

Uniqueness results in geometric tomography

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1 Overview of the Field

The research of the group was focused on problems in geometric tomography. The latter area is concerned with the study of geometric properties of convex bodies based on information about sections or projections of these bodies. Geometric tomography has important applications to many areas of mathematics and science, in general. The book “Geometric Tomography” [4] by Gardner gives an excellent account of various problems and techniques that arise in geometric tomography. Of paramount importance are questions about unique determination of convex bodies from the size of their sections or projections. For many years the dominating tools for proving uniqueness were those involving spherical harmonics and direct geometric methods. In recent years, we have seen a rapid development of new methods, based on Fourier analysis, which allowed to solve many open problems in convex geometry. The general idea is to express geometric characteristics of a body in terms of the Fourier transform and then use methods of harmonic analysis to solve geometric problems. This approach has led to several results including Fourier analytic solutions of the Busemann-Petty and Shephard problems, characterizations of intersection and projection bodies, extremal sections and projections of certain classes of bodies. These developments are described in the books “Fourier Analysis in Convex Geometry” [8] by Koldobsky and “The Interface between Convex Geometry and Harmonic Analysis” [9] by Koldobsky and Yaskin. The most recent results include solutions of several longstanding uniqueness problems, and the discovery of stability in volume comparison problems and its connection to hyperplane inequalities.

2 Recent Developments and Open Problems

One of the classical results in the area (attributed to Funk and Minkowski; see [4]) is that an origin-symmetric star body in \mathbb{R}^n is uniquely determined by $(n - 1)$ -dimensional volumes of its central hyperplane sections. More precisely, if K and L are origin-symmetric convex bodies in \mathbb{R}^n such that

$$|K \cap \xi^\perp| = |L \cap \xi^\perp|$$

for every $\xi \in S^{n-1}$, then $K = L$. Here and below, $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$.

Note that this result is false without the symmetry assumption. It is natural to ask what information is needed to uniquely determine convex bodies that are not necessarily origin symmetric. One of the results in this direction was obtained by Falconer [2] and Gardner [3], who proved that in order to determine a

non-symmetric convex body one needs to know volumes of hyperplane sections through two interior points. However, the following problem is still open, even in \mathbb{R}^2 .

Problem 1. *Let K and L be convex bodies in \mathbb{R}^n containing a sphere of radius t in their interiors. Suppose that for every hyperplane H tangent to the sphere we have $|K \cap H| = |L \cap H|$. Does this mean that $K = L$?*

The problem in this form was stated by Barker and Larman [1], though a similar question on the sphere was considered earlier by Santaló [14]. Let us give an overview of partial results pertaining to this problem. In their paper Barker and Larman have shown that if a planar body has chords of constant length at distance t from the origin, then the body is a Euclidean disk. It is shown by Yaskin [16] that the answer to the problem is affirmative in the class of convex polytopes in \mathbb{R}^n . Recently, the following modification of Problem 1 was considered by Yaskin and N. Zhang; see [17].

Let K and L be convex bodies in \mathbb{R}^2 and let D_1 and D_2 be two disks in the interior of $K \cap L$. Assume that neither of D_1 or D_2 is contained in the other. If the chords $K \cap H$ and $L \cap H$ have equal length for all H supporting either D_1 or D_2 , then $K = L$.

In this context let us also recall several results related to maximal sections. Let K be a convex body in \mathbb{R}^n . The *inner section function* m_K is defined by

$$m_K(\xi) = \max_{t \in \mathbb{R}} |K \cap (\xi^\perp + t\xi)|,$$

for $\xi \in S^{n-1}$.

