AFFINE THREEFOLDS ADMITTING G_a-ACTIONS

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R.V. Gurjar, M. Koras, K. Masuda, M. Miyanishi, P. Russell had the wonderful opportunity to meet at BIRS during August 23-30, 2015, for the BIRS event Unipotent Geometry 15frg175. Some portion of our manuscript, including the Introduction of our manuscript below, describes some of the results we proved during our stay. Since we had only one week for discussions the Theorems 2.4 and 2.5 can be considered as the most complete results we proved during our stay. These results were previously not known and we believe that they will be appreciated by other experts interested in G_a actions. We also constructed several examples of G_a action on smooth affine 3-folds. These examples are significant since there is a general paucity of such examples in the literature.

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We still need to work out some details of proofs of results in other parts of the manuscript.

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Abstract. Affine varieties of dimension greater than two can be explored their structures with the help of fibrations by the affine line or plane and quotient morphisms by G_a-actions. We consider G_a-actions on affine threefolds and discuss the structure and the singularities of the quotient surface as well as the singular fibers of the quotient morphism. Relative G_a and G_m-actions on $\mathbb{A}^4$ or on an affine 4-fold having $\mathbb{A}^3$-fibrations over $\mathbb{A}^1$ are discussed when the actions leave one variable invariant or the fibration invariant.

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Introduction

Let X be a smooth affine threefold defined over an algebraically closed field k of characteristic zero. If X has a nontrivial G_a-action, there exists the quotient morphism $q : X \to Y = X//G_a$ and Y has dimension two. Since the G_a-action is defined by a locally nilpotent k-derivation δ on the coordinate ring A of X, the quotient surface is defined by $Y = \operatorname{Spec} B$ with $B = \operatorname{Ker} \delta$. Then there exists an open set U of Y such that $U = \operatorname{Spec} B[1/u]$ with $u \in B$ and $q^{-1}(U) \cong U \times \mathbb{A}^1$. This implies that general fibers of q are G_a-orbits, but q has, in general, singular fibers outside the open set U. In fact, topological or algebro-geometric properties of X are greatly affected by the singular fibers of q and the algebraic quotient surface Y, and *vice versa*. In Theorem 1.1, we consider the case where X is, in addition, factorial and given two independent G_a-actions. Here we say that two G_a-actions on X are independent if the orbits by two G_a-actions have independent tangential directions at some point of X. Then it is shown that the quotient surface Y by one G_a-action is isomorphic to the affine plane $\mathbb{A}^2$ or an affine hypersurface $x^2 + y^3 + z^5 = 0$ which is isomorphic to the quotient $\mathbb{A}^2/\Gamma$, where Γ is the binary icosahedral subgroup of $\mathrm{SL}(2, k)$.

Furthermore, Y is isomorphic to $\mathbb{A}^2$ if and only if X is simply connected. If one weakens the factoriality condition to the $\mathbb{Q}$-factoriality, there are examples for which $Y \cong \mathbb{A}^2/\Gamma$, where Γ is now an arbitrary finite subgroup of $\mathrm{SL}(2, k)$.

Some basic description of singular fibers of the quotient morphism $q : X \to Y$ is given in [23]. Among other things, it is notable that if a singular fiber F_0 of q is one-dimensional, then every irreducible component of F_0 is a contractible curve ($k = \mathbb{C}$), and it is an interesting problem to ask whether it is always isomorphic to the affine line $\mathbb{A}^1$. In Theorem 2.1, it is shown that if an irreducible component has multiplicity one, i.e., it is reduced, then the component is isomorphic to $\mathbb{A}^1$. If the G_a-action is given by a homogeneous locally nilpotent derivation on the polynomial ring $k[x, y, z]$ with a positive grading (*quasi-homogeneous case*), then the quotient morphism $q : \mathbb{A}^3 \to \mathbb{A}^2$, where $X = \mathbb{A}^3$ and $Y = \mathbb{A}^2$ has only the affine line as the (unique) irreducible component of the (unique) singular fiber. This is shown by a detailed analysis of the singular fiber in Lemma 2.2 and Theorem 2.3. In the general case with only assumption that Y be smooth, every irreducible fiber component is isomorphic to $\mathbb{A}^1$ in Theorem 2.5. The singular locus $\mathrm{Sing}(q)$ of the quotient morphism q is also observed. It is shown in Theorem 2.8 that if X and Y are smooth and the G_a-action is analytically reduced everywhere on X then the union of multiple components of q coincide with the fixedpoint locus X^{G_a}.

In the third section, we consider some of well-established results about G_a-actions acting on the affine space defined over k in the relative setting where we add one free variable to the polynomial ring and the G_a-action keeps the additional variable invariant. We consider Kaliman's theorem on a free G_a-action on $\mathbb{A}^3$ and Rentschler's theorem on the normalization of a G_a-action on $\mathbb{A}^2$ in the relative setting (see Kaliman's Theorem and Example 3.5). Kaliman [27] has recently proved that a fixed point free, proper G_a-action on $\mathbb{A}^4$ is a translation after a suitable change of variables. We analyzed this result from our point of view.

In the fourth section, we consider a G_m-action on an affine 4-fold fibered over $\mathbb{A}^1$ whose every fiber is isomorphic to $\mathbb{A}^3$ so that G_m acts on each fiber and the parameter space invariant. Theorem 4.1 shows that the 4-fold is actually trivial, i.e., isomorphic to $\mathbb{A}^3 \times Y$ and the G_m-action is diagonalized.

In the fifth section, we treat a smooth affine surface $V(m, 1)$ with $m \geq 1$ which has an $\mathbb{A}^1$-fibration $\rho : V(m, 1) \to \mathbb{A}^1$ such that $\rho^{-1}(\mathbb{A}^1_*) \cong \mathbb{A}^1_* \times \mathbb{A}^1$ and $\rho^{-1}(0)$ consists of two reduced irreducible components isomorphic to $\mathbb{A}^1$. If $m = 1$, $V(1, 1)$ is isomorphic to the hypersurface

$xy = z^2 - 1$, which is one of the Danielewski surfaces. However, if $m \geq 2$, $V(m, 1)$ is not a hypersurface. In fact $V(m, 1) \not\cong V(m', 1)$ if $m \neq m'$. Meanwhile, we have an isomorphism $V(m, 1) \times \mathbb{A}^1 \cong V(m', 1) \times \mathbb{A}^1$. Hence these surfaces provide a new kind of counterexamples to the cancellation problem (see Theorem 5.3). The surface $V(m, 1)$ is constructed as the quotient variety of a G_m-action with G_m acting on the hypersurface threefold $X(m, 1) = \{xy - z^m u = 1\}$.

In the sixth section, various generalizations of the Nori exact sequence of the fundamental groups of a fibration $f : X \to Y$ with connected fibers such that each fiber contains at least one *reduced* irreducible component (see [43]). This exact sequence is very effective when we use topological arguments for affine varieties. Hence its generalizations have been considered, and the section gathers together those generalizations. Applications of generalized results are also observed.

In the seventh section, we defined an $\mathcal{L}$-twisted additive group scheme $G_{a,\mathcal{L}}$ over a scheme Y for an invertible sheaf $\mathcal{L}$ on Y. This enables us to treat $\mathbb{A}^1$-fibrations (even defined over a complete variety) from the viewpoint of the additive group scheme action. The idea is originally due to Dubouloz [11].

We denote by $R^{[n]}$ a polynomial ring over a ring R in n variables whenever we do not specify variables of the polynomial ring. A locally nilpotent derivation is aften abbreviated as *lnd*.

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1. The quotient space of smooth affine threefolds by G_a-actions

We are interested in a smooth, factorial, affine threefold X with G_a-actions and its quotient surface Y by one of the G_a-actions. Central problems are if Y is smooth or what kind of singularity it admits provided Y is singular. Given two G_a-actions on X, we say that the two actions are *independent* if two respective G_a-orbits passing a point P of X have independent tangential directions. If X admits two independent G_a-actions, then we have the following result.

Theorem 1.1. *Let X be a smooth affine factorial threefold with two independent G_a-actions, say $G_1,\ G_2$. Assume further that $\Gamma(X, \mathcal{O}_X)^* =$*

k^*. Then $X//G_1$ is isomorphic to either $\mathbb{A}^2$ or an affine hypersurface $x^2 + y^3 + z^5 = 0$. Furthermore, X is simply connected if and only if $X//G_1 \cong \mathbb{A}^2$.

Proof. Let $q_1 : X \to Z := X//G_1$ be the quotient morphism. For a general point $p \in Z$, let $C_p = q_1^{-1}(p)$ and Y_p be the closure of $\bigcup_{P \in C_p} G_2 P$. Then $C_p \cong \mathbb{A}^1$ and Y_p is an affine surface equipped with an $\mathbb{A}^1$-fibration $\{G_2 P \mid P \in C_p\}$. Note that $G_2 P = G_2 P'$ for distinct points P, P' when the orbit $G_2 P$ meets C_p in the point P'. But note that $G_2 P \cap G_2 P' \neq \emptyset$ implies $G_2 P = G_2 P'$. Consider the following two cases (i) and (ii) separately.

(i) Suppose that $q_1 |_Y : Y \to Z$ is dominant. Choosing the point p generally on Z, we may assume that $q_1 |_Y$ is quasi-finite[1]. Let Z° be the smooth part of the normal surface Z. Then $q_1^{-1}(Z \setminus Z^\circ)$ is a finite set and lies on a union of finitely many orbits $G_2 P$ with $P \in C_p$. Further, by eliminating finitely many G_2-orbits, we have a dominant morphism from a smooth affine surface with an $\mathbb{A}^1$-fibration to Z°. Hence $\overline{\kappa}(Z^\circ) = -\infty$. If Z^0 has an $\mathbb{A}^1$-fibration, it extends to Z. Otherwise, Z° contains an open set U of the form $(\mathbb{A}^2/\Gamma)^\circ$, where Γ is a finite group, $(\mathbb{A}^2/\Gamma)^\circ$ is the smooth part of $\mathbb{A}^2/\Gamma$ and $Z^\circ \setminus U$ is a disjoint union of the curves isomorphic to $\mathbb{A}^1$ which are called the *half-point attachments*. Hence the curves in $Z^\circ \setminus U$ give rise to the independent classes of $\mathrm{Pic}\,(Z^\circ)$. Meanwhile, since Z is factorial, it follows that $\mathrm{Pic}\,(Z^\circ) = 0$. This implies that $Z^\circ = U \cong (\mathbb{A}^2/\Gamma)^\circ$. Then $Z = \mathbb{A}^2/\Gamma$ because $Z \setminus Z^\circ$ is a finite set. Since Z is factorial, Γ must be the binary icosahedral group of $\mathrm{SL}(2, k)$. Hence Z is an affine hypersurface $x^2 + y^3 + z^5 = 0$.

Suppose that X is simply connected. Let $\rho : \mathbb{A}^2 \to \mathbb{A}^2/\Gamma$ be the quotient morphism by the Γ-action on $\mathbb{A}^2$. Then $\rho^\circ : \mathbb{A}^2_* := \mathbb{A}^2 \setminus \{0\} \to (\mathbb{A}^2/\Gamma)^\circ$ is a universal covering with Galois group Γ. Let $X^\circ := X \setminus q^{-1}(\mathrm{Sing}(\mathbb{A}^2/\Gamma)) = q^{-1}((\mathbb{A}^2/\Gamma)^\circ)$. Note that $q^{-1}(\mathrm{Sing}(\mathbb{A}^2/\Gamma))$ is a closed set of codimension ≥ 2 in X. Hence X° is simply connected. Then $q^\circ := q |_{X^\circ} : X^\circ \to Z^\circ$ is factored by $\mathbb{A}^2_*$ as $q^\circ : X^0 \xrightarrow{\pi^\circ} \mathbb{A}^2_* \xrightarrow{\rho^\circ} Z^0$. Then a general fiber of q (and hence q°) is a disjoint union of as many affine lines as the order of Γ. This is a contradiction. Hence the case $Z \cong \mathbb{A}^2/\Gamma$ does not occur.

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[1]The actions of G_1, G_2 are associated with locally nilpotent derivations δ_1, δ_2. We may replace δ_1, δ_2 by $a_1^{-1}\delta_1, a_2^{-1}\delta_2$ if necessary with $a_1 \in \mathrm{Ker}\,\delta_1, a_2 \in \mathrm{Ker}\,\delta_2$. Thus we may assume that the fixed-point locus of G_1 is contained in $q_1^{-1}(S_1)$ with a finite set S_1 of Z. Similarly, the fixed-point locus is contained in $q_2^{-1}(S_2)$ with a finite set S_2 of $X//G_2$, where $q_2 : X \to X//G_2$ is the quotient morphism. Then, for a general point p of Z, every point P of C_p has the one-dimensional orbit $G_2 P$. If $G_2 P$ is contained in the orbit of q_1, it must be C_p itself. This is impossible.

(ii) Suppose that $q_1 |_Y \colon Y \to Z$ is not dominant. Then the image of $q_1 |_Y$ is a rational curve B_p with one place at infinity which passes through the point p. The assumption implies that for any point $P \in C_p$ the orbit G_2P is mapped surjectively onto B_p. If $B_p \cap B_{p'} \neq \emptyset$ for distinct points p, p' of Z, then $G_2P \cap G_2P' \neq \emptyset$ for $P \in C_p$ and $P' \in C_{p'}$. Since G_2P, G_2P' are the G_2-orbits, it follows that $G_2P = G_2P'$ and hence $B_p = B_{p'}$. Thus the family $\{B_p\}_{p \in Z}$ has no base points. This implies that a general member B_p is smooth, hence isomorphic to $\mathbb{A}^1$. So, Z has an $\mathbb{A}^1$-fibration. Since Z is factorial and $\Gamma(Z, \mathcal{O}_Z)^* = k^*$, it follows that $Z \cong \mathbb{A}^2$.

By the above argument, we have shown that Z is isomorphic to either $\mathbb{A}^2$ or $\mathbb{A}^2/\Gamma$. Finally, we prove that if $Z \cong \mathbb{A}^2$ then X is simply connected. Let S be the closed curve such that for each general point $p \in S$ the fiber $q^{-1}(p)$ is not isomorphic to $\mathbb{A}^1$ in the scheme-theoretic sense. Let S_1 be an irreducible component of S and let f be a prime element of $\Gamma(Z, \mathcal{O}_Z) \cong k^{[2]}$ such that $S_1 = V(f)$. Since f is a prime element of $\Gamma(X, \mathcal{O}_X)$, the surface $T_1 := q^{-1}(S_1)$ is an irreducible surface. Considering the Stein factorization of $q |_{T_1} \colon T_1 \to S_1$ we know that the general fibers consist of reduced irreducible components. Now let Z° be the open set of Z such that for every point $p \in Z^\circ$ the fiber $q^{-1}(p)$ is reduced. Then $Z \setminus Z^\circ$ is a finite set. Let $X^\circ = q^{-1}(Z^\circ)$ and $q^\circ = q |_{X^\circ}$. We apply Nori's exact sequence of the fundamental groups [43] (see Lemma 6.1 of the present article)

$$\pi_1(F) \to \pi_1(X^\circ) \to \pi_1(Z^\circ) \to (1),$$

where F is a general fiber of q°. Since $X \setminus X^\circ$ and $Z \setminus Z^\circ$ have codimension larger than one, we have $\pi_1(X^\circ) = \pi_1(X)$ and $\pi_1(Z^\circ) = \pi_1(Z) = (1)$. Hence $\pi_1(X) = (1)$. □

We do not know if the case $X/\!/G_1 \cong \mathbb{A}^2/\Gamma$ with $\Gamma \neq (1)$ can occur. But if we drop the assumption that X is factorial, such an example exists.

Example 1.2. *Let $\widetilde{X}$ be an affine quadric hypersurface defined by $xz - yu = 1$ in $\mathbb{A}^4 = \mathrm{Spec}\, k[x, y, z, u]$. Then the following assertions hold.*

(1) *$\widetilde{X}$ is a smooth factorial affine threefold with a G_a-action defined by a locally nilpotent derivation $\widetilde{\delta}$ such that*

$$\widetilde{\delta}(x) = \widetilde{\delta}(y) = 0,\ \widetilde{\delta}(z) = y,\ \widetilde{\delta}(u) = x.$$

(2) *Let ι be the involution on $\widetilde{X}$ defined by*

$$\iota(x) = -x,\ \iota(y) = -y, \iota(z) = -z,\ \iota(u) = -u.$$

Then ι has no fixed point on $\widetilde{X}$, and hence the quotient threefold $X := \widetilde{X}/\langle\iota\rangle$ is a smooth affine threefold. Furthermore, $\widetilde{\delta}$ commutes with ι, i.e., $\widetilde{\delta}\iota = \iota\widetilde{\delta}$.

