Multi-norms, Banach lattices, and the Fourier algebra

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1 Overview of the Field

The theory of multi-norms has been developed over several years.

We first recall the definition of a multi-normed space. Let E be a Banach space, and denote by E^n the n-fold Cartesian product of E; $\mathfrak{S}_n$ is the set of permutations of $\{1. \dots, n\}$

A *multi-norm* based on E is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n and such that the following Axioms (A1)–(A4) are satisfied for each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$:

(A1) $\left\|(x_{\sigma(1)}, \dots, x_{\sigma(n)})\right\|_n = \|(x_1, \dots, x_n)\|_n \quad (\sigma \in \mathfrak{S}_n)$;
(A2) $\|(\alpha_1 x_1, \dots, \alpha_n x_n)\|_n \leq (\max_{i=1,\dots,n} |\alpha_i|) \|(x_1, \dots, x_n)\|_n \quad (\alpha_1, \dots, \alpha_n \in \mathbb{C})$;
(A3) $\|(x_1, \dots, x_n, 0)\|_{n+1} = \|(x_1, \dots, x_n)\|_n$;
(A4) $\|(x_1, \dots, x_{n-1}, x_n, x_n)\|_{n+1} = \|(x_1, \dots, x_n)\|_n$.

This notion was introduced by Dales and Polyakov in [**?**]; it was used in [**?**] to resolve a long-standing question of Barry Johnson on the injectivity of Banach left $L^1(G)$-modules (where G is a locally compact group and $L^1(G)$ is the group algebra based on G); there is a substantial theory of the equivalences of multi-norms in [**?**] and, more recently, in [**?**]; the notion has been extended, in particular in connection with Banach lattices, in [**?**], where various representation theorems are obtained. The definition of a multi-norm can be reformulated in terms of tensor products, and so our theory can be regarded as a study of certain Banach space tensor products. The theory is also related to that of absolutely summing operators. A survey of the theory is given in [**?**].

In the above references, many of the multi-norms that are studied are based on spaces of the form $L^r(\Omega)$, where Ω is a measure space, and especially on spaces ℓ^r, where $r \geq 1$. In particular, there is a study of 'standard t-multi-norms for $1 \leq r \leq t$' based on $L^r(\Omega)$ in [**?**] and continued in the other references.

A particular case of the spaces $L^r(\Omega)$ are the group algebras $L^1(G)$, and the standard t-multi-norms are of particular interest in this setting.

Let G be a locally compact group. We recall that the group algebra $L^1(G)$ is a Banach algebra (see [**?**] for the theory of Banach algebras); when G is abelian, $L^1(G)$ is identified via the Fourier transform with the Fourier algebra $A(\Gamma)$, where Γ is the dual group of G. There is a generalization of the definition of $A(\Gamma)$ to the case where Γ is an arbitrary (non-abelian) locally compact group; see [**?**] for the seminal paper. There are several related algebras, including $B(\Gamma)$, the non-abelian analogue of the measure algebra on a locally compact group. For recent papers on these algebras, see [**?**, **?**], for example. One of our goals is to form a deeper connection between the theories of multi-norms and of Fourier algebras.

A new notion that we are considering is that of a 'multi-Banach algebra'. Thus let A be a Banach algebra, and suppose that $(\|\cdot\|_n)$ is a multi-norm based on A (with $\|\cdot\|_1$ the given norm on A). Then $(A^n, (\|\cdot\|_n))$

is a *multi-Banach algebra* if there is a constant $C > 0$ such that

$$\|(a_1b_1,\ldots,a_nb_n)\|_n \leq C\,\|(a_1,\ldots,a_n)\|_n\,\|(b_1,\ldots,b_n)\|_n$$

for each $n \in \mathbb{N}$ and each $a_1,\ldots,a_n,b_1,\ldots,b_n \in A$.

2 Recent Developments and Open Problems

As mentioned, a recent development in the theory of multi-norms is the memoir [**?**]; this develops the theory of 'p–multi-norms', defined for $p \in [1,\infty]$. The case where $p = \infty$ recovers multi-norms themselves; the case where $p = 1$ recovers the theory of 'dual multi-norms', already developed in [**?**]; the notion in the case where $1 < p < \infty$ is the new feature of [**?**], and this leads to the ideas of 'strong p–multi-norms', for example. The memoir [**?**] contains a rather extensive study of p-Banach spaces, including famous theorems of Herz and Kwapień; it concludes with various representation theorems generalizing one of Pisier given in [**?**].

Our week of 'Research in Teams' was devoted to the study of some problems involving multi-norms and, in particular, their connection with the Fourier algebras $A(\Gamma)$ and the theory of multi-Banach algebras.

3 Presentation Highlights

Since this was a workshop for three people assembled for 'Research in Teams', there were no formal presentations.

4 Scientific Progress Made

We made progress in four related areas.

1) In our proposal we asked if there are useful multi-norms defined on the Fourier algebra $A(\Gamma)$ of an arbitrary locally compact group Γ that reduce to standard (and maybe other) known multi-norms in the special case that Γ is abelian.

In fact we found that there is a more general method of constructing multi-norms than was previously realised. This involves the theory of L-decompositions of a Banach space, as discussed at length in the books [**?**, **?**]. The new method recovers as a special case all the standard t-multi-norms based on spaces $L^r(\Omega)$. It remains to be seen whether this method or a variant of it can generate various other well-known multi-norms.

2) In particular, let us consider the theory of L-decompositions of a von Neumann algebra M. We used the theory of M-ideals in C^*-algebras to show that three reasonable versions of multi-norms that are defined on the pre-dual M_* of M in fact give the same multi-norm - and we regard this as the canonical multi-norm defined on M_*. We continue to examine its properties.

3) Let Γ be a locally compact group. Then the dual of $A(\Gamma)$ is the von Neumann algebra $VN(\Gamma)$ [**?**, **?**]. The above construction, using L-decompositions of $VN(\Gamma)$, gives a multi-norm based on $A(\Gamma)$ that indeed generalizes the standard 1-multi-norm based on $L^1(G)$, where Γ is the dual group to G, in the case where Γ is abelian. We can also specifically identify this multi-norm in the case where Γ is $\mathbb{F}_2$, the free group on two generators, but the nature of this multi-norm in more general cases remains to be explored. We expect that variations of the method will generate an analogue on general Fourier algebras of the standard t-multi-norm based on $L^1(G)$ for each $t \geq 1$.

4) We can show that the new multi-norms do indeed give multi-Banach algebras in some special cases. We expect that further work will show that this is true much more generally.

5 Outcome of the Meeting

The three participants are continuing to work on the mentioned problems, and we expect our work to lead to a publication in due course.

We expect to meet again to continue our work. For example, we shall all attend the next ‘Banach algebra’ conference, to be held at the Fields Institute in Toronto in August 2015. Dales and Lau will also attend the next ‘Harmonic analysis’ conference, to be held in Halifax, Nova Scotia, also in August 2015.

We aspire to return again to the benign location of BIRS to continue our work as a ‘Research in Teams’ next year.

Professor Lau has been appointed as a ‘Distinguished Faculty Visitor’ at Lancaster University for one month in 2016, and we hope to make substantial progress on our work in that month.

It is a pleasure to thank BIRS for the opportunity to advance our work in such a pleasant setting.

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