An old question of Klee dating back to 1969 asks whether a convex body is uniquely determined (up to translation and reflection in the origin) by its inner section function. Recently, this question was answered in the negative by Gardner, Ryabogin, Yaskin, and Zvavitch [6]. Klee also asked whether a convex body in \mathbb{R}^n , $n \geq 3$, whose inner section function is constant, must be a ball. Nazarov, Ryabogin, and Zvavitch disproved this conjecture; see [10] and [11]. They also constructed a counterexample (in even dimensions) to a much older question of Bonnesen asking whether a convex body is determined by its inner section function and its brightness function. However, the odd-dimensional case is still unresolved.

Problem 2. *Let K and L be convex bodies in \mathbb{R}^n , n is odd, such that $m_K(\xi) = m_L(\xi)$ and the projections $K|\xi^\perp$ and $L|\xi^\perp$ have equal $(n-1)$ -dimensional volumes. Does this information guarantee that $K = L$ (up to translation and reflection)?*

Another version of this question is open in all dimensions.

Problem 3. *Let K be a convex body in \mathbb{R}^n , $n \geq 3$, such that $m_K(\xi) = \text{const}$ and all projections $K|\xi^\perp$ have the same $(n-1)$ -dimensional volume. Is K necessarily a Euclidean ball?*

Instead of areas of sections and projections, one can also consider other intrinsic volumes. For example, the following problem from [4] has attracted a lot of interest recently.

Problem 4. *Let K and L be origin-symmetric convex bodies in \mathbb{R}^3 such that the sections $K \cap \xi^\perp$ and $L \cap \xi^\perp$ have equal perimeters for every $\xi \in S^2$. Does it follow that $K = L$?*

Of course, one can also consider higher dimensional versions of this problem (instead of perimeters it is natural to take the surface area of sections). There is still very little progress done toward the solution of this problem. In particular, it is not known whether the uniqueness holds if one of the bodies is the Euclidean ball. That is, if K is an origin-symmetric convex body in \mathbb{R}^n such that the surface area of $K \cap \xi^\perp$ is independent of ξ , does this mean that K is a ball? It was shown by Howard, Nazarov, Ryabogin and Zvavitch [7] the latter problem has an affirmative answer in the class of C^1 star bodies of revolution. Rusu [13] settled an infinitesimal version of this problem for one-parameter analytic deformations of the ball. Yaskin [15] solved Problem 1 in the class of origin-symmetric convex polytopes.

Discrete tomography is a related area, where instead of convex bodies one deals with lattice sets. A finite subset A of \mathbb{Z}^n is called a *convex lattice set* if $A = (\text{conv} A) \cap \mathbb{Z}^n$. For such sets one can ask questions similar to those in geometric tomography. For example, Gardner, Gronchi and Zong [5] studied an analogue of Alexandrov's projection theorem in discrete settings.

Problem 5. *Let A and B be origin-symmetric convex lattice sets in \mathbb{Z}^n such that for each $u \in \mathbb{Z}^n$ the projections $A|u^\perp$ and $B|u^\perp$ have the same number of points. Is then A necessarily the same as B ?*

Gardner, Gronchi and Zong have shown that the answer is negative if $n = 2$. However, the counterexample constructed does not provide a complete understanding of the 2-dimensional case. Are there any other

counterexamples? Is it possible to impose a mild condition that would make the answer positive for $n = 2$? In the 3-dimensional case the problem is completely open.

Some progress in this direction has been recently obtained by Ryabogin, Yaskin and N. Zhang [12].

3 Outcome of the Meeting

During the meeting we discussed the problems described in the previous section, as well as other related questions. In particular, we tried to attack Problem 4 using recent advances in Fourier analytic methods. A. Koldobsky gave an overview of his results concerning stability and separation in problems on sections and projections. We identified possible new problems and directions. There are many questions similar in spirit to Problem 5 and it is worth looking at possible applications of Fourier methods to such discrete problems. In fact, it looks like a good idea to try to organize a BIRS workshop with the goal of bringing together researchers with analytic background and discrete background. This would help to facilitate collaboration between these two communities of convex geometers, and hopefully to lead to a better understanding of such problems.

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