(3) *Let $\widetilde{R} := \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ and let $R = \Gamma(X, \mathcal{O}_X)$. Then R is the ι-invariant subring and $\widetilde{\delta}$ defines a locally nilpotent derivation δ on R.*

(4) *Let $\widetilde{\sigma}$ and σ be the G_a-actions on $\widetilde{X}$ and X respectively defined by $\widetilde{\delta}$ and δ. Then $\widetilde{X}//G_a \cong \mathbb{A}^2$ and $X//G_a \cong \mathbb{A}^2/\Gamma$, where $\Gamma = \mathbb{Z}/2\mathbb{Z}$.*

(5) $\mathrm{Pic}\,(X) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. The assertions (1), (2) and (3) are easy to show. As for the assertion (4), note that $\mathrm{Ker}\,\delta = \mathrm{Ker}\,\widetilde{\delta} \cap R$ and $\mathrm{Ker}\,\widetilde{\delta} = k[x, y]$. Hence $\mathrm{Ker}\,\delta = k[x^2, xy, y^2]$ which is the coordinate ring of $\mathbb{A}^2/\Gamma$ with $\Gamma = \mathbb{Z}/2\mathbb{Z}$.

(5) Let $\mathfrak{p}$ be a prime ideal of height 1 of R. Then $\mathfrak{p}\widetilde{R}$ is a height 1 ideal of $\widetilde{R}$. Hence it is written as $\mathfrak{p}\widetilde{R} = f\widetilde{R}$ with $f \in \widetilde{R}$ since $\widetilde{R}$ is factorial. Since $\iota^*(\mathfrak{p}\widetilde{R}) = \mathfrak{p}\widetilde{R}$, we have $\iota(f) = fh$ with $h \in \widetilde{R}$ such that $\iota(h)h = 1$. Since $\widetilde{R}^* = k^*$, we have $h \in k$ and $\iota(h) = h$. Then $h = \pm 1$. If $h = 1$, then $f \in R$ and $\mathfrak{p} = fR$. If $h = -1$, then $\iota(f) = -f$. In particular, $f^2 \in R$, and if $f_1, f_2 \in \widetilde{R}$ satisfy $\iota(f_1) = -f_1, \iota(f_2) = -f_2$ then $f_1 f_2 \in R$. It follows from these observations that $\mathrm{Pic}(X) \cong \mathbb{Z}/2\mathbb{Z}$, and it is generated by the prime ideal $x\widetilde{R} \cap R$. □

The following example generalizes the above example. Namely, Example 1.2 is the case $n = 2$ in the example below.

Example 1.3. *Let $T_n = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \,\middle|\, \xi^n = 1 \right\}$ be a cyclic subgroup of* $\mathrm{SL}(2)$ *of order n. Let T_n act on* $\mathrm{SL}(2)$ *by the left multiplication and let $X = T_n \backslash \mathrm{SL}(2)$. Let $U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \,\middle|\, t \in k \right\}$ be the additive subgroup acting on* $\mathrm{SL}(2)$ *by the right multiplication. Since the actions of T_n and U commute each other, there is a right G_a-action on X. Let $Y := X//G_a$ and let $q : X \to Y$ be the quotient morphism. Then we have :*

(1) *X is a smooth affine threefold with* $\mathrm{Pic}\,(X) \cong \mathbb{Z}/n\mathbb{Z}$.

(2) *$Y \cong T_n \backslash \mathbb{A}^2$ with the action ${}^{\xi}(x, y) = (\xi x, \xi^{-1} y)$* [2].

[2] To avoid the notation $T_n \backslash \mathbb{A}^2$, we can consider the left action of the lower triangular subgroup $\left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \,\middle|\, t \in k \right\}$ as a G_a-action and the right action of T_n.

Proof. (1) Since X is a homogeneous space, it is clear that X is a smooth affine threefold. The computation of Pic (X) is the same as in Example 1.2. If we write a general matrix of SL(2) as $\begin{pmatrix} x & u \\ y & z \end{pmatrix}$, x is transformed to ξx by the left action of $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ on $\begin{pmatrix} x & u \\ y & z \end{pmatrix}$. Hence the prime ideal $\mathfrak{p} = x\widetilde{R} \cap R$ gives rise to a generator of Pic (X), where $\widetilde{R}$ is the coordinate ring of SL(2) and R is the T_n-invariant subring of $\widetilde{R}$.

(2) The quotient morphism $q : X \to Y$ is induced by the quotient morphism $\widetilde{q} : \mathrm{SL}(2) \to \mathrm{SL}(2)/U$ which is given as $\begin{pmatrix} x & u \\ y & z \end{pmatrix} \mapsto (x, y)$. (Take the quotients of SL(2) and SL(2)$/U$ with respect to the left T_n-multiplications.) Hence $Y \cong T_n\backslash\mathbb{A}^2$. $\square$

By considering other finite subgroups of SL(2), we can produce $\mathbb{A}^2/\Gamma$ as the quotient surface of a smooth affine threefold by a G_a-action, where Γ is any finite subgroup of SL(2). Furthermore, note that $X =$ SL(2) admits other G_a-actions. For example, the right multiplication by the lower triangular matrices $\left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \;\middle|\; t \in k \right\}$ gives another G_a-action. So, the assumption that X be factorial seems to be crucial to conclude that $X//G_1$ is smooth.

Concerning the smoothness of the quotient surface $X//G_a$, the following problem is interesting, where the assumption that X be simply-connected and rational is new.

Problem 1.4. *Let X be a smooth, factorial, simply-connected, rational, affine threefold with a nontrivial G_a-action. Is $Y := X//G_a$ then smooth?*

Proposition 1.5. *With the notations in Problem 1.4, Y is a factorial, rational, affine surface such that*

(1) $Y - \mathrm{Sing}(Y)$ *is simply-connected, and*
(2) *a point $P \in Y$ has quotient singularity at worst of E_8-type provided $q^{-1}(P) \neq \emptyset$.*

Proof. Since X is factorial, the quotient morphism $q : X \to Y$ does not contain codimension 1 fiber components. Hence $q^{-1}(\mathrm{Sing}(Y))$ has codimension larger than 1. This implies that $\pi_1(X - q^{-1}(\mathrm{Sing}(Y))) =$ (1). Let $p : Z \to Y - \mathrm{Sing}(Y)$ be the universal covering. Then the restriction of q onto $X - q^{-1}(\mathrm{Sing}(Y))$ is factored by the mapping p. This implies that p is a finite covering and p is the identity since the general fibers of q are connected. So, $\pi_1(Y - \mathrm{Sing}(Y)) = (1)$. This proves the assertion (1). If $q^{-1}(P) \neq \emptyset$, then the singularity of Y at P is at most quotient singularity. Since $\mathcal{O}_{Y,P}$ is factorial, the singularity is at worst of E_8-type. This verifies the assertion (2). □

When we ask if a factorial, rational, affine surface Y is smooth provided $\pi_1(Y - \mathrm{Sing}(Y)) = (1)$, we have the following counterexample. Note that if we assume additionally that Y is contractible, then Y is smooth by an affine Mumford theorem [21, Theorem 3.6].

Example 1.6. (1) *Let V be an affine surface constructed in* [21, Proposition 3.8]. *Then V is a factorial, rational, affine surface with an E_8-singularity and $\pi_1(V - \mathrm{Sing}(V)) = (1)$. This implies that a factorial, rational, affine surface Y with $\pi_1(Y - \mathrm{Sing}(Y)) = (1)$ is not necessarily smooth.*

(2) *Let a, b, c be mutually coprime positive integers. Then the affine hypersurface $x^a + y^b + z^c = 0$ is factorial, though it has a non-quotient singularity for a suitable choice of a, b, c. We refer to* [41].

As a corollary of Theorem 1.1, we prove the following result.

Theorem 1.7. *Let R be a factorial affine domain of dimension two and let $R[x]$ be a polynomial ring in one variable x over R. Let $X =$ Spec $R[x]$, $Y =$ Spec R and $p : X \to Y$ be the projection. Then the following three conditions are equivalent.*

(1) *The Makar-Limanov invariant* $\mathrm{ML}(X)$ *is equal to k. Namely, there are three independent G_a-actions on X.*
(2) *Y is isomorphic to $\mathbb{A}^2$.*
(3) *X is isomorphic to $\mathbb{A}^3$.*

Proof. It suffices to show that the condition (1) implies the condition (2). The existence of three independent G_a-actions on X implies that there is a dominant morphism from $G_a \times G_a \times G_a$ to X. Hence the unit group of $R[x]$, which is equal to R^*, is k^*. Consider an lnd δ of $R[x]$ such that $\delta(R) = 0$ and $\delta(x) = 1$. Then δ gives rise to a G_a-action along the fibers of the projection p. Namely, p is the quotient morphism under the G_a-action. Note that $R[x]$ is factorial. By Theorem 1.1, the surface Y is isomorphic to the affine plane $\mathbb{A}^2$ or the hypersurface $x^2+y^3+z^5 = 0$ in $\mathbb{A}^3$. Suppose that Y is isomorphic to the hypersurface $x^2 + y^3 + z^5 = 0$. Since the singularity of Y is not a cyclic singularity, $\mathrm{ML}(Y) = R$ by [39]. By Crachiola and Makar-Limanov [7], we then have $\mathrm{ML}(X) = \mathrm{ML}(Y) = R$. This contradicts the assumption. Hence Y is isomorphic to $\mathbb{A}^2$. □

Generalizing Theorem 1.7, we raise the following problem.

Problem 1.8. *Let R be an affine domain of dimension n over k and let $R[x]$ be a polynomial ring in a variable x over R. Suppose that* $\mathrm{ML}(R[x]) = k$. *Is* $\mathrm{ML}(R)$ *then equal to k?*

But the problem has negative answers. We give two counterexamples. In the first example, $\dim R = 2$ but R is not factorial. In the second example, R is factorial but $\dim R = 3$. This question was treated, for example, by Bandman-Makar-Limanov [5].

Example 1.9. *For an integer $n \geq 1$, let S_n be the Danielewski surface $x^n y = z^2 - 1$. Then* $\mathrm{ML}(S_n) = k$ *if $n = 1$ and* $\mathrm{ML}(S_n) = k[x]$ *if $n \geq 2$ [38]. Then $S_1 \times \mathbb{A}^1 \cong S_n \times \mathbb{A}^1$ for $n > 1$ and* $\mathrm{ML}(S_1 \times \mathbb{A}^1) = \mathrm{ML}(S_n \times \mathbb{A}^1) = k$, *but* $\mathrm{ML}(S_n) = k[x]$ *if $n > 1$.*

The following example is due to Dubouloz [13], which is also a counterexample to the conjecture in [5, p.209].

Example 1.10. *Let X be Koras-Russell cubic threefold $x^2 y + x + z^2 + t^3 = 0$, which is a hypersurface in $\mathbb{A}^4$. Let R be the coordinate ring of X. Then R is factorial,* $\mathrm{ML}(X \times \mathbb{A}^1) = k$ *and* $\mathrm{ML}(X) = k[x]$.

2. Singular fibers of the quotient morphism by a G_a-action

In [23, Lemma 3.5], we proved that if X is a smooth factorial affine threefold with a G_a-action and the quotient morphism $q : X \to Y := X/\!/G_a$, a singular fiber is a disjoint union of contractible curves, and asked if the singular fiber is indeed a disjoint union of the affine lines. In the present section, we prove three results on the smoothness of fiber components of the morphism q, Theorem 2.1, Theorem 2.3 and Theorem 2.5 among which the last result is the most general and contains the former two as partial cases. We think that the arguments used in the proofs of these results have independent interest. This is why we do not throw the first two results and retain only the last one. We prove first the following result.

Theorem 2.1. *Let X be a smooth factorial affine threefold with a G_a-action. Assume that the quotient surface $Y := X/\!/G_a$ is isomorphic to the affine plane. Let F be a singular fiber of the quotient morphism $q : X \to Y$ and let F_0 be an irreducible component with multiplicity 1, i.e., a reduced component. Then F_0 is isomorphic to the affine line.*

Proof. (1) Let $B := \Gamma(Y, \mathcal{O}_Y) = k[f, g]$, where $f - \alpha, g - \beta$ are prime elements in $A := \Gamma(X, \mathcal{O}_X)$ for all $\alpha, \beta \in k$. Let $Q = q(F)$. We may assume that $f = 0$ at Q. Let $L = \{f = 0\}$ in Y and let $Z = q^{-1}(L)$.

Then Z is an irreducible and reduced surface with the induced G_a-action. A general fiber of $q|_Z\colon Z \to L$ is not necessarily irreducible. If reducible, by a linear change of coordinates f, g on Y, we may assume that L passes through a point of Y which is the image of a general fiber of q. Then a general fiber of $q|_Z$ is isomorphic to $\mathbb{A}^1$. Then the morphism $q_Z := q|_Z\colon Z \to L$ is the quotient morphism of Z relative to the induced G_a-action.

(2) We consider a singular fiber F on the affine surface Z. Though Z is not necessarily normal, it has the quotient curve $L = \operatorname{Spec} k[g]$ with F defined by $g = 0$. Let F_0 be a reduced irreducible component of F. We then replace Z by $(Z \setminus F) \cup F_0$, which is an affine surface with the induced G_a-action. Suppose that F_0 is not isomorphic to $\mathbb{A}^1$. Then F_0 is a G_a-stable, singular, contractible curve. Hence F_0 is contained in the fixed-point locus Z^{G_a}. Let δ be the locally nilpotent derivation on the ring $R := \Gamma(Z, \mathcal{O}_Z)$. Since F_0 is defined by $g = 0$ and $F_0 \subseteq Z^{G_a}$, the induced locally nilpotent derivation on R/gR is trivial. In fact, every maximal ideal $\mathfrak{m}$ of R/gR is mapped into $\mathfrak{m}$ itself. If we fix a maximal ideal $\mathfrak{m}$, every element $a \in R/gR$ has some constant $\alpha \in k$ such that $a - \alpha \in \mathfrak{m}$. Then $\delta(a) = \delta(a - \alpha) \in \mathfrak{m}$. So, $\delta(a) \in \cap_m \mathfrak{m}$, where $\mathfrak{m}$ runs through over the set of all maximal ideals of R/gR. Since R/gR is reduced by assumption, we have $\delta(a) \in \cap_m \mathfrak{m} = (0)$. This implies that $\delta(R) \subset gR$. Since $g \in \operatorname{Ker} \delta$, $g^{-1}\delta$ is a locally nilpotent derivation of R such that the associated G_a-action has the same quotient morphism $q_Z : Z \to L$. If F_0 is still contained in the fixed-point locus Z^{G_a}, we repeat the same process. But it is impossible that δ is divisible by g infinitely many times. Hence we eventually reach to the situation $F_0 \not\subseteq Z^{G_a}$. Then F_0 must be isomorphic to $\mathbb{A}^1$. This is a contradiction. Hence F_0 has no singular points, and hence isomorphic to $\mathbb{A}^1$. $\square$

We consider the following quasi-homogeneous case. We need some observations.

Lemma 2.2. *We assign weights a, b, c to the variables x, y, z of a polynomial ring $k[x, y, z]$, where a, b, c are pairwise coprime positive integers. Suppose that δ is a homogeneous locally nilpotent derivation with respect to this grading. Let F, G be the generators of* $\operatorname{Ker}\delta$, *i.e.,* $\operatorname{Ker}\delta = k[F, G]$. *Then we have:*

(1) *We may assume that*

$$\delta = \Delta_{(F,G)} := \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix} \frac{\partial}{\partial x} + \begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix} \frac{\partial}{\partial y} + \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} \frac{\partial}{\partial z} .$$

Furthermore, the coefficients of $\Delta_{(F,G)}$ has no non-constant common divisors in $k[x, y, z]$.

