

Low-lying zeros of quadratic Dirichlet L -functions

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August 9-16, 2015

1 Background

Prime numbers form the building blocks of the natural numbers. As such, in number theory our goal is to improve our understanding of their behavior. One of the most important unresolved questions in number theory is the Riemann Hypothesis, first conjectured by Riemann in 1859. In simple terms the Riemann Hypothesis tells us that the distribution of prime numbers amongst the natural numbers is as nice as possible. The original statement was given in terms of the zeros of a function called the Riemann zeta function, but this conjecture has since been generalized to similar statements for many other so-called L -functions.

In 1973, Montgomery [9] noticed that certain statistics of the zeros of the Riemann zeta function bear a striking similarity to statistics coming from random unitary matrices in $U(N)$ in the large N limit. In recent years, such similarities were also seen to be present for certain families of L -functions. Depending on the family, one might have to replace $U(N)$ with one of the groups $O(N)$, $SO(2N + 1)$, $SO(2N)$, $Sp(N)$; this is what Katz and Sarnak coined the symmetry type of the family.

The 1-level density is a central statistic, which analyzes the low-lying zeros of members of a family. It has the advantage of allowing one to isolate the symmetry type, while being quite versatile and tractable under certain restrictions on the involved test function. It should be noted that low-lying zeros of L -functions are of central importance in many number-theoretical problems, and their thorough understanding could lead to the solution of several longstanding problems.

The family that we will discuss in this report is that of Dirichlet L -functions of real primitive Dirichlet characters of modulus $8d$, i.e. the family of L -functions attached to the quadratic characters

$$\chi_{8d}(\cdot) := \left(\frac{8d}{\cdot} \right).$$

More precisely, we consider

$$\mathcal{F}(X) := \{L(s, \chi_{8d}) : 1 \leq |d| \leq X, d \text{ is odd and square-free}\}.$$

This family is known to have significant advantages over that of all real characters [12, 5], and will allow us to obtain precise results. We introduce an even Schwartz test function ϕ , which is assumed to be real-valued. Given a positive number X , the 1-level density for the single L -function $L(s, \chi_{8d})$ is defined as the sum

$$D_X(\chi_{8d}; \phi) := \sum_{\gamma_{8d}} \phi\left(\gamma_{8d} \frac{L}{2\pi}\right),$$

with $\gamma_{8d} := -i(\rho_{8d} - \frac{1}{2})$, where ρ_{8d} runs over the nontrivial zeros of $L(s, \chi_{8d})$. Here,

$$L := \log \left(\frac{X}{2\pi e} \right). \quad (1)$$

We introduce the following weighted 1-level density

$$\mathcal{D}^*(\phi; X) := \frac{1}{W(X)} \sum_{d \text{ odd}}^* w \left(\frac{d}{X} \right) D_X(\chi_{8d}; \phi),$$

where \sum^* indicates that the summation is restricted to square-free numbers, $w(t)$ is an even nonnegative Schwartz test function which is not identically zero (which essentially restricts the sum to $d \ll X$) and

$$W(X) := \sum_{d \text{ odd}}^* w \left(\frac{d}{X} \right).$$

Note that in the literature the weight w is often taken to be a sharp cutoff. However, it is more convenient to use a smooth function, especially when computing lower order terms.

After a detailed analysis of a corresponding function field family, Katz and Sarnak made the following prediction:

$$\lim_{X \rightarrow \infty} \mathcal{D}^*(\phi; X) = \widehat{\phi}(0) - \frac{1}{2} \int_{-1}^1 \widehat{\phi}(x) dx. \quad (2)$$

In fact they predict that this family is symplectic. Note that there is a transition when the supremum of the support of $\widehat{\phi}$ reaches 1. It is therefore particularly interesting to obtain results for test functions having larger support and to describe this transition also with respect to lower order terms.

Katz and Sarnak have shown [6] that their prediction holds under the restriction $\sup(\text{supp}(\widehat{\phi})) < 1$, and under the additional assumption of GRH (the Generalized Riemann Hypothesis) they were able to relax this condition to $\sup(\text{supp}(\widehat{\phi})) < 2$. Rubinstein [10] has extended their unconditional result to the n -level density ($n \geq 1$) for test functions with suitably restricted support. As for the GRH result, it was extended by Gao [5] to the n -level density; again with the admissible support doubled. Note that for several years it was not known how to match Gao's asymptotic with the random matrix theory predictions. However, this was recently established for $n \leq 7$ by Levinson and Miller [7], and for all remaining n by Entin, Roditty-Gershon and Rudnick [2].

2 Recent progress and results

While the symmetry type of a family is expected to predict an asymptotic for the 1-level density of any reasonable family, it can not predict lower order terms. Such terms of order $1/\log X$ have been found in many families so far (see, e.g., [13, 4] in the case of families of elliptic curves), and depend on the arithmetic properties of the family under consideration. In the family we are considering, such terms have been isolated by Miller [8] under the restriction $\sup(\text{supp}(\widehat{\phi})) < 1$. As noted above, this restriction hides the transition at 1, and therefore one might believe that the global picture is quite different.

