where $N_f^s(T)$ denote the number of simple zeros of $L_f(s)$ up to height T in the critical strip. Since Conrey and Ghosh’s work [5] it has been desirable to obtain lower bounds for $N_f^s(T)$ for other modular forms or Maass forms f . Cho [4] established the analogue of (2) for a few Maass forms f . Furthermore, Milinovich and Ng [17] have shown conditionally on the generalized Riemann Hypothesis for $L_f(s)$ where f is any holomorphic newform that for any $\varepsilon > 0$, $N_f^s(T) \geq T(\log T)^{-\varepsilon}$, for T sufficiently large. Recently, in a major breakthrough, Booker [3] proved the qualitative estimate that $N_f^s(T)$ is unbounded for all holomorphic newforms f . Despite these results, it is still currently unknown how to prove (3) or any other quantitative estimate for the number of simple zeros of $L_f(s)$ for all holomorphic newforms f .

3 Scientific Progress Made

The main goals of this meeting were to establish a general subconvexity result for modular form L -functions $L_f(s)$ and to provide lower bounds for $N_f^s(T)$, the number of simple zeros of the L -function $L_f(s)$ whose imaginary parts lie in $[0, T]$. During this meeting we established the following results.

Theorem 1. *Let $f \in S_k(\Gamma_1(N))^{new}$ be a normalized Hecke eigenform. Then*

$$|L_f(\frac{1}{2} + it)| \ll_{f,\varepsilon} (1 + |t|)^{\frac{1}{3} + \varepsilon},$$

with polynomial dependence on N and k .

For level 1 modular forms, this was previously established by Good [10]. Our proof of Theorem 1 follows an approach of Jutila [12] which was later refined by Huxley [11]. Their argument makes a very clever use of Farey fractions and the Voronoi summation formula. In addition exponential sum techniques and a large sieve inequality are required. A key difficulty in establishing the generalization given in Theorem 1 is that we needed to prove a new version of Voronoi's summation formula that has a number of new restrictions on the set of "admissible" Farey fractions. These restrictions do not appear in Jutila and Huxley's argument, making the proof of Theorem 1 quite a bit more involved. We were able to overcome these restrictions and succeeded in making the Jutila-Huxley argument work in this more general setting.

Using the subconvexity estimate in Theorem 1 along with ideas from the important papers of Conrey and Ghosh [5] and Booker [2, 3], we were able to establish the following quantitative result on the number of simple zero zeros of a Dirichlet twist of a modular L -function.

Theorem 2. *Let $f \in S_k(\Gamma_1(N))^{new}$ be a normalized Hecke eigenform. There is a primitive character χ such that, for any $\varepsilon > 0$, there exists a sufficiently large T with $N_{f \times \chi}^s(T) \geq T^{\frac{1}{6} - \varepsilon}$.*

In other words, given a modular form f with L -function $L_f(s)$ there exists a 'twisted' L -function which has many simple zeros. Furthermore, we have been able to show that if quasi Riemann hypothesis holds for $L_f(s)$ then it is possible to obtain a result like (2) with an exponent which depends on the width of the zero-free region.

4 Outcome of the Meeting and Future directions

In this meeting we succeeded in proving a subconvexity bound in the "t-aspect" for any modular form L -functions. Our result is of the same quality as Hardy and Littlewood's classical bound for the the Riemann zeta function, namely $|\zeta(\frac{1}{2} + it)| = O(|t|^{1/6 + \varepsilon})$, for any $\varepsilon > 0$. This generalized an argument of Jutila [12] and Huxley [11] to all modular form L -functions. One future direction would be extend this result to all Maass form L -functions. If this were possible, then it would follows that all degree two L -functions $L_f(s)$ should satisfy a subconvexity bound $|L_f(\frac{1}{2} + it)| = O(|t|^{1/3 + \varepsilon})$. We were interested in subconvexity bounds since they have applications to the number of simple zeros of an L -function.

A key open question that still remains is to prove a good quantitative lower bound for the number of simple zeros of any degree two L -function. We were unable to show a good quantitative lower bound for the number of simple zeros of any fixed modular form L -function $L_f(s)$. However, we could show that some twist of $L_f(s)$ has many simple zeros. It is still desirable to establish (2) for any holomorphic newform f .

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