(2) *The fixed-point locus* $(\mathbb{A}^3)^{G_a}$ *has dimension* ≤ 1.
(3) *Let* C *be a singular curve which is a fiber component of the quotient morphism* $q : X \to Y = X//G_a$, *where* $X = \mathbb{A}^3$ *and* $Y = \operatorname{Spec} k[F, G]$. *Then* $C = (\mathbb{A}^3)^{G_a}$. *Hence such a curve* C *is uniquely determined.*
(4) *Let* $\pi : X \setminus \{(0, 0, 0)\} \to \Pi$ *be the quotient morphism by the* G_m-*action induced by the grading, where* Π *is the weighted projective plane. Let* $V_+(F)$ *and* $V_+(G)$ *be the curves defined by* $F = 0$ *and* $G = 0$ *in the weighted projective plane* Π. *Let* Q_0 *be the unique point of intersection* $V_+(F) \cap V_+(G)$ (see [9, Theorem] or [19, Theorem 5.28]). *Then* $\pi^{-1}(Q_0)$ *is contained in* $(\mathbb{A}^3)^{G_a}$. *If* C *is a singular curve as in the assertion (3), then* $C = \pi^{-1}(Q_0) \cup \{(0, 0, 0)\}$.

Proof. (1) In [8] or [19, Theorem 5.6], it is shown that the derivation $\Delta_{(F,G)}$ is locally nilpotent, and $\delta = H\Delta_{(F,G)}$ for $H \in \operatorname{Ker} \delta$. So, we can take δ to be $\Delta_{(F,G)}$. If A is a common divisor of the coefficients of $\Delta_{(F,G)}$, then $A \in \operatorname{Ker} \delta$ [3] and $A^{-1}\delta$ is a locally nilpotent derivation giving rise to the same kernel as $\operatorname{Ker} \delta$. Then by the cited result [*ibid.*], we have $A^{-1}\delta = B\delta$ with $\delta = \Delta_{(F,G)}$ and $B \in \operatorname{Ker} \delta$. Hence $AB = 1$ and A is a constant.

(2) Suppose that $\dim(\mathbb{A}^3)^{G_a} = 2$. Then $(\mathbb{A}^3)^{G_a}$ contains an irreducible component of codimension one which is defined by an element H of $\operatorname{Ker} \delta$. Since the defining ideal of $(\mathbb{A}^3)^{G_a}$ is the ideal generated by the coefficients of $\Delta_{(F,G)}$, the polynomial H is a common factor of the coefficients of $\Delta_{(F,G)}$. This is a contradiction to the assertion (1).

(3) We know that C is contractible by [23]. Let P_0 be a singular point of C. Then $P_0 \in (\mathbb{A}^3)^{G_a}$ because C is G_a-stable and the singular point of C does not move under the G_a-action. If P is a point of C other than P_0, the G_a-orbit G_aP contains P_0 if $\dim G_aP = 1$, whence P_0 would be a smooth point. Hence $C \subseteq (\mathbb{A}^3)^{G_a}$. Furthermore, the curve C is G_m-stable with respect to the G_m-action associated to the positive grading. In fact, the assumption that the locally nilpotent derivation δ is quasi-homogeneous derivation of degree d implies that there is a $G_a \rtimes G_m$-action on X, where $(0, \lambda^{-1})(t, 1)(0, \lambda) = (\lambda^d t, 1)$ for $\lambda \in G_m$ and $t \in G_a$. Hence, if $P \in (\mathbb{A}^3)^{G_a}$, we have $(t, 1)(0, \lambda)P = (0, \lambda)(0, \lambda^{-1})(t, 1)(0, \lambda)P = (0, \lambda)(\lambda^d t, 1)P = (0, \lambda)P$. This implies that the point $(0, \lambda)P$ is a G_a-fixed point. If $(0, \lambda)P \notin C$, then the G_m-translate G_mC will give a surface after taking its closure.

[3]Suppose that $\Delta_{(F,G)} = AD$ for a derivation D. There exists then an element $\xi \in k[x, y, z]$ such that $\Delta_{(F,G)}(\xi)$ is a nonzero element of $\operatorname{Ker} \Delta_{(F,G)}$. Then a factor A of an element of $\operatorname{Ker} \Delta_{(F,G)}$ is also in the same kernel.

This leads to a common factor of the coefficients of $\Delta_{(F,G)}$ which is a contradiction by the assertion (1). Hence $(0,\lambda)P \in C$. So, C is G_m-stable. Since the closure of a G_m-orbit can have a singular point only at the origin $(0,0,0)$, the point P_0 is the origin $(0,0,0)$. If $(\mathbb{A}^3)^{G_a}$ contains another irreducible component, say C', then C' is a curve passing through the origin and an irreducible component of the fiber $q^{-1}(q(C))$. But the irreducible components are connected components in the fiber $q^{-1}(q(C))$. Hence there are no other irreducible components in $(\mathbb{A}^3)^{G_a}$ other than C.

(4) Let $P \in \pi^{-1}(Q_0)$. Then $F((t,1)P) = (t,1)^*F(P) = F(P) = 0$ because $F \in \mathrm{Ker}\delta$. Similarly, $G((t,1)P) = 0$. Hence $(t,1)P \in \pi^{-1}(Q_0)$. If $(t,1)P \neq P$, then there is a dominant morphism $\mathbb{A}^1 \to \pi^{-1}(Q_0) \cong \mathbb{A}^1_*$. This is a contradiction. Hence $(t,1)P = P$ and $P \in (\mathbb{A}^3)^{G_a}$. □

Theorem 2.3. *With the notations and assumptions as in Lemma 2.2, we further assume that the integers a,b,c in the triple (a,b,c) are greater than 1. Then there are no singular curves which are fiber components of the quotient morphism $q : X \to Y$.*

Proof. We make essential use of the following two facts.

1. Π is a projective normal surface at worst with cyclic quotient singularities (see [10, Proposition 1.3.3]). Let Q_1, Q_2, Q_3 be the vertices of Π, i.e., $Q_1 = \pi(1:0:0), Q_2 = \pi(0:1:0), Q_3 = \pi(0:0:1)$. Then these three points are singular points of Π.
2. $\Pi \setminus (V_+(F) \cup V_+(G))$ is isomorphic to $\mathbb{P}^2 \setminus (\ell_1 \cup \ell_2)$ (see [9, Theorem] or [19, Theorem 5.28]).

We assume that there exists a singular curve C which is an irreducible component of the quotient morphism q as in Lemma 2.2 and show that this assumption leads to a contradiction. By (2) above, the singular points of Π lie on the curves $V_+(F) \cup V_+(G)$. Let $V = \Pi \setminus (V_+(F) \cup V_+(G))$. By (2) above, V is an affine surface with an $\mathbb{A}^1$-fibration. In fact, $V \cong \mathbb{A}^1 \times \mathbb{A}^1_*$. Note that F and G are irreducible polynomial in $k[x,y,z]$. In fact, suppose that $F = F_1F_2$. Then $F_1, F_2 \in k[F,G]$ because $\mathrm{Ker}\,\delta$ is factorially closed in $k[x,y,z]$. We can write $F_1 = \Phi_1(F,G)$ and $F_2 = \Phi_2(F,G)$. Then $F = \Phi_1(F,G)\Phi_2(F,G)$ with two non-constant polynomials $\Phi_1(F,G), \Phi_2(F,G)$. This is a contradiction. Hence the curves $V_+(F)$ and $V_+(G)$ are irreducible. If $Q_0 \notin \{Q_1,Q_2,Q_3\}$, then two of Q_1,Q_2,Q_3 lie on one of $V_+(F), V_+(G)$. Suppose $Q_1, Q_2 \in V_+(F)$. Consider an affine surface $W = \Pi \setminus V_+(G)$. Then W is a normal affine surface with an $\mathbb{A}^1$-fibration whose fibers are all irreducible. But this is a contradiction because an irreducible fiber component of an A^1-fibration on a normal affine surface can carry at most one cyclic singular point of the surface by [39]. Hence $Q_0 \in \{Q_1, Q_2, Q_3\}$. This implies that $C = \pi^{-1}(Q_0) \cup \{(0,0,0)\}$ is a line and hence $C \cong \mathbb{A}^1$. This is a contradiction. □

Now we will prove the following general result which has applications to the study of singular fibers of the quotient morphism $X \to X//G_a$, where X is a smooth affine threefold and $X//G_a$ is a smooth affine surface. This result is much more general than Theorem 2.1 and Theorem 2.3.

Theorem 2.4. *Let $f : V \to Y$ be a projective morphism from a smooth threefold V onto a smooth surface Y. Assume that a general fiber of f is isomorphic to $\mathbb{P}^1$. For a point $P \in Y$, let C be a one-dimensional component of the fiber $F_P := f^{-1}(P)$. Then C is isomorphic to $\mathbb{P}^1$.*

Proof. It is proved in Kollàr [33, p. 107, (2.8.6)] that with f, V, Y as above we have $R^i f_*(\mathcal{O}_V) = 0$ for $i > 0$. We can assume that Y is affine. Then, by a standard spectral sequence argument, we have $H^i(V, \mathcal{O}_V) = 0$ for $i > 0$.

Now we will use the Theorem on Formal Functions [25, Chapter III, Theorem 11.1].

Let $\mathcal{O}$ be the local ring of Y at P and let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. As usual let $V_n := V \times_Y \mathrm{Spec}\,(\mathcal{O}/\mathfrak{m}^n)$ and $\mathcal{O}_n := \mathcal{O}/\mathfrak{m}^n$, considered as the structure sheaf of V_n. For each $n \geq 0$ we have natural morphisms $R^i f_*(\mathcal{O}_V) \otimes \mathcal{O}/\mathfrak{m}^n \to H^i(V_n, \mathcal{O}_n)$. As n varies we have two inverse systems, inducing a natural morphism which is an isomorphism by the Theorem of Formal functions

$$R^i f_*(\mathcal{O})^\wedge_P \xrightarrow{\sim} \varprojlim H^i(V_n, \mathcal{O}_n).$$

For $i > 0$ the LHS is 0, hence we get $\varprojlim H^i(V_n, \mathcal{O}_n) = 0$.

We write $V_n = U_{1n} \cup U_{2n}$, where U_{1n} is a suitable open neighborhood in V_n of the union of all one-dimensional components of V_n and U_{2n} is an open neighborhood in V_n of the union of all the irreducible components of dimension > 1. We can assume that $U_{1n} \cap U_{2n}$ is a disjoint union of connected (non-reduced) Stein spaces.

We denote the sheaves of abelian groups $\mathcal{O}_{1n}, \mathcal{O}_{2n}$ on U_{1n}, U_{2n} resp. which are just restrictions of $\mathcal{O}_n$ to U_{1n}, U_{2n} resp.

In this siuation there is a Mayer-Vietoris sequence (obtained by using sheaves of discontinuous sections to obtain cohomology) [1, p. 236]

$$\cdots \to H^1(V_n, \mathcal{O}_n) \to H^1(U_{1n}, \mathcal{O}_{1n}) \oplus H^1(U_{2n}, \mathcal{O}_{2n}) \\ \to H^1(U_{1n} \cap U_{2n}, \mathcal{O}_n) \to \cdots .$$

Since $U_{1n} \cap U_{2n}$ is a disjoint union of finitely many connected Stein spaces, the last cohomology group is trivial. As n varies, we get Mayer-Vietoris sequences with maps from the groups in the $(n+1)$ st sequence to the corresponding groups in the n th sequence making all the diagrams commute.

Since $\varprojlim H^i(V_n, \mathcal{O}_n) = 0$, we deduce that $\varprojlim H^i(U_{1n}, \mathcal{O}_{1n}) = 0$. Let I_n be the ideal sheaf of U_{1n} in $U_{1(n+1)}$.

¿From the exact sequence $0 \to I_n \to \mathcal{O}_{1(n+1)} \to \mathcal{O}_{1n} \to 0$, and using the fact that U_{1n} is a non-compact 2-dimensional complex space without 2-dimensional compact components so that $H^2(U_{1(n+1)}, I_n) = (0)$, we get that the natural maps $H^1(U_{1(n+1)}, \mathcal{O}_{n+1}) \to H^1(U_{1n}, \mathcal{O}_n)$ are surjections for each n. Since $\varprojlim H^i(U_{1n}, \mathcal{O}_{1n}) = 0$ we deduce that $H^1(U_{1n}, \mathcal{O}_n) = 0$ for $n > 0$.

The reduced curve $C \subset F_p$ is a closed subscheme of U_{1n} for each n. By the same argument as above we deduce that $H^1(U_{1n}, \mathcal{O}_n) \to H^1(C, \mathcal{O}_C)$ is a surjection. This means that $H^1(C, \mathcal{O}_C) = 0$, proving that $C \cong \mathbb{P}^1$. □

As a corollary, we obtain the following result.

Theorem 2.5. *Let G_a act on a smooth affine threefold X. Let Y be the affine variety corresponding to the ring of invariants. Assume that Y is smooth* [4]*. Let $q : X \to Y$ be the induced morphism. If C_0 is a one-dimensional component of a fiber of π, then $C_0 \cong \mathbb{A}^1$.*

Proof. We can embed $X \subset V$ as a Zariski-open subvariety where V is a smooth quasi-projective threefold such that π extends to a proper morphism $f : V \to Y$. Then the closure C of C_0 in X is a one-dimensional component of the fiber containing C_0. As already mentioned, C_0 is contractible. By Theorem 2.4, we have $C \cong \mathbb{P}^1$. It follows that $C_0 \cong \mathbb{A}^1$. □

Remark 2.6. It seems reasonable to conjecture that Theorem 2.4 (Theorem 2.5) is true for $\mathbb{P}^1$-fibrations (resp. $\mathbb{A}^1$-fibrations) on n-folds for $n > 3$. □

Let $q : X \to Y$ be an $\mathbb{A}^1$-fibration of normal affine varieties. Let Sing(q) be the set of points $P \in Y$ such that the fiber F_P, i.e., the

[4] This condition is not a serious restriction. Since $A := \Gamma(X, \mathcal{O}_X)$ is regular, Ker δ is normal. Hence Y has only isolated singular points. If one consider the fiber F_P and its fiber component C_0 over a smooth point P of Y, we replace Y by a smooth affine open neighborhood U of P and replace X by $q^{-1}(U)$. Since q is an affine morphism, the restriction $q|_{q^{-1}(U)} : q^{-1}(U) \to U$ satisfies the conditions of the theorem.

scheme-theoretic inverse image $X \times_Y \operatorname{Spec} k(P)$ is not isomorphic to $\mathbb{A}^1$, where $k(P)$ is the residue field of P in Y. We call $\operatorname{Sing}(q)$ the *singular locus* or *degeneracy locus* of q. We do not know in general if $\operatorname{Sing}(q)$ is a closed set in Y. However we know the following partial result.

Lemma 2.7. (1) *Assume that q is a flat morphism and $\overline{Y - q(X)}$ has codimension greater than one in Y. Then either $\operatorname{Sing}(q) = \emptyset$ or has pure codimension one. If $\operatorname{Sing}(q) = \emptyset$ then $q : X \to q(X)$ is an $\mathbb{A}^1$-bundle.*

(2) *Assume that q is the quotient morphism of a smooth affine threefold X equipped with a G_a-action. Suppose that q is equi-dimensional. Then $\operatorname{Sing}(q)$ is a closed set. Furthermore, if the fiber F_P contains a reduced irreducible component, then the point P is smooth in Y.*

Proof. For (1), see [23, Lemma 1.15], and for (2), see [22, Lemma 3.1]. □

Hereafter until Proposition 2.9, we assume that $X = \operatorname{Spec} A$ is a smooth affine variety of dimension n equipped with a G_a-action σ which corresponds to an lnd δ of A and that $q : X \to Y$ is the quotient morphism by σ, hence $B := \Gamma(Y, \mathcal{O}_Y) = \operatorname{Ker} \delta$ is finitely generated over k. Assume that the quotient morphism $q : X \to Y := X//G_a$ has equi-dimension one. The fixed point locus X^{G_a} is then a union of fiber components of q by [23, Corollary 3.2], which is valid for all $n \geq 2$. An irreducible fiber component E is a *multiple* component of q if the Artin local ring $\mathcal{O}_{F_P,\xi}$ is not a field, where ξ is the generic point of E. The length of $\mathcal{O}_{F_P,\xi}$ is the *multiplicity* of E.

Given the G_a-action σ on $X = \operatorname{Spec} A$, let δ be the corresponding lnd of A. If Q is a point of X then δ induces the k-derivation δ_Q of the local ring $\mathcal{O}_{X,Q}$ as well as the k-derivation $\widehat{\delta}_Q$ of the completion $\widehat{\mathcal{O}}_{X,Q}$. We say that σ (or δ) is *analytically reducible* at Q if $\widehat{\delta}_Q = h\Delta$ for an element h of the maximal ideal $\widehat{\mathfrak{m}}_{X,Q}$ and a k-derivation Δ of $\widehat{\mathcal{O}}_{X,Q}$. Otherwise we call δ is *analytically reduced* at Q. The fixed point locus X^{G_a} is defined by the ideal I of A generated by $\delta(A) := \{\delta(a) | a \in A\}$. Write δ_Q as

$$\delta_Q = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n},$$

where $\{x_1, \ldots, x_n\}$ is a regular system of parameters of $\mathcal{O}_{X,Q}$. Then the following conditions are equivalent.

(i) $Q \in X^{G_a}$.
(ii) $f_1(Q) = \cdots = f_n(Q) = 0$.
(iii) $f_1, \ldots, f_n \in I\mathcal{O}_{X,Q}$.

(iv) $(f_1, \ldots, f_n)\mathcal{O}_{X,Q} = I\mathcal{O}_{X,Q}$.

(v) $(f_1, \ldots, f_n)\widehat{\mathcal{O}}_{X,Q} = I\widehat{\mathcal{O}}_{X,Q}$.

We say that σ is *reducible* if $\delta = h\delta'$ for a k-derivation δ' of A. If this is the case, $h \in \operatorname{Ker}\delta$ and δ' is an lnd (see [19, p.33]). Otherwise, δ is *reduced.* If $\delta_Q = h\delta'$ with $h \in \mathfrak{m}_{X,Q}$ and a k-derivation δ' of $\mathcal{O}_{X,Q}$, we say that δ_Q is reducible. This is equivalent to saying that $I\mathcal{O}_{X,Q}$ has a height one prime divisor since $\mathcal{O}_{X,Q}$ is factorial. Similarly, $\widehat{\delta}_Q$ is analytically reducible if and only if $I\widehat{\mathcal{O}}_{X,Q}$ has a height one prime divisor. Then δ is reducible at Q if and only if $\widehat{\delta}_Q$ is analytically reducible. In fact, the "only if" part is clear. For the "if" part, suppose that $I\widehat{\mathcal{O}}_{X,Q}$ has a height one prime divisor $\widehat{\mathfrak{p}}$. Since $\widehat{\mathcal{O}}_{X,Q} = k[[x_1, \ldots, x_n]]$, where we may assume that $x_1, \ldots, x_n \in \mathfrak{m}_{X,Q}$, we have $\widehat{\mathfrak{p}} = (\widehat{h})$ with $\widehat{h} \in k[[x_1, \ldots, x_n]]$. By the Weierstrass Preparation Theorem it follows that $\widehat{h} = hu$ with $h \in k[x_1, \ldots, x_n]$ and a unit $u \in k[[x_1, \ldots, x_n]]$. Then $h \in I\widehat{\mathcal{O}}_{X,Q} \cap \mathcal{O}_{X,Q} = I\mathcal{O}_{X,Q}$ and $f_i h^{-1} = (f_i\widehat{h}^{-1})u \in \widehat{\mathcal{O}}_{X,Q} \cap Q(\mathcal{O}_{X,Q}) = \mathcal{O}_{X,Q}$ for every i. Hence $I\mathcal{O}_{X,Q}$ has a height one prime divisor and δ is reducible at Q.

Consider the following conditions.

(1) δ is reduced.

(2) δ is reduced at every closed point Q of X.

(3) δ is analytically reduced at every closed point Q of X.

The conditions (2) and (3) are equivalent by the previous argument, and they are equivalent to the condition that I has no height one prime divisors. Hence the conditions (2) and (3) imply the condition (1). If X is factorial, the three conditions are equivalent.

Theorem 2.8. *Every multiple component E of q is contained in the fixed point locus X^{G_a}. Conversely, assume that $n := \dim X = 3$ and Y is smooth. If σ is analytically reduced everywhere on X, then X^{G_a} is the union of multiple components of q.*

Proof. Let $Q \in X$ be a closed point and let $(\mathcal{O}, \mathfrak{m})$ be the local ring $\mathcal{O}_{X,Q}$. Then Q is a G_a-fixed point if and only if $\delta(\mathfrak{m}) \subset \mathfrak{m}$. If $Q \notin X^{G_a}$, then there exists an element $x \in \mathfrak{m}$ such that $\delta(x)$ is a unit of $\mathcal{O}$. Let $\widehat{\mathcal{O}}$ be the $\mathfrak{m}$-adic completion of $\mathcal{O}$. By the well-known result of Zariski, we have $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_0[[x]]$. This implies that the morphism $q : X \to Y$ is complex-analytically a product $Z \times \mathbb{C}$ near the point Q, where Q corresponds to the point $(Q_0, 0)$ with $Q_0 \in Z$ and $0 \in \mathbb{C}$ such that the local ring of Z at Q_0 is $\widehat{\mathcal{O}}_0$. Hence the germ (Z, Q_0) is analytically isomorphic to $(Y, q(Q))$. This shows that the fiber F_P with $P := q(Q)$ is reduced at the point Q. Taking the contrapositive of the assertion, we conclude that if the fiber F_P is a multiple fiber then $\delta(\mathfrak{m}) \subset \mathfrak{m}$. Namely, $Q \in X^{G_a}$.

Conversely, let $Q \in X^{G_a}$ and let $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_{X,Q}$. Then $\widehat{\delta}_Q(\widehat{\mathfrak{m}}) \subset \widehat{\mathfrak{m}}$, where $\widehat{\mathfrak{m}}$ is the maximal ideal of $\widehat{\mathcal{O}}$. Then $\widehat{\mathcal{O}}$ is not smooth over $\widehat{\mathcal{O}}_0 := \mathrm{Ker}\, \widehat{\delta}_Q$. Indeed, otherwise $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_0[[t]]$ with a fiber parameter t. Hence $\widehat{\mathcal{O}}_0$ is a complete regular local ring, i.e., $\widehat{\mathcal{O}}_0 = k[[u_1, \ldots, u_{n-1}]]$ and $\widehat{\delta}_Q = h(\partial/\partial t)$ with $h \in \widehat{\mathfrak{m}}$. Then δ is analytically reducible at Q. This contradicts the hypothesis. So, the component of $q : X \to Y$ passing through Q is not reduced, for otherwise the component is reduced and isomorphic to $\mathbb{A}^1$ by Theorem 2.5, hence q is smooth at Q. here we used the assumption that $n = 3$ and Y is smooth. This implies that Q is in a multiple component. □

Let $X = \mathbb{A}^3$. Given a G_a-action σ on $\mathbb{A}^3$, it is known by [40] that $Y := \mathbb{A}^3/\!/G_a$ is isomorphic to $\mathbb{A}^2$. The G_a-action σ is said to be *triangularizable* if the corresponding lnd is written as

$$\delta = a(x)\frac{\partial}{\partial y} + b(x, y)\frac{\partial}{\partial z}$$

with respect to a suitable system of variables $\{x, y, z\}$, where $a(x) \in k[x]$ and $b(x, y) \in k[x, y]$. Assume that δ is nontrivial and reduced. Let $q : X \to Y$ be the quotient morphism, which is equi-dimensional and surjective by [6]. If either $a(x)$ or $b(x, y)$ is zero then $b(x, y) \in k^*$ or $a(x) \in k^*$ and f is a trivial $\mathbb{A}^1$-bundle. Hence we exclude these cases in the following result and assume that both $a(x)$ and $b(x, y)$ are nonzero polynomials.

Proposition 2.9. *With the above notations and assumptions we have the following assertions.*

1. $\mathrm{Ker}\,\delta = k[\xi, \eta]$, *where* $\xi = x$ *and* $\eta = a(x)z - \int b(x, y)dy$.
2. $\mathrm{Sing}(q)$ *is defined by* $a(x) = 0$. *Hence* $\mathrm{Sing}(q)$ *consists of parallel lines if it is not the empty set.*
3. X^{G_a} *is defined by* $a(x) = b(x, y) = 0$.

Proof. It is clear that $k[\xi, \eta] \subseteq \mathrm{Ker}\,\delta$. Let $Z = \mathrm{Spec}\, k[\xi, \eta]$ and let $p : X \to Z$ (resp. $\pi : Y \to Z$) be the morphism defined by the inclusion $k[\xi, \eta] \hookrightarrow A$ (resp. $k[\xi, \eta] \hookrightarrow \mathrm{Ker}\,\delta$). Then the quotient morphism q factors p as $p = \pi \circ q$. Let $U = \{a(x) \neq 0\}$ be the open set of Z. Then $p : p^{-1}(U) \to U$ is an $\mathbb{A}^1$-bundle because δ extends to an lnd of $A[a(x)^{-1}]$ for which $ya(x)^{-1}$ is a slice. This implies that $\pi : \pi^{-1}(U) \to U$ is an isomorphism. Hence π is birational. Let $\alpha \in k$ be a root of $a(x) = 0$ and let L_α be the line $\xi = \alpha$ on Z. Let $\widetilde{b}(x, y) = \int b(x, y)dy$. Let $\beta \in k$. Then $p^{-1}(\alpha, \beta) = \coprod C(\alpha, \gamma)$, where $C(\alpha, \gamma)$ is the line

$\{(\alpha, \gamma, z) | z \in k\}$ and γ runs over the roots of $\widetilde{b}(\alpha, y) + \beta = 0$. Hence the morphism $\pi : Y \to Z$ is quasi-finite over $\coprod L_\alpha$, where α runs over the roots of $a(x) = 0$. By Zariski Main Theorem, π is an isomorphism. Hence $\operatorname{Ker} \delta = k[\xi, \eta]$. The line $C(\alpha, \gamma)$ is a multiple component if and only if γ is a multiple root of $\widetilde{b}(\alpha, y) + \beta = 0$, i.e., $b(\alpha, \gamma) = 0$. The above observations verify all the assertions. □

The following result is somewhat surprising.

Lemma 2.10. *Let X be a smooth affine threefold with a G_a-action such that $Y := X//G_a$ is isomorphic to a quotient $\mathbb{A}^2/\Gamma$, where Γ is a non-trivial finite group of linear automorphisms of $\mathbb{A}^2$ without non-trivial pseudo-reflections. Let $q : X \to Y$ be the quotient morphism. Let P be the unique singular point of $\mathbb{A}^2/\Gamma$. Assume that q is an $\mathbb{A}^1$-bundle over $Y \setminus \{P\}$. Then $q^{-1}(P)$ cannot contain a one-dimensional component.*

Proof. Only finitely many fibers of q can contain divisorial components. If X_0 is the complement in X of the union of these two-dimensional fiber components then X_0 is again affine, G_a-stable and $X_0//G_a = Y$. In view of this we can assume that no fiber of q contains a two-dimensional component. By Theorem 2.5 the fiber over P is a disjoint union of curves isomorphic to $\mathbb{A}^1$. We assume that $q^{-1}(P) \neq \emptyset$.

Consider the natural morphism $\mathbb{A}^2 \to \mathbb{A}^2/\Gamma$ which is unramified outside P. Let X' be the normalized fiber product $X \times_Y \mathbb{A}^2$. Then the G_a-action extends to a G_a-action on X' such that $X'//G_a = \mathbb{A}^2$. The quotient morphism $q' : X' \to \mathbb{A}^2$ has the property that outside one point P' in $\mathbb{A}^2$ it is an $\mathbb{A}^1$-bundle. By a result of A.K. Dutta [16], q' is an $\mathbb{A}^1$-bundle. This is a contradiction since the inverse image of P' has at least two irreducible components since $\mathbb{A}^1$ is simply-connected. □

We will observe below various examples of the quotient morphism $q : X \to Y := X//G_a$, where X is a smooth affine threefold with a G_a-action.

Example 2.11. Let X be as in Example 1.2. Then the quotient surface Y is isomorphic to $\mathbb{A}^2/\Gamma$ with $\Gamma = \mathbb{Z}/2\mathbb{Z}$, and the quotient morphism q is induced by the projection $\widetilde{q} : \mathrm{SL}(2) \to \mathbb{A}^2$ defined by

$$\begin{pmatrix} x & u \\ y & z \end{pmatrix} \mapsto (x, y)$$

which is the quotient morphism $\mathrm{SL}(2) \to \mathrm{SL}(2)/G_a$ with G_a acting on $\mathrm{SL}(2)$ from the right via the upper triangular unipotent matrices. Then $\widetilde{q}^{-1}(0, 0)$ is the empty set. Hence the fiber of q over the singular point of Y is the empty set. The other fibers of q are all isomorphic to $\mathbb{A}^1$. □

Example 2.12. Let X be a smooth hypersurface in $\mathbb{A}^4 = \mathrm{Spec} k[x, y, z, u]$ defined by $xu - y^2 z = y$. Then X has a G_a-action defined by an lnd

$$\delta = x\frac{\partial}{\partial z} + y^2\frac{\partial}{\partial u}.$$

Then $\mathrm{Ker}\, \delta = k[x, y]$ and the quotient morphism $q : X \to \mathbb{A}^2$ is given by $(x, y, z, u) \mapsto (x, y)$. Hence $q^{-1}(0, 0) = \mathbb{A}^2 = \mathrm{Spec}\, k[z, u]$ and $q^{-1}(\alpha, \beta) \cong \mathbb{A}^1$ if $(\alpha, \beta) \neq (0, 0)$. The ideal $I = (y, u)A$ is G_a-invariant, where $A = \Gamma(X, \mathcal{O}_X)$, and SL(2) is obtained from an affine transformation of X with respect to the ideal I and an element y. □

Example 2.13. Let X be the Koras-Russell threefold $x + x^2 y + z^2 + t^3 = 0$. Since every G_a-action on X makes x invariant, there are two independent G_a-actions which correspond to the following lnds:

$$\delta_1 = -2z\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}$$
$$\delta_2 = -3t^3\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial t}.$$

The quotient morphism is given by $q_1 : X \to \mathbb{A}^2 = \mathrm{Spec}\, k[x, t]$ and $q_2 : X \to \mathbb{A}^2 = \mathrm{Spec}\, k[x, z]$. Then we have

$$q_1^{-1}(\alpha, \beta) = \begin{cases} \mathbb{A}^1 & \text{if } \alpha \neq 0 \\ \mathbb{A}^1 + \mathbb{A}^1 & \text{if } \alpha = 0,\ \beta \neq 0 \\ 2\mathbb{A}^1 & \text{if } \alpha = \beta = 0 \end{cases}$$
$$q_2^{-1}(\alpha, \beta) = \begin{cases} \mathbb{A}^1 & \text{if } \alpha \neq 0 \\ \mathbb{A}^1 + \mathbb{A}^1 + \mathbb{A}^1 & \text{if } \alpha = 0,\ \beta \neq 0 \\ 3\mathbb{A}^1 & \text{if } \alpha = \beta = 0 \end{cases}$$

□

We give one example in the case $\dim X = 4$. It is due to Winkelmann [47].