An analogous question over function fields was studied by Rudnick [11]. He studied the symplectic family of zeta functions of hyperelliptic curves $y^2 = Q(X)$ defined over $\mathbb{F}_q[x]$, where $Q(x)$ is a monic square-free polynomial of degree $2g + 1$. The hyperelliptic curve has genus g , and Rudnick considered the associated 1-level density when q is fixed and $g \rightarrow \infty$ (this is a more direct analogue to number fields than the $q \rightarrow \infty$ limit). His result is the following estimate for this quantity, under the restriction $\sup(\text{supp}(\widehat{\phi})) < 2$:

$$\int_{\text{USp}(2g)} Z_\phi(U) dU + \frac{1}{g} \left[\widehat{\phi}(0) \sum_{P \text{ monic irr.}} \frac{\deg P}{q^{2 \deg P - 1}} - \frac{\widehat{\phi}(1)}{q - 1} \right] + o \left(\frac{1}{g} \right). \quad (3)$$

Here, $Z_\phi(U)$ is the 1-level density of small eigenvalues of the matrix U (and hence it is a purely random matrix theoretical object). Note that the term involving $\widehat{\phi}(1)$ is only present when the supremum of the support of $\widehat{\phi}$ reaches one, and hence this confirms the existence of a transition at this point.

The result we obtained during our stay in Banff is an asymptotic expression for $\mathcal{D}^*(\phi; X)$ in descending powers of $\log X$, under the assumption of GRH and the restriction $\sigma := \sup(\text{supp}(\widehat{\phi})) < 2$. As with Rudnick's result (3), our result uncovers the transition at 1, and therefore is expected to hold regardless of the support of $\widehat{\phi}$. In fact, the first lower order term we obtain has a striking similarity to that obtained by Rudnick, since it involves both $\widehat{\phi}(0)$ and $\widehat{\phi}(1)$. We now give a more precise statement of our result.

Theorem 1. Fix $\varepsilon > 0$. Let w be a nonnegative Schwartz function on \mathbb{R} which is not identically zero and let ϕ be an even Schwartz function on \mathbb{R} whose Fourier transform satisfies $\sigma = \sup(\text{supp}(\widehat{\phi})) < 2$. Then, assuming GRH, the 1-level density for the zeros of the family of L -functions attached to real primitive Dirichlet characters of modulus $8d$, where d is odd and square-free, is given by

$$\mathcal{D}^*(\phi; X) = \widehat{\phi}(0) - \frac{1}{2} \int_{-1}^1 \widehat{\phi}(u) du + \frac{1}{\log X} [C_{w,1}\widehat{\phi}(0) + C_{w,2}\widehat{\phi}(1)] + O\left(\frac{1}{(\log X)^2}\right),$$

where $C_{w,1}$ and $C_{w,2}$ can be given explicitly in terms of integrals involving the weight function w .

Using similar techniques, we can also study the low-lying zeros in the related family

$$\mathcal{F}_1(X) := \{L(s, \chi_d) : 1 \leq |d| \leq X\}.$$

In this case we observe that the family contains an abundance of repetitions, that is, the L -functions in $\mathcal{F}_1(X)$ appear with certain multiplicities. There are several known examples in the literature where allowing repetitions in a family lead to a more manageable analysis and also to more precise results (cf. [14, 3, 4]). This turns out to be true also in the present case, where we can prove a result of the same flavor as Theorem 1, but with better control of the involved error terms. The improved quality of the error terms is particularly interesting in comparison with the predictions made for the 1-level density by the powerful L -functions Ratios Conjecture of Conrey, Farmer and Zirnbauer [1].

References

- [1] J. B. Conrey, D. W. Farmer, and M. R. Zirnbauer, Autocorrelation of ratios of L -functions, *Commun. Number Theory Phys.* **2** (2008), no. 3, 593–636.
- [2] A. Entin, E. Roditty–Gershon, and Z. Rudnick, Low-lying zeros of quadratic Dirichlet L -functions, hyper-elliptic curves and random matrix theory, *Geom. Funct. Anal.* **23** (2013), no. 4, 1230–1261.
- [3] D. Fiorilli and S. J. Miller, Surpassing the Ratios Conjecture in the 1-level density of Dirichlet L -functions, *Algebra Number Theory* **9** (2015), no. 1, 13–52.
- [4] D. Fiorilli, J. Parks, and A. Södergren, Low-lying zeros of elliptic curve L -functions: Beyond the ratios conjecture, preprint, arXiv:1405.5110.
- [5] P. Gao, n -level density of the low-lying zeros of quadratic Dirichlet L -functions, *Int. Math. Res. Not. IMRN* 2014, no. 6, 1699–1728.
- [6] N. M. Katz and P. Sarnak, Zeros of zeta functions, their spacings and their spectral nature, preprint, 1997.
- [7] J. Levinson and S. J. Miller, The n -level densities of low-lying zeros of quadratic Dirichlet L -functions, *Acta Arith.* **161** (2013), no. 2, 145–182.
- [8] S. J. Miller, A symplectic test of the L -functions ratios conjecture, *Int. Math. Res. Not. IMRN* 2008, no. 3, Art. ID rmm146, 36 pp.
- [9] H. L. Montgomery, The pair correlation of zeros of the zeta function. In *Analytic number theory* (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), 181–193, American Mathematical Society, Providence, RI, 1973.

- [10] M. Rubinstein, Low-lying zeros of L -functions and random matrix theory, *Duke Math. J.* **109** (2001), no. 1, 147–181.
- [11] Z. Rudnick, Traces of high powers of the Frobenius class in the hyperelliptic ensemble, *Acta Arith.* **143** (2010), no. 1, 81–99.
- [12] K. Soundararajan, Nonvanishing of quadratic Dirichlet L -functions at $s = \frac{1}{2}$, *Ann. of Math. (2)* **152** (2000), no. 2, 447–488.
- [13] M. P. Young, Lower-order terms of the 1-level density of families of elliptic curves, *Int. Math. Res. Not.* (2005), no. 10, 587–633.
- [14] M. P. Young, Low-lying zeros of families of elliptic curves, *J. Amer. Math. Soc.* **19** (2006), no. 1, 205–250.