Example 2.14. Let $X = \mathbb{A}^4 = \mathrm{Spec}\, k[x_1, x_2, x_3, x_4]$ equipped with a G_a-action defined by

$$\delta = x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} + (x_2^2 - 2x_1 x_3 - 1)\frac{\partial}{\partial x_4}.$$

Then Ker $\delta = k[\xi_1, \xi_2, \xi_3, \xi_4]$, where

$$\begin{aligned}\xi_1 &= x_1\\ \xi_2 &= x_2^2 - 2x_1x_3\\ \xi_3 &= x_1x_4 - x_2(x_2^2 - 2x_1x_3 - 1)\\ \xi_4 &= x_1x_4^2 - 2x_2x_4(x_2^2 - 2x_1x_3 - 1) + 2x_3(x_2^2 - 2x_1x_3 - 1)^2\end{aligned}$$

and $Y := X//G_a$ is a hypersurface $\xi_1\xi_4 = \xi_3^2 - \xi_2(\xi_2 - 1)^2$ in $\mathbb{A}^4 = \mathrm{Spec}\, k[\xi_1, \xi_2, \xi_3, \xi_4]$. Then Y has a unique singular point $(\xi_1, \xi_2, \xi_3, \xi_4) = (0, 1, 0, 0)$, and the fiber $q^{-1}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is given as follows:

$$\begin{cases} \mathbb{A}^1 & \text{if} \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \neq (0, 1, 0, 0)\\ \mathbb{A}^2 + \mathbb{A}^2 & \text{if} \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 1, 0, 0)\end{cases}$$

□

3. Relative fixpoint free G_a-actions on $\mathbb{A}^4$ over $\mathbb{A}^1$

Let $X = \mathrm{Spec}\, A$ be an affine variety defined over a field k of characteristic zero which we assume to be not necessarily algebraically closed in this section. We say that X is a *k-form of* $\mathbb{A}^3$ if there exists a finite algebraic extension K/k such that $X_K := \mathrm{Spec}\, A \otimes_k K$ is K-isomorphic to $\mathbb{A}^3_K$. Let $\sigma : G_a \times X \to X$ be a G_a-action which corresponds to an lnd δ of A. Then the action σ is fixpoint free if and only if the ideal $\sum_{a \in A} \delta(a)A$ is the unit ideal. Now we prove the following result.

Lemma 3.1. *Let $X = \mathrm{Spec}\, A$ be a k-form of $\mathbb{A}^3$ equipped with a fixed point free G_a-action. Then X is k-isomorphic to $\mathbb{A}^3$.*

Proof. With the above notations, let $B = \mathrm{Ker}\delta$. Then $A_K := A \otimes_k K$ has an lnd $\delta_K := \delta \otimes_k K$ such that $\sum_{a \in A_K} \delta_K(a)A_K = A_K$. Since $X_K \cong \mathbb{A}^3_K$, it follows that $B_K := B \otimes_k K = \mathrm{Ker}\, \delta_K$ is isomorphic to $K[x, y]$. In fact, this is so over an algebraic closure $\overline{K}$ of K by [40]. Let $Y = \mathrm{Spec}\, B$. Then Y is a k-form of $\mathbb{A}^2$. Then $Y \cong \mathbb{A}^2_k$ by Kambayashi [31]. Let $q : X \to Y$ be the morphism defined by the inclusion $B \hookrightarrow A$, which is the quotient morphism by the action σ. Since $q_{\overline{K}} : \mathbb{A}^3_{\overline{K}} \to Y_{\overline{K}}$ is an $\mathbb{A}^1$-bundle as the action $\sigma_{\overline{K}}$ is a translation by Kaliman [26], the morphism $q : X \to Y$ is an $\mathbb{A}^1$-bundle. Since $Y \cong \mathbb{A}^2_k$ and an $\mathbb{A}^1$-bundle over $\mathbb{A}^2$ is trivial, we have $X \cong \mathbb{A}^3$ and the action σ is a translation. □

Let $X = \mathrm{Spec}\, A$ be an affine variety over k equipped with a G_a-action σ. We say that σ is a *translation* if $X = Y \times \mathbb{A}^1$, i.e., $A = B[x]$, and the action σ is given by $t \cdot b = b$ for every element $b \in B$ and $t \cdot x = x + c_0t$ with $c_0 \in B \setminus \{0\}$. The argument in the above proof implies the following result.

Corollary 3.2. *Let X be an affine variety defined over k equipped with a fixed point free G_a-action $\sigma : G_a \times X \to X$. Let K/k be a finite algebraic extension such that $\sigma_K : G_a \times X_K \to X_K$ is a translation. Then σ is a translation.*

Proof. Let $X = \text{Spec}\, A$ and let δ be the lnd associated with the action σ. Let $B = \text{Ker}\,\delta$. Then we have $B_K := B \otimes_k K = \text{Ker}\,\delta_K$ with $\delta_K = \delta \otimes_k K$. By the assumption, we have $A_K = B_K[x]$ for a variable x. Let $Y = \text{Spec}\, B$ and let $q : X \to Y$ be the quotient morphism. Then $q_K : X_K \to Y_K$ is an $\mathbb{A}^1$-bundle, hence $q : X \to Y$ is an $\mathbb{A}^1$-bundle as in the proof of Lemma 3.1. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open covering of Y such that $U_i = \text{Spec}\, B_i$ and $q^{-1}(U_i) = \text{Spec}\, B_i[x_i]$. For every pair (i, j), we have $x_j = b_{ji} x_i + c_{ji}$, where $b_{ji} \in \Gamma(U_i \cap U_j, \mathcal{O}_Y^*)$ and $c_{ji} \in \Gamma(U_i \cap U_j, \mathcal{O}_Y)$. The lnd δ extends to an lnd (denoted by the same letter) of $B_i[x_i]$. Let $s_i = \delta(x_i)$, which is an element of B_i since $\delta^2(x_i) = 0$. Then we have $s_j = b_{ji} s_i$. Hence $\{s_i\}_{i \in I}$ determines a section in $\Gamma(Y, \mathcal{L}^{-1})$, where $\mathcal{L}$ is an invertible sheaf on Y determined by transition functions $\{b_{ji}\}$ with respect to $\mathcal{U}$. Note that G_a acts on each fiber of q without fixed points. This implies that $s_i \in B_i^*$ and hence $\mathcal{L} \cong \mathcal{O}_Y$. Replacing x_i by $s_i x_i$, we may assume that $x_j = x_i + c_{ji}$. Hence the obstruction $\{c_{ji}\}$ for q to be trivial lies in $H^1(Y, \mathcal{O}_Y)$. Since Y is affine, it follows that $X \cong Y \times \mathbb{A}^1$. This implies that σ is a translation. $\square$

We are interested in generalizing Kaliman's theorem on a fixpoint-free G_a-action on $\mathbb{A}^4$ with one coordinate invariant. Kaliman's theorem states that a fixpoint-free G_a-action on $\mathbb{A}^3 = \text{Spec}\, k[x, y, z]$ is a translation. Namely, after a suitable change of coordinates, the G_a-action is given by $t \cdot (x, y, z) = (x, y, z + t)$. Recently, Kaliman [27] proved the following result.

Kaliman's Theorem. *Let $X := \mathbb{A}^4 = \text{Spec}\, k[u, x, y, z]$ and let $\sigma : G_a \times X \to X$ be a proper, fixpoint-free action preserving one of the coordinates, say u, invariant. Then σ is a translation.*

The G_a-action σ is called *proper* if the morphism $\Phi : G_a \times X \to X \times X$ defined by $(t, Q) \mapsto (\sigma(t, Q), Q)$ is a proper (hence finite) immersion. The example of Winkelmann (see Example 2.14) is given by a triangular lnd

$$\delta = u\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (x^2 - 2uy - 1)\frac{\partial}{\partial z}.$$

But the action is not a translation. Otherwise, the quotient threefold $Y := X//G_a$ exists and is isomorphic to $\mathbb{A}^3$. However, as shown in

Example 2.14, Y has a unique singular point. So, Winkelmann's G_a-action is not proper. If the action σ is furthermore triangularizable, the above theorem of Kaliman was proved by Dubouloz-Finston-Jaradat [15]. A crucial point of their arguments is stated as follows. Let $\mathcal{Y}$ be the geometric quotient of $\mathbb{A}^4$ by G_a in the category of algebraic spaces (or Deligne-Mumford stacks). In fact, $\mathcal{Y}$ exists by a general theory. The algebraic quotient $Y := \mathbb{A}^4/\!/G_a$ exists as $\operatorname{Spec}\operatorname{Ker}\delta$ since the ring $\operatorname{Ker}\delta = k[u,x,y,z]^{G_a}$ is finitely generated over k provided u is G_a-invariant [3]. If the G_a-action is fixpoint-free, triangular and proper, it is shown that the canonical morphism $\pi : \mathcal{Y} \to \mathbb{A}^1 = \operatorname{Spec} k[u]$ has all fibers isomorphic to $\mathbb{A}^2$ and the quotient morphism $\rho : \mathbb{A}^4 \to \mathcal{Y}$ is an $\mathbb{A}^1$-bundle in the étale topology of $\mathcal{Y}$. A key result is to show that $\mathcal{Y} \cong \mathbb{A}^4/\!/G_a$ and it is an $\mathbb{A}^2$-bundle over $\mathbb{A}^1$ in the Zariski topology. Hence $\operatorname{Ker}\delta = k[u,x,y]$ after a suitable change of coordinates.

Mostly for a technical reason, we need to assume that the ground field k has infinite transcendence degree over the algebraic closure of the prime field $\mathbb{Q}$. So, we assume that k is the complex field $\mathbb{C}$.

Lemma 3.3. *Consider a fixpoint-free G_a-action on $X := \mathbb{A}^4$. Assume that the G_a-action fixes invariant a variable u of $A := \Gamma(X, \mathcal{O}_X)$. Then there exists $w \in k[u]$ such that $A_w := A[1/w]$ is G_a-isomorphic to $k[u]_w[x,y,z]$ with a G_a-action $t \cdot (u,x,y,z) = (u,x,y,z+t)$.*

Proof. Consider the given G_a-morphism $f : X = \mathbb{A}^4 \to \mathbb{A}^1 = \operatorname{Spec} k[u]$. Let $g : \operatorname{Spec} k[u,x,y,z] \to \mathbb{A}^1 = \operatorname{Spec} k[u]$ be the G_a-morphism with the action as a translation $t\cdot(u,x,y,z) = (u,x,y,z+t)$. We will compare these two G_a-actions. By Kaliman's theorem [26], the fibers of f and g over any closed point of $\operatorname{Spec} k[u]$ are G_a-isomorphic.

By the Generic Equivalence Theorem [37] (where we use the assumption that k has infinite transcendence degree over $\overline{\mathbb{Q}}$), there exists a finite Galois extension $L \supset K = k(u)$ and a G_a-isomorphism over L

$$L \otimes_{k[u]} A \xrightarrow{\sim} L[x,y,z].$$

Let Γ be the Galois group of L/K. We have

$$(K \otimes_{k[u]} A)^{G_a} = K \otimes_{k[u]} A^{G_a}$$

and

$$(L \otimes_{k[u]} A)^{G_a} = L \otimes_{k[u]} A^{G_a} = L \otimes_K (K \otimes_{k[u]} A^{G_a}).$$

Since $L \otimes_{k[u]} A^{G_a} \cong L[x,y]$ and forms of $K[x,y]$ are trivial [31], there exist $x_1, y_1 \in (K \otimes_{k[u]} A)^{G_a}$ such that $(K \otimes_{k[u]} A)^{G_a} = K[x_1,y_1] \cong K^{[2]}$. Let z_1 be a variable for $L \otimes_{k[u]} A$ over $L[x_1,y_1]$. For $\gamma \in \Gamma$, $\gamma(z_1) = a_\gamma z_1 + b_\gamma$ with $a_\gamma \in L^*$ and $b_\gamma \in L[x_1,y_1]$. By Hilbert's Theorem 90, the multiplicative version, we have $\alpha_\gamma = a(a^\gamma)^{-1}$ for some $a \in L^*$ and all $\gamma \in \Gamma$. Free to replace z_1 by az_1, we can assume $a_\gamma = 1$ for all γ. Then $b_\gamma = b - b^\gamma$ for some $b \in L[x_1, y_1]$ by Hilbert's Theorem 90, the additive version. (Note that Γ acts on the coefficients of b only.) Free to replace z_1 by $z_1 + b$, we can assume $z_1 \in K \otimes_{k[u]} A$, i.e., $K \otimes_{k[u]} A = K[x_1, y_1, z_1]$. We have $t \cdot z_1 = z_1 + bt$ with $b \in K^*$. Since we can replace z_1 by $z_1 b^{-1}$, we can assume $t \cdot z_1 = z_1 + t$.

By the above argument, there exists $w \in k[u]$ such that $A_w := A[1/w]$ is G_a-isomorphic to $k[u]_w[x, y, z]$. After multiplying variables by powers of w if necessary, we have $A_w = k[u]_w[x_1, y_1, z_1]$ with $x_1, y_1, z_1 \in A$, x_1, y_1 fixed by G_a and $t \cdot z_1 = z_1 + ct$ for $c \in (k[u]_w)^*$. □

Remark 3.4. Let X and the G_a-action be the same as in Lemma 3.3. Let $Y := X//G_a$ and let $q : X \to Y$ be the quotient morphism. *Suppose that q is a flat morphism.* Then $B := \Gamma(Y, \mathcal{O}_Y)$ is a regular, factorial, affine domain of dimension three with $B^* = k^*$ and $B[w^{-1}] = k[u, w^{-1}, x_1, y_1]$ with $w \in k[u]$. Hence the partial derivatives $(\partial/\partial x_1)$ and $(\partial/\partial y_1)$ multiplied by powers of w extend to the lnds δ_1 and δ_2 which define independent G_a-actions σ_1 and σ_2 on Y. By the definition, σ_1 and σ_2 commute. Furthermore, it is easy to show that Y is simply connected. In fact, any topological covering $\widetilde{Y} \to Y$ factors the morphism q as $X \to \widetilde{Y} \to Y$ because $X = \mathbb{A}^4$ is simply connected. Since the general fibers of q is irreducible and reduced, $\widetilde{Y} \to Y$ is birational and hence biregular.

Let $R_i = \operatorname{Ker} \delta_i$ and $Z_i = \operatorname{Spec} R_i$ for $i = 1, 2$. By Theorem 1.1, it follows that $Z_1 \cong Z_2 \cong \mathbb{A}^2$. Since $\delta_i(u) = 0$ for $i = 1, 2$, the quotient morphisms $q_i : Y \to Z_i$ are morphisms over $\mathbb{A}^1 = \operatorname{Spec} k[u]$. Since σ_1 and σ_2 commute, σ_2 induces a G_a-action on Z_1 such that $Z_1//G_a = \operatorname{Spec} k[u]$. This is the case with Z_2. Hence there exists a morphism $\rho := (q_1, q_2) : Y \to Z_1 \times_{\mathbb{A}^1} Z_2 \cong \mathbb{A}^3$. By Zariski's Main Theorem, ρ is an isomorphism if and only if two $\mathbb{A}^1$-fibrations q_1 and q_2 share no fiber components.

As a matter of fact, the Koras-Russell threefold, which we denote by Y for the sake of consistency of the notations, is a smooth factorial affine threefold equipped with two commuting G_a-actions (cf. Example 2.13). The quotients of Y by these G_a-actions are isomorphic to $\mathbb{A}^2$. Hence there is a birational morphism $\rho : Y \to \mathbb{A}^3$. But the quotient morphisms q_1 and q_2 share fiber components. So, we can pose here a question. *Does there exists a dominant morphism form $\mathbb{A}^4$ to the Koras-Russell threefold?* In other words, *under the present situation where Y is obtained as the quotient threefold $\mathbb{A}^4//G_a$, does the phenomenon of sharing fiber components of both $\mathbb{A}^1$-fibrations occur?*

The quotient morphism $q : X \to Y$ is equi-dimensional by the flatness condition. By Theorem 2.8, q has no multiple components. It seems that all reducible fibers of q are disjoint unions of irreducible components isomorphic to $\mathbb{A}^1$. Suppose, for example, that $Y \cong \mathbb{A}^3$. For any point P of Y, we take a general linear plane H passing through P. Then $X_H := q^{-1}(H)$ is a smooth affine threefold admitting a fixed point free G_a-action and having H as the quotient by the induced G_a-action. By Theorem 2.5, the fiber $q^{-1}(P)$ is a disjoint union of the $\mathbb{A}^1$. Suppose that $\mathrm{Sing}(q) \neq \emptyset$. Then $\mathrm{Sing}(q)$ has pure codimension one by Lemma 2.7, (1). Let S be an irreducible component of $\mathrm{Sing}(q)$ passing through P. Since Y is factorial, S is defined by $s = 0$ with $s \in B := \Gamma(Y, \mathcal{O}_Y)$. Since s is a prime element of A as well, all irreducible components of $q^{-1}(P)$ are contained in an irreducible subvariety $q^{-1}(S)$ of codimension one in X. Namely, each irreducible component of $q^{-1}(P)$ should have a moduli in $X = \mathbb{A}^4$. □

The following example shows that Rentschler's theorem does not hold in the relative case. The theorem states that if δ is a locally nilpotent derivation on a polynomial ring $k[x, y]$, then, after a suitable change of variables, we have $\mathrm{Ker}\,\delta = k[x]$ and the associated G_a-action is given by $t \cdot (x, y) = (x, y + f(x)t)$ with $f(x) \in k[x]$.

Example 3.5. Let δ be a locally nilpotent derivation on $A = k[x, y, z]$ defined by

$$\delta = -2z\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

Then the associated G_a-action on $\mathbb{A}^3 = \mathrm{Spec}\,A$ is given by $t \cdot (x, y, z) = (x, y - 2zt - xt^2, z + xt)$, and the G_a-invariant subring is $A^{G_a} = k[x, xy + z^2]$. Let $q : \mathbb{A}^3 \to \mathbb{A}^2 = \mathrm{Spec}\,A^{G_a}$. Then q has a fiber of multiplicity 2 over the point of origin $x = xy + z^2 = 0$. If Rentschler's theorem holds with this example, one can choose variables x, y, z so that $A^{G_a} = k[x, y]$ and the G_a-action is given by $t \cdot z = z + f(x, y)t$ with $f(x, y) \in k[x, y]$. Then the quotient morphism has no multiple fibers. □

Remark 3.6. In Example 3.5, the G_a-fixed point locus coincides with the multiple component $\{x = z = 0\}$. If the G_a-action on $\mathbb{A}^3$ is fixed-point free, then one can take $\mathrm{Ker}\delta = k[x, y]$ and $t \cdot Z = z + t$ by Kaliman [26] after a suitable change of coordinates $\{x, y, z\}$ with x unchanged. So, the relative Rentschler's Theorem holds. *Does it hold if the G_a-action is proper?* □

4. Relative effective G_m-actions on $\mathbb{A}^4$ over $\mathbb{A}^1$

It is not clear to us how to generalize the results of the previous section to more general G_a-actions since not much appears to be known about their deformations. Actions of G_m, however, are very rigid and a very complete answer can be given. We will tacitly assuume $k = \mathbb{C}$ in some places. It is not difficult to remove this assumption *a posteriori*, see [36].

Theorem 4.1. *Let $X = \operatorname{Spec} A$ be a smooth affine variety and*

$$q : X \to Y := \mathbb{A}^1 = \operatorname{Spec} k[u]$$

a morphism with every closed fiber isomorphic to $\mathbb{A}^3$. Suppose that X admits an effective relative G_m-action with trivial action on Y. Then X is equivariantly isomorphic to $\mathbb{A}^3 \times Y$ with diagonal G_m-action on $\mathbb{A}^3$.

Proof. The G_m-action on X has a fixpoint, in fact at least one in each fiber of q. As is well known, the induced action on the tangent space at a fixpoint is diagonalizable with weights independent of the fixpoint chosen, see [32]. The induced actions on the fibers of q are diagonalizable, see [30], and it follows that the weights are the same for all fibers. We denote them by a, b, c. We assume $a \geq b \geq c$. We will have to distinguish three cases.

(1) $a, b, c \geq 0$.
(2) $a > 0, b = 0, c < 0$.
(3) $a > 0, b > 0, c < 0$.

Let $Z = \operatorname{Spec} k[\xi, \eta, \zeta] \cong \mathbb{A}^3$ be the G_m-variety with ξ, η, ζ given weights a, b, c. We will compare q to the G_m-morphism $q' : Z \times Y \to Y$. By the Generic Equivalence Theorem [37] (where we again use the assumption that k has infinite transcendence degree over $\overline{\mathbb{Q}}$) and the theorem on the absence of non-trivial forms of G_m-actions on $\mathbb{A}^3$ [36], we obtain:

(∗) there exists a dense open set $U \subset Y$ such that $q^{-1}(U)$ is G_m-isomorphic with $Z \times U$.

Since $q^{-1}(U)$ is factorial and closed fibers of q are irreducible, we have by Nagata's lemma that

(∗∗) A is factorial.

(I) Assume that we have case (1) above. Then the G_m-action on X is fixpointed and X is a vector bundle over the fixpoint set $T = X^{G_m}$, see [4] and [32]. We consider the case $a > 0, b = c = 0$, the other cases are similar, and easier. Consider $\pi = q|_T : T \to Y$. By (∗) we have

$\pi^{-1}(U) \cong \mathrm{Spec} k[\eta,\zeta]\times U$. Moreover, for each $P \in Y$ we have $\pi^{-1}(P) \cong \mathrm{Spec}\, k[\eta,\zeta]$. By Sathaye's theorem [46] we obtain $T \cong \mathrm{Spec}\, k[\eta,\zeta]\times Y$ and $X \cong \mathrm{Spec}\, k[\eta,\zeta]\times Y \times \mathbb{A}^1$, with action of G_m on the last factor.

(II) Assume that we have case (2) above. We consider X^+ (resp. X^-), the locus of $Q \in X$ such that $\lim_{t\to 0} t\cdot Q, t \in G_m$ (resp. $\lim_{1/t\to 0} t\cdot Q, t \in G_m$) exists. [5] Note that in $Z \times Y$ the $(+)$- and $(-)$-locus are defined by $\zeta = 0$ and $\xi = 0$ respectively. By $(*)$ we can find $x, y, z \in A$ so that $q^{-1}(U) = \mathrm{Spec}\, S[x,y,z]$, S a localization of $k[u]$, and $q^{-1}(U)^+$ and $q^{-1}(U)^-$ defined by $z = 0$ and $x = 0$. The zero locus of z on X has X^+ as an irreducible component. Any other irreducible component will be a fiber $u = \lambda$ of q, i.e., we have $z \in (u-\lambda)A$ and can replace z by $z/(u-\lambda)$. We may assume therefore that X^+ is the zero-locus of z and X^- the zero-locus of x. Finally, note that $\eta \in k[u,\xi,\eta,\zeta]^{G_m} \subset k[u,\xi,\eta,\zeta]$ is the defining equation for the fixpoint set inside the quotient. By $(*)$ we can assume that y defines $q^{-1}(U)^{G_m}$ inside $q^{-1}(U)//G_m$. Appealing once more to $(*)$ and Sathaye's theorem we find that $q^{-1}(U)//G_m \cong \mathbb{A}^3$. If the zero-locus of y in $X//G_m$ has a a component other than X^{G_m}, it will be a fiber of $X//G_m \to Y$ and y is divisible by some $u-\lambda$ in A^{G_m}, hence in A. Again we may assume this does not happen. Now in each $A/(u-\lambda)A$ the images $\overline{x}, \overline{y}, \overline{z}$ of x, y, z define, respectively, the $(-)$-locus, the fixpoint set inside the quotient and the $(+)$-locus. They then generate $A/(u-\lambda)A$. It follows that $A = k[u,x,y,z]$.

(III) Assume that we have case (3) above.

We have $X^+ \cong \mathrm{Spec}\, k[u]\times\mathbb{A}^2$ and $X^- \cong \mathrm{Spec}\, k[u]\times\mathbb{A}^1$ with $X^{G_m} \cong \mathrm{Spec}\,(k[u])$. We have:

> $X^+ = \mathrm{Spec}\, k[A/zA]$, where $z \in \mathbb{A}$ is an irreducible semi-invariant of negative weight, unique up to a constant.

As in $(*)$ we can find homogeneous $\xi,\eta,\zeta \in A$ of weights $a \geq b > 0$ and $c < 0$ and $h \in k[u], h \neq 0$ so that

$$A_h = k[u, h(u)^{-1}, \xi, \eta, \zeta].$$

After factoring out factors from $k[u]$ we can assume

$$\zeta = z.$$

We now follow the line of argument in [35, Proposition 1.8] to reconstruct X from $X_1 = \mathrm{Spec}\, A/(z-1)A$ and the G_m-action. Note that X_1 is irreducible and has a $\Omega := \mathbb{Z}/c\mathbb{Z}$-action relative to $\mathrm{Spec}\,(k[u])$.

[5] It is straightforward to give a purely algebraic definition of X^+ and X^-.

CLAIM 1: $X_1 \cong \mathbb{A}^3$.

In fact, consider $k[u] \subset B = A/(z-1)A$. We have $B_h \cong k[u]_h^{[2]}$ and $B/(u-\lambda)B \cong k^{[2]}$ for all $\lambda \in k$. The result follows from Sathaye's theorem [46].

CLAIM 2: We have an Ω-isomorphism

$$\varphi : X_1 = \operatorname{Spec} B \to \operatorname{Spec} k[u, x, y],$$

where Ω acts trivially on u and with weights a, b modulo c on x, y. Any such φ induces a G_m-isomorphism $\Phi : X \setminus X_1 = \operatorname{Spec} A_z \to \operatorname{Spec} k[u, v, w, z, z^{-1}]$, where u, v, w, z have weights $0, a, b, c$.

In fact, φ exists by CLAIM 1 and since finite group actions on $\mathbb{A}^3$ that fix a variable are linearizable [37]. Φ is defined by

$$\Phi(t \cdot p) = t \cdot \varphi(p) \text{ for } p \in X_1.$$

See [35, 36] for details.

To simplify notation, we identify $k[u, x, y]$ with a subring of B and $k[u, v, w, z]$ with a subring of A_z. We can, and will, arrange x and y so that they vanish on $X^- \cap X_1$. This is automatic if $a \neq 0$ mod c and $b \neq 0$ mod c.

CLAIM 3: For each point $p_0 \in \operatorname{Spec} k[u]$ there exists a Zariski open neighborhood U so that $q^{-1}(U)$ is equivariantly isomorphic to $U \times \mathbb{A}^3$.

By an equivariant version of the theorem of Bass-Connell-Wright [2], proving this claim will prove the result.

We assume for simplicity that $u(p_0) = 0$ in the proof of CLAIM 3. We have $p_0 \in X^{G_m} = X^+ \cap X^-$. By [34] there exists a G_m-invariant analytic open neighborhood of p_0 in X analytically and equivariantly isomorphic to an analytic $\mathbb{C}^*$-neighborhood of $0 \in \mathbb{C}^4$ with linear action of $\mathbb{C}^*$. We may assume that the isomorphism is given by u, F, G, z, where F, G are homogeneous analytic functions on X of weights a, b. We can assume that $x_0 = F|_{X^+}, y_0 = G|_{X^+}$ are algebraic on X^+, where u, x_0, y_0 are homogeneous coordinates on $X^+ = \operatorname{Spec}(A/z)$. Moreover $u, f = F|_{X_1}, g = G|_{X_1}$ are local analytic coordinates at $p_1 = q^{-1}(p_0) \cap X^-$.

Let the notation be as in CLAIM 2. Let us consider $x = x(u, f, g)$ and $y = y(u, f, g)$ as power series in u, f, g, homogeneous of weights a and b mod c. Let t be a parameter for G_m. Then v and w are characterized by

$$t^a v = x(u, t^a f, t^b g) \text{ and } t^b w = y(u, t^a f, t^b g).$$

Consider a term $u^\ell f^m g^n$ appearing in x (resp. y). We have $am + bn = a$ mod c (resp. $am + bn = b$ mod c). We obtain a contribution $u^\ell F^m G^n z^s$ to v (resp. w) as series in u, F, G, z, where $am + bn - a = -cs$ (resp. $am + bn - b = -cs$). Negative powers of z occur only if $am + bn < a$ (resp. $am + bn < b$). For v this occurs for terms $u^\ell g^n$ with $bn < a$, and since $b \leq a$ this does not occur for w.

The linear forms of u, x, y are linearly independent. Since x, y vanish for u in a neighbourhood of 0, those for x, y are free of u. Write them as

$$l_1 = a_{11} f + a_{12} g, l_2 = a_{21} f + a_{22} g.$$

We obtain:

CLAIM 4: (i) w is a power series in u, F, G, z with linear form

$$L_2 = a_{21} F + a_{22} G \text{ in case } a = b \text{ and } a_{22} G \text{ otherwise.}$$

(ii) L_2 depends on F, G only and w is regular at p_0 if and only if $L_2 \neq 0$.

If f and g have different weights mod c, i.e., if $a \neq b$ mod c, then $a_{12} = a_{21} = 0, a_{11} \neq 0, a_{22} \neq 0$. On the other hand, if $a = b$ mod c, we can change φ by any linear automorphism of $\mathbb{A}^3$ that fixes u. So we can arrange (with a new choice of y if necessary) that $a_{22} \neq 0$.

Since w is homogeneous for G_m we obtain:

CLAIM 5: w is regular and $dw \neq 0$ at each point of a G_m-invariant Zariski-neighborhood W of p_0 in X, where we can assume $W = q^{-1}(U)$ with U an open Zariski-neighborhood of 0 in $Spec(k[u])$.

Write $U = \text{Spec}\,(R), R = k[u, 1/h], h \in k[u], h(0) \neq 0$, and put $D = A_h$, $K = R[w, z] \subset D$. We will show that $D = K^{[1]}$.

CLAIM 6: w is a variable in D/z, i.e., $D/z = R[w]^{[1]}$.

In fact, let $p_0' \in U$. We can repeat the above discussion with new local analytic coordinates $u', F', G', z, u'(p_0')$ and u', f', g'. We can consider D/z, with Spec $(D/z) = W^+$, as $\text{Sym}_R(I/I^2)$, where I is the ideal in D/z of the fixpoint set W^{G_m}. By CLAIM 5 and and CLAIM 4 (applied to the new analytic coordinates) we find that w induces a nowhere zero homogeneous element of the (free) R-module I/I^2.

Let an overline denote images mod $u - \lambda, \lambda = u(p'_0)$. Then $\overline{w}$ is obtained via the procedure in CLAIM 2 by setting $u = \lambda$. We have $\overline{w} \in \overline{D}$ and $d\overline{w} \neq 0$ at each point of $q^{-1}(p_0) \cong \mathbb{A}^3$ by CLAIM 5 and CLAIM 4. As explained in [35, 1.8, 1.10], we can, by a modification of $\overline{\varphi}$ that keeps ly unchanged, find a homogeneous $v' \in \overline{D}$ so that $\overline{D} = k[v', \overline{w}, z]$. We therefore have:

CLAIM 7: For each maximal ideal $\mathfrak{m} = (u - \lambda)R \in \text{Spec}\,(R)$ we have $D/\mathfrak{m}D = k[\overline{w}, z]^{[1]}$.

Consider $K \subset D$ and $S = \{1, z, z^2, \ldots\}$. Using CLAIM 2 we can write $S^{-1}D = (S^{-1}K)[\tau]$ with $\tau \in D$ and homogeneous. Using CLAIM 6 and CLAIM 7 it is straightforward to verify that there exist homogeneous $\kappa \in K$ and $\sigma \in S$ so that

$$\tau' = (\tau - \kappa)/\sigma \in D$$

and

$$D = K[\tau'].$$

According to [45, Theorem 2.3.1], we have to check that K is S-inert in D, i.e.,

(i) $D \cap S^{-1}K = K$,

(ii) Q, the field of quotients of $\overline{K} = K/(zD \cap K) \subset D/zD$, is algebraically closed in the field of quotients of D/zD,

(iii) $Q \cap D/zD = \overline{K}$.

Now (i) is readily deduced from CLAIM 6, and (ii) and (iii) are clear by CLAIM 7. □

Remark 4.2. It is an open question whether the conclusion $X \cong \mathbb{A}^3 \times Y$ is correct without reference to any group action, as it is for $\mathbb{A}^2$-fibers, see [29], or [37] for an argument in the spirit of the present paper.

The following results are immediate consequences of Theorem 4.1.

Theorem 4.3. *Suppose $F \in A = k^{[4]}$ is invariant under an effective G_m-action on A and for all $\lambda \in k$ we have $A/(F - \lambda)A \cong k^{[3]}$. Then F is a variable in A.*

Theorem 4.4. *A G_m-action on $\mathbb{A}^4$ that fixes a variable is linearizable. Any variable that is fixed by the action is part of a system of variables that diagonalizes the action.*

5. Danielewski surfaces of non-hypersurface type

The Danielewski surfaces are the hypersurfaces $X(P)$ in $\mathbb{A}^3 = \mathrm{Spec}\, k[x,y,z]$ defined by $x^m y = P(z)$, where $m \geq 1$ and $P(z) \in k[z]$ with $\gcd(P(z), P'(z)) = 1$. We consider here the simplest case X with $m = 1$ and $P(z) = z^2 - 1$. The following properties are well known about X.

Lemma 5.1. *Let $X = \{xy = z^2 - 1\}$ in $\mathbb{A}^3$. Then we have:*

(1) *X has two $\mathbb{A}^1$-fibrations $\rho_1 : X \to \mathbb{A}^1$ and $\rho_2 : X \to \mathbb{A}^1$ defined by $\rho_1(x,y,z) = x$ and $\rho_2(x,y,z) = y$. These two $\mathbb{A}^1$-fibrations are obtained as orbits of the G_a-actions σ_1 and σ_2 which are defined respectively by the locally nilpotent derivations δ_1 and δ_2 such that $\delta_1(x) = 0, \delta_1(y) = 2p_1(x)z, \delta_1(z) = p_1(x)x$ with $p_1(x) \in k[x]$ and $\delta_2(x) = 2p_2(y)z, \delta_2(y) = 0, \delta_2(z) = p_2(y)y$.*
(2) *$X = \mathbb{F}_0 \setminus D$, where D is the diagonal of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.*
(3) *X is simply connected, $H_1(X;\mathbb{Z}) = 0$ and $H_2(X;\mathbb{Z}) = \mathbb{Z}$.*
(4) *An algebraic subgroup of $\mathrm{Aut}(X)$ is either $G_a = \{\exp(\delta_i t) \mid t \in k\}$ for $i = 1, 2$ or $\mathrm{PGL}(2)$ which is generated by $\exp(\delta_1 t)$ and $\exp(\delta_2 s)$, where $p_1(x) = p_2(y) = 1$.*

The following results have been known to the experts (see [24]), and exhibit possible directions for the Danielewski surfaces to be generalized in the non-hypersurface case and the higher-dimesnional case.

Lemma 5.2. *Let X be as in Lemma 5.1. Then we have:*

(1) *$X \cong T\backslash\mathrm{PGL}(2)$, where T is a maximal torus of $\mathrm{PGL}(2)$.*
(2) *The embedding $X \hookrightarrow \mathbb{F}_0$ is $\mathrm{PGL}(2)$-equivariant, where $\mathrm{PGL}(2)$ acts on $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ diagonally.*
(3) *Write $T\backslash\mathrm{PGL}(2) = T'\backslash\mathrm{SL}(2)$, where T' is the maximal torus $\left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \;\middle|\; t \in k^* \right\}$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{SL}(2)$. Then the left action of T' on $\mathrm{SL}(2)$ is given by $(a,b,c,d) \mapsto (ta, tb, t^{-1}c, t^{-1}d)$, and hence the T'-invariant subring A of the coordinate ring $B := k[a,b,c,d]/(ad - bc = 1)$ of $\mathrm{SL}(2)$ is generated over k by $x = ac,\ y = bd,\ w = ad$ and $z = bc$. The relation among x, y, w and z are $w = z + 1$ and $xy = wz$. Eliminating w, we have $xy = z(z+1)$, which is a defining equation of the Danielewski surface after a suitable change of coordinates. We employ this equation as the defining equation of X.*
(4) *The G_a-action σ_1 on X corresponding to the locally nilpotent derivation δ_1 with $p_1(x) = 1$ in Lemma 5.1 is given by the right multiplication of $\left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \;\middle|\; u \in k \right\}$. The $\mathbb{A}^1$-fibration*

$\rho_1 : X \to \mathbb{A}^1$ which is the quotient morphism by σ_1 is given by $(x, y, z) \mapsto x$ and has the following splitting data with the section $T\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ for $x \in \mathbb{A}^1$

$$T\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = T\begin{pmatrix} 1 & u \\ x & ux+1 \end{pmatrix} \qquad (*)$$

where $y = u(ux + 1),\ w = ux + 1,\ z = ux$. If $a \neq 0$, then the T-residue class $T\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is written in the form $(*)$ as shown in the following computation:

$$\begin{aligned} T\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= T\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= T\begin{pmatrix} 1 & a^{-1}b \\ ac & ad \end{pmatrix} = T\begin{pmatrix} 1 & u \\ x & w \end{pmatrix}, \end{aligned}$$

with $x = ac,\ u = a^{-1}b, w = ad$ and $z = ad - 1$. If $a = 0$, then $bc = -1$ and

$$\begin{aligned} T\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} &= T\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & bd \end{pmatrix} \\ &= T\begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix} = T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where $y = bd$. Thus the fiber of ρ_1 over the point $x = 0$ consists of two G_a-orbits passing through $T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $T\begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix}$, respectively.

(5) *The G_a-action σ_2 on X corresponding to the locally nilpotent derivation δ_2 with $p_2(y) = 1$ is given by the right multiplication of $\left\{\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \middle| \ v \in k\right\}$. The $\mathbb{A}^1$-fibration $\rho_2 : X \to \mathbb{A}^1$ which is the quotient morphism by σ_2 is given by $(x, y, z) \mapsto y$ and has the following splitting data with the section $T\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ for $y \in \mathbb{A}^1$*

$$T\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} = T\begin{pmatrix} 1+vy & y \\ v & 1 \end{pmatrix} \qquad (**)$$

where $x = v(1 + yv),\ w = 1 + vy$ and $z = vy$. If $d \neq 0$, then the T-residue class $T\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is written in the form $(**)$

$$
\begin{aligned}
T\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= T\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= T\begin{pmatrix} ad & bd \\ cd^{-1} & 1 \end{pmatrix} = T\begin{pmatrix} w & y \\ v & 1 \end{pmatrix},
\end{aligned}
$$

where $y = bd,\ v = cd^{-1}$ and $z = ad - 1$. If $d = 0$, then $bc = -1$ and

$$
\begin{aligned}
T\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} &= T\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\begin{pmatrix} -ac & 1 \\ -1 & 0 \end{pmatrix} \\
&= T\begin{pmatrix} -x & 1 \\ -1 & 0 \end{pmatrix} = T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix},
\end{aligned}
$$

where $x = ac$. Thus the fiber of ρ_2 over the point $y = 0$ has two G_a-orbits passing through $T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $T\begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix}$, respectively.

(6) *Two $\mathbb{A}^1$-fibrations ρ_1 and ρ_2 are transformed to each other by the action of the Weyl group:*

$$
\begin{aligned}
&\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T\begin{pmatrix} 1 & u \\ x & ux+1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ x & ux+1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= T\begin{pmatrix} ux+1 & -x \\ -u & 1 \end{pmatrix}.
\end{aligned}
$$

□

The group SL(2) is a quadric hypersurface $xy - zu = 1$ in $\mathbb{A}^4$. A slight modification of the defining equation of SL(2) will give similar but interesting results.

Theorem 5.3. *Let $X(m,1)$ be the hypersurface $xy - z^m u = 1$ in $\mathbb{A}^4 =$* Spec $k[x,y,z,u]$*, where $m \geq 2$. Then the following assertions hold.*

(1) *$X(m,1)$ is factorial and simply-connected.*
(2) *$X(m,1)$ has a locally nilpotent derivation (simply* lnd*) δ defined by*

$$\delta(x) = \delta(u) = 0,\ \delta(y) = mz^{m-1}u \text{ and } \delta(z) = x.$$

(3) *$X(m,1)$ has a G_m-action defined by ${}^t(x,y,z,u) = (tx, t^{-1}y, tz, t^{-m}u)$. Let $R = k[x,y,z,u]/(xy - z^m u = 1)$ be the coordinate*

ring of $X(m,1)$. Then the G_m-invariant subring $A = R^{G_m}$ is given by

$$A = k[xy, yz, x^{m-i}z^i u \ (0 \le i \le m)].$$

Let B be an affine k-domain defined by

$$B = k[X, Y_0, Y_1, \ldots, Y_m] \Big/ \left(\begin{array}{l} XY_{i-1} = Y_i(1+Y_m),\ 1 \le i \le m \\ Y_iY_j = Y_{i'}Y_{j'},\ i+j = i'+j' \end{array} \right)$$

Then there exists an isomorphism $\theta : B \xrightarrow{\sim} A$ defined by $\theta(X) = yz,\ \theta(Y_i) = x^{m-i}z^i u$ for $0 \le i \le m$.

(4) *Let $V(m,1) = \operatorname{Spec} A = X(m,1)//G_m$. Then $V(m,1)$ is a smooth affine surface. The lnd δ on R induces an lnd $\overline{\delta}$ on A such that $\overline{\delta}(X) = (m+1)Y_m + 1$ and $\overline{\delta}(Y_i) = iY_{i-1}$ for $0 \le i \le m$. Hence $\operatorname{Ker} \overline{\delta} = k[Y_0]$. Let $\rho_2 : V(m,1) \to \mathbb{A}^1$ be the quotient morphism by the G_a-action associated to $\overline{\delta}$, where $\mathbb{A}^1 = \operatorname{Spec} k[Y_0]$. If $Y_0 \ne 0$ then*

$$X = \frac{1}{Y_0^m} Y_1(Y_0^{m-1} + Y_1^m) \text{ and } Y_i = \frac{Y_1^i}{Y_0^{i-1}}\ (2 \le i \le m).$$

Hence $\rho_2^{-1}(\mathbb{A}^1_*) \cong \mathbb{A}^1_* \times \mathbb{A}^1$, and $\rho_2^{-1}(0)$ is a disjoint union of two reduced irreducible components isomorphic to $\mathbb{A}^1$ defined by $(Y_0 = \cdots = Y_{m-1} = 0, Y_m = -1)$ and $(Y_0 = \cdots = Y_{m-1} = Y_m = 0)$ respectively.

(5) *Let $\rho_1 : V(m,1) \to \mathbb{A}^1 = \operatorname{Spec} k[X]$ be defined by the inclusion $k[X] \hookrightarrow A$. If $X \ne 0$ then*

$$Y_i = \frac{Y_m}{X^{m-i}}(1+Y_m)^{m-i}\ (0 \le i \le m) \qquad (*)$$

whence $\rho_1^{-1}(\mathbb{A}^1_*) \cong \mathbb{A}^1_* \times \mathbb{A}^1$, and $\rho_1^{-1}(0)$ is a disjoint union of two reduced irreducible components isomorphic to $\mathbb{A}^1$, for one of which $Y_1 = \cdots = Y_m = 0$ and Y_0 is a variable, and another of which $Y_m = -1$ and

$$Y_{m-i} = (-1)^{i+1} Y_{m-1}^i\ (0 \le i \le m). \qquad (**)$$

(6) *The minimal closed embedding of $V(m,1)$ into the affine space is $V(m,1) \hookrightarrow \mathbb{A}^{m+2}$, which is given by the expression of B in terms of $X, Y_0, \ldots, Y_m$. Hence if $m \ne m'$, $V(m,1)$ is not isomorphic to $V(m',1)$. In particular, $V(m,1) \not\cong V(1,1)$ if $m \ge 2$, where $V(1,1)$ is the hypersurface Danielewski surface $xy = z^2 - 1$.*

(7) *For $m \ne m'$, we have $V(m,1) \times \mathbb{A}^1 \cong V(m',1) \times \mathbb{A}^1$.*

Proof. (1) The subvariety $V(x)$ of $X(m,1)$ defined by $x=0$ is isomorphic to $C\times\mathbb{A}^1$, where C is the irreducible plane curve $z^m u+1=0$. Further, $D(x):=X(m,1)\setminus V(x)\cong\mathbb{A}^1_*\times\mathbb{A}^2$, which is factorial. Hence $X(m,1)$ is factorial by Nagata [42]. Next apply Nori's exact sequence (see Lemma 6.2 below) to the projection $p:X(m,1)\to\mathbb{A}^1,\ (x,y,z,u)\mapsto x$. We then have an exact sequence

$$\pi_1(\mathbb{A}^2)\to\pi_1(X(m,1))\to\pi_1(\mathbb{A}^1)\to(1).$$

This shows that $X(m,1)$ is simply connected.

(2) Straightforward.

(3) We show that $R^{G_m}=k[xy,yz,x^{m-i}z^iu\ (0\le i\le m)]$. Under the given G_m-action, a monomial $x^ay^bz^cu^d$ is G_m-invariant if and only if $a+c=b+md$, where a,b,c,d are non-negative integers. We consider 3 cases separately.

(i) If $0\le c<d$, then

$$x^ay^bz^cu^d=(xy)^b(x^{m-1}zu)^c(x^mu)^{d-c}.$$

(ii) If $id<c<(i+1)d$ for some $1\le i<m$, then

$$x^ay^bz^cu^d=(xy)^b(x^{m-i}z^iu)^{(i+1)d-c}(x^{m-i-1}z^{i+1}u)^{c-id}.$$

(iii) If $md\le c$, then $b\ge a$ and

$$x^ay^bz^cu^d=(xy)^a(yz)^{b-a}(z^mu)^d.$$

Hence R^{G_m} is generated by xy,yz and $x^{m-i}z^iu$ for $0\le i\le m$.

It is clear that the homomorphism $\theta:B\to A$ is well-defined and surjective. We show that it is an isomorphism. If $X\ne 0$ then

$$Y_i=\frac{Y_m}{X^{m-i}}(1+Y_m)^{m-i}\quad(0\le i\le m).$$

Hence $\dim B=2$, and hence θ is birational. Consider the case where $X=0$. Since $XY_{m-1}=Y_m(1+Y_m)$, we have $Y_m=0$ or -1. Further, since $XY_i=Y_{i+1}(1+Y_m)$ for $0\le i<m$, we have $Y_1=\cdots=Y_{m-1}=0$ if $Y_m=0$. Hence θ induces an isomorphism between $B/(X,Y_m)\cong k[Y_0]$ and $k[x^mu]$. Suppose that $Y_m=-1$. We show by induction that if $Y_{m-1}\ne 0$ then Y_{m-i} $(1\le i\le m)$ is a monomial in Y_{m-1} with constant coefficient and positive degree, which we denote $Y_{m-i}\sim Y_{m-1}^{\alpha_i}$ with $\alpha_i>0$. In fact, $Y_mY_{m-2}=Y_{m-1}^2$ implies $Y_{m-2}=-Y_{m-1}^2$ with $\alpha_2=2$. Suppose that $Y_{m-i+2}\sim Y_{m-1}^{\alpha_{i-2}}$ and $Y_{m-i+1}\sim Y_{m-1}^{\alpha_{i-1}}$ with $0<\alpha_{i-2}<\alpha_{i-1}$ for $i\ge 2$. Since $Y_{m-i+2}Y_{m-i}=Y_{m-i+1}^2$, we have $Y_{m-i}\sim Y_{m-1}^{\alpha_i}$ with $\alpha_i=2\alpha_{i-1}-\alpha_{i-2}>0$. Since $Y_{m-i-1}Y_{m-i+1}=Y_{m-i}^2$, we have $Y_{m-i-1}\sim Y_{m-1}^{\alpha_{i+1}}$ with $\alpha_{i+1}=2\alpha_i-\alpha_{i-1}=(4\alpha_{i-1}-2\alpha_{i-2})-\alpha_{i-1}=3\alpha_{i-1}-2\alpha_{i-2}$, where $\alpha_{i+1}-\alpha_i=\alpha_{i-1}-\alpha_{i-2}>0$. So, we are done.

Suppose that $X = Y_{m-1} = 0$ and $Y_m = -1$. We show that $Y_i = 0$ for $0 \le i < m$. In fact, since $Y_0 Y_m = Y_1 Y_{m-1}$, we have $Y_0 = 0$. Similarly, if $0 < i < m-1$, we have $Y_i Y_m = Y_{i+1} Y_{m-1}$. Hence $Y_i = 0$. Thus we know that Spec $B/(X, Y_m + 1) \cong \mathbb{A}^1_* \cup \{\text{one point}\}$, where $\mathbb{A}^1_* \cong \mathrm{Spec} k[Y_{m-1}, Y_{m-1}^{-1}]$ and θ induces an isomorrphism between $B[Y_{m-1}^{-1}]/(X, Y_m + 1) \cong k[Y_{m-1}, Y_{m-1}^{-1}]$ and $k[xz^{-1}, x^{-1}z]$. By the Serre criterion of normality, B is a normal domain, and we know that θ is, in fact, an isomorphism because A is regular by the assertion (4) below.

(4) Since the G_m-action on $X(m, 1)$ is fixpoint free, the quotient surface $V(m, 1)$ is smooth by Luna's étale slice theorem. We show that $\rho_2^{-1}(0)$ consists of two reduced irreducible components isomorphic to $\mathbb{A}^1$. The rest of the assertion is straightforward. By (3) above, we can identify X, Y_i with the elements $yz, x^{m-i}z^i u$ respectively. Hence $Y_0 = x^m u = 0$ implies that either $x = 0$ or $u = 0$. If $x = 0$ then $Y_0 = \cdots = Y_{m-1} = 0$ and $Y_m = -1$. Then the corresponding component of $\rho_2^{-1}(0)$ is the affine line Spec $k[X]$. If $u = 0$ then $Y_0 = Y_1 = \cdots = Y_m = 0$. Hence the corresponding component of $\rho_2^{-1}(0)$ is also the affine line Spec $k[X]$.

(5) The equality $(*)$ follows from the relations $XY_{i-1} = Y_i(1 + Y_m)$ for $1 \le i \le m$. Suppose $X = 0$. Then $Y_m = 0$ or -1 because $XY_{m-1} = Y_m(1+Y_m)$. If $Y_m = 0$ then $XY_{i-1} = Y_i(1+Y_m)$ implies $Y_i = 0$ for $1 \le i \le m$. Then Y_0 is a variable of the irreducible component of $\rho_1^{-1}(0)$ with $Y_m = 0$. Suppose $Y_m = -1$. By (3), one can show that $Y_m^{i-1} Y_{m-i} = Y_{m-1}^i$ for $0 \le i \le m$, whence follows the equality $(**)$. So, the other irreducible component of $\rho_1^{-1}(0)$ with $Y_m = -1$ is isomorphic to $\mathbb{A}^1 = \mathrm{Spec}\, k[Y_{m-1}]$.

(6) It is clear that $\{xy, yz, x^{m-i}z^i u \ (0 \le i \le m)\}$ is a minimal set of generators of A. Hence follows the rest of the assertion (6).

(7) Note that by the assertion (4), $V(m, 1)$ has a G_a-bundle structure over a non-separated scheme $\mathbb{A}^1_*$ with two points added instead of the point of origin. Then the assertion can be proved by the same argument as in Dubouloz [12, Example 0.1]. □

Remark 5.4. In [14], Dubouloz-Finston treated a hypersurface $X(m, n) = \{x^m y - z^n u = 1\}$ in $\mathbb{A}^4 = \mathrm{Spec} k[x, y, z, u]$ which is quasi-homogeneous with respect to the grading ${}^t(x, y, z, u) = (tx, t^{-m}y, tz, t^{-n}u)$ and showed that $V(m, n) := X(m, n)//G_m$ is a Danilov-Gizatullin surface of degree $d := m+n$. Namely $V(m, n)$ is obtained as $\mathbb{F}_n \setminus C$, where $\mathbb{F}_n$ is a Hirzebruch surface of degree n and C is an ample section of the canonical $\mathbb{P}^1$-fibration of $\mathbb{F}_n$. This generalises the fact that the Danielewski surface $X(1, 1)$ gives rise to $V(1, 1)$ which is isomorphic to $\mathbb{F}_0 \setminus \Delta$, where Δ is the diagonal.

It is shown in [18] that a Danilov-Gizatullin surface V has a G_a-action such that $V//G_a \cong \mathbb{A}^1$ and the quotient morphism $q : V \to \mathbb{A}^1$ has only one reducible fiber with two irreducible components with respective multiplicity one.

The following result shows that the threefolds $X(m,1)$ cannot be distinguished from each other by means of the ordinary topological quantities.

Theorem 5.5. *Let $X_m := X(m,1)$ be an affine hypersurface $xy - z^m u = 1$. Then the following assertions hold.*

(1) *$H_1(X_m;\mathbb{Z}) = H_2(X_m;\mathbb{Z}) = 0$ and $H_3(X_m;Z) = \mathbb{Z}$.*
(2) *Let N_m be the boundary 5-manifold at infinity, i.e., the boundary of a tubular neighborhood of the boundary divisor at infinity with respect to a suitable smooth normal compactification of X_m. Then the homology groups $H_i(N_m;\mathbb{Z})$ is independent of m, where $0 \le i \le 5$.*

Proof. (1) As explained in Theorem 5.3, X_m has a fixpoint free G_m-action. Hence X_m is a $\mathbb{C}^*$-fiber bundle over $V_m := X_m//G_m$ in the sense of C^∞-topology. Hence we have a homotopy exact sequence

$$\pi_2(\mathbb{C}^*) \to \pi_2(X_m) \to \pi_2(V_m) \to \pi_1(\mathbb{C}^*) \to \pi_1(X_m) \to \pi_1(V_m),$$

where $\pi_2(X_m) \cong H_2(X_m;\mathbb{Z})$ and $\pi_2(V_m) \cong H_2(V_m;\mathbb{Z})$ by Hurewicz's isomorphism theorem since $\pi_1(X_m) \cong \pi_1(V_m) = (1)$. Note that $H_2(V_m;\mathbb{Z}) \cong \mathbb{Z}$ because $\chi(V_m) = 2$ and $H_2(V_m;\mathbb{Z})$ has no torsion by Hamm's theorem. Hence $H_2(X_m;\mathbb{Z}) = 0$ because $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$ and $\pi_2(\mathbb{C}^*) \cong \pi_2(S^1) = (1)$. On the other hand, let Y_1 be the subvariety of X_1 defined by $z = 0$. Then the morphism $\psi : X_m \to X_1$ defined by $(x,y,z,u) \mapsto (x,y,z^m,u)$ is a ramified covering which is totally ramified over Y_1 and unramified over $X_1 \setminus Y_1$. Here $Y_1 \cong \mathbb{A}^1_* \times \mathbb{A}^1$, whence $\chi(Y_1) = 0$. Since $\chi(X_1) = 0$, we have $\chi(X \setminus Y_1) = 0$ and hence we have

$$\begin{aligned} \chi(X_m) &= \chi(X_m \setminus \psi^{-1}(Y_1)) + \chi(\psi^{-1}(Y_1)) \\ &= m\chi(X_1 \setminus Y_1) + \chi(Y_1) \\ &= m\,(\chi(X_1 \setminus Y_1) + \chi(Y_1)) = 0. \end{aligned}$$

Since $H_3(X_m;\mathbb{Z})$ has no torsion again by Hamm's theorem, we have $H_3(X_m;\mathbb{Z}) \cong \mathbb{Z}$.

(2) By a change of coordinates, we write the defining equation of X_m as $x^2 + y^2 + z^m u = 1$. Let N_m be as stated above. It is also the boundary of a big closed ball taken inside X_m. Then N_m is a ramified covering of N_1 which is totally ramified over $F_1 := N_1 \cap \{z = 0\}$ and unramified outside F_1.

Note that X_1 has a natural compactification $\overline{X}_1 = \{X^2+Y^2+ZU = V^2\}$ which is a quadric hypersurface in $\mathbb{P}^4$ and $\overline{X}_1 \setminus X_1$ is the quadric surface $Q = \{X^2 + Y^2 + ZU = 0\}$ in $\mathbb{P}^3$. Since N_1 is the boundary of a tubular neighborhood of Q which is smooth, N_1 is an S^1-bundle over $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. The complement $N_1 \setminus F_1$ is a trivial S^1-bundle over $\mathbb{A}^2 = Q \setminus \{Z = 0\}$, whence $N_1 \setminus F_1$ is isomorphic to $\mathbb{A}^2 \times S^1$.

Now F_1 is embedded into N_m since the mapping $N_m \to N_1,\ (x, y, z, u) \mapsto (x, y, z^m, u)$, is totally ramified over F_1. Using the relative cohomology sequence with integral coefficients for the pair (N_m, F_1), we have an exact sequence

$$\begin{aligned} 0 &\to H^0(N_m, F_1) \to H^0(N_m) \to H^0(F_1) \\ &\to H^1(N_m, F_1) \to H^1(N_m) \to H^1(F_1) \to H^2(N_m, F_1) \\ &\to H^2(N_m) \to H^2(F_1) \to H^3(N_m, F_1) \to H^3(N_m) \\ &\to H^3(F_1) \to H^4(N_m, F_1) \to H^5(N_m)\ , \end{aligned}$$

where

$$\begin{aligned} &H^i(N_m, F_1) \cong H_{5-i}(\mathbb{A}^2 \times S^1; \mathbb{Z}) = 0\ (i = 1, 2, 3) \\ &H^4(N_m, F_1) \cong H_1(\mathbb{A}^2 \times S^1; \mathbb{Z}) \cong \mathbb{Z} \end{aligned}$$

by the Lefschetz duality and by noting that the homology groups of $\mathbb{A}^2 \times S^1$ are the same as those of S^1 since $\mathbb{A}^2$ is contractible to one point. Hence it follows that $H^i(N_m) \xrightarrow{\sim} H^i(F_1)$ for $i = 0, 1, 2$. Since N_m is simply connected, $H^5(N_m) \cong H_1(N_m; \mathbb{Z}) = 0$. So, $H^3(N_m) \cong \mathrm{Ker}\,(H^3(F_1) \to \mathbb{Z})$.

By the above observation and the Poincaré duality, the homology groups $H_{5-i}(N_m; \mathbb{Z})$ is independent of m for $0 \le i \le 3$. Since $H_1(N_m; \mathbb{Z}) = 0$ and $H_0(N_m; \mathbb{Z}) \cong \mathbb{Z}$, all integral homology groups of N_m are independent of m. □

Remark 5.6. The independence of $H_i(N_m; \mathbb{Z})$ of m makes a clear difference from the case of the Danielewski surfaces $Z_m := \{x^m y = z^2 - 1\}$, where $H_1(Z_m; \mathbb{Z}) \cong \mathbb{Z}/2m\mathbb{Z}$ by Fieseler [17].

Generalizing the results in Lemma 5.2, we ask the following problem.

Problem 5.7. *Let $X = T\backslash \mathrm{PGL}(n)$ with a maximal torus T. Then X is a smooth affine variety of dimension $n(n-1)$.*

(1) *Clarify the $\mathbb{A}^{\frac{1}{2}n(n-1)}$-fibrations on X. Those $\mathbb{A}^{\frac{1}{2}n(n-1)}$-fibrations are given by the right actions of maximal unipotent subgroups of $\mathrm{PGL}(n)$ and transferred to each other by the action of the Weyl group. Especially, clarify all the singular fibers. Describe the connected componets of the singular fibers in terms of the*

algebraic group. Show that X does not have any $\mathbb{A}^r$-fibration with $r > \frac{1}{2}n(n-1)$.

(2) *X is simply connected as it contains $\mathbb{A}^{n(n-1)}$ as an open set. Use the above clarifications of singular fibers to compute the homology groups of X.*
(3) *Compute the Picard group* $\mathrm{Pic}(X)$ *and the Picard number $\rho(X)$.*
(4) *If possible, prove that X does not have the cancellation property.*

6. Some results about $\mathbb{A}^1$- and $\mathbb{P}^1$-fibrations

In this section, by a component of a variety W, we will mean an irreducible component of W. We will describe some elementary results about singular fibers of an $\mathbb{A}^1$-fibration (a $\mathbb{P}^1$-fibration) on a smooth affine (resp. a projective) 3-fold. Some of these arguments have been already used in [23].

Let $f : X \to Y$ be a morphism from a smooth affine 3-fold X to a normal affine surface Y. We do not assume that f is surjective. We can embed X as a Zariski-open subset of a smooth 3-fold V such that f extends to a proper morphism $\widetilde{f} : V \to Y$. The set of points $y \in Y$ such that the fiber $\widetilde{f}^{-1}(y)$ is scheme-theoretically not isomorphic to $\mathbb{P}^1$ is called a *singular value* of $\widetilde{f}$. For simplicity we call the divisorial part of the set of singular values the *bad curve* of $\widetilde{f}$, denoted by $\mathbb{B}(\widetilde{f})$. The fibers $f^{-1}(y)$ or $\widetilde{f}^{-1}(y)$ are denoted by $f^*(y)$ or $\widetilde{f}^*(y)$ to emphasize the scheme-theoretic fibers.

We will use the following general result of Nori [43].

Lemma 6.1. *Let $f : X \to Y$ be a surjective morphism between smooth algebraic varieties such that a general fiber F of f is irreducible. Assume that there is a closed subvariety S of Y such that S has codimension > 1 in Y and for any $y \in Y \setminus S$ the fiber $f^{-1}(y)$ has a reduced component. Then the natural sequence of homomorphisms $\pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to (1)$ is exact.*

We will need a slight extension of this result as follows [48, §1].

Lemma 6.2. *Let $f : X \to C$ be a surjective morphism with X a smooth surface, C a smooth curve and a general fiber F of f irreducible. For a point $y \in C$ the greatest common divisor of the multiplicities of the components of $f^*(y)$ is called the* multiplicity *of the fiber $f^*(y)$. Let $m_1F_1, .., m_\ell F_\ell$ exhaust all the fibers of f with multiplicities $m_i > 1$. Suppose that C has r places at infinity and let g be the geometric genus of C. Then there is an exact sequence*

$$\pi(F) \to \pi_1(X) \to \Gamma \to (1),$$

where Γ *is the group with generators* $a_1, b_1, ..., a_g, b_g, c_1, .., c_r, e_1, .., e_\ell$ *and relations*

$$[a_1, b_1] \cdots [a_g, b_g] c_1 \cdot c_2 \cdots c_r = 1 = e_1^{m_1} = \cdots = e_\ell^{m_\ell}.$$

Now assume that $f : X \to Y$ is as in Lemma 6.1, where $\dim Y > 1$. For a general hyperplane section Z of Y, both Z and $f^{-1}(Z)$ are smooth and the restricted morphism $f^{-1}(Z) \to Z$ has an irreducible general fiber. By a relative version of Lefschetz hyperplane section theorem there is an isomorphism $\pi_1(f^{-1}(Z)) \cong \pi_1(X)$. We apply this to the proper morphism $\widetilde{f} : \widetilde{V} \to Y$, where the hyperplane section is a smooth curve C on Y. Now we can use Lemma 6.2.

Lemma 6.3. *Let* C_0 *be a component of* C. *Then there is no integer* $m > 1$ *with the following property.*

Let $y \in C_0$ *be a general (smooth) point of both* C *and* Y. *Let* z_1, z_2 *be local analytic coordinates at* y *on* Y *such that* $C_0 = \{z_1 = 0\}$. *For any point* $x \in V$ *such that* $f(x) = y$ *the function* z_1 *is expressed as* $z_1 = h^m$ *for a holomorphic function* h *on a neighborhood of* x *in* V.

Namely, the multiplicity of a fiber of $\widetilde{f}$ *is* 1 *except for a finite set of singular values of* $\widetilde{f}$.

Proof. Suppose the result is false. Then there exists some integer $m > 1$ which has the property described in the statement. We choose another small disc D in $\mathbb{C}^2$ with local holomorphic coordinates w_1, w_2 on D. Let $\tau : D \to Y$ be the holomorphic map $\tau(w_1, w_2) = (w_1^m, w_2)$ such that the image of the origin in D is the point $y \in Y$ and $w_1^m = z_1$ and $w_2 = z_2$. Consider the normalization $\widetilde{V}$ of the fiber product $D \times_Y V$. (We call $\widetilde{V}$ the *normalized fiber product.*) Using purity of branch loci, we check easily that $\widetilde{V}$ is a complex (smooth) manifold and the analytic map $\widetilde{V} \to V$ is finite and unramified. Further, every fiber of the induced $\mathbb{P}^1$-fibration $\widetilde{V} \to D$ except possibly the fiber over the origin has multiplicity 1. Since the punctured disc $D - \{(0, 0)\}$ is simply-connected, by Lemma 6.1, we have an exact sequence

$$\pi_1(\mathbb{P}^1) \to \pi_1(\widetilde{V} - \widetilde{F_0}) \to (1).$$

Here $\widetilde{F_0}$ is the fiber of $\widetilde{V} \to D$ over the origin. This shows that $\widetilde{V} - \widetilde{F_0}$, and hence also $\widetilde{V}$, are simply-connected. Since $\widetilde{F_0}$ is a strong deformation retract of $\widetilde{V}$, we infer that $\widetilde{F_0}$ is simply-connected. Denoting the fiber of $V \to Y$ over y by F_0, we now know that F_0 is a quotient of $\widetilde{F_0}$ modulo the finite cyclic group $\mathbb{Z}/m\mathbb{Z}$. By Lemma 6.4 below, F_0 is also simply-connected. The morphism $\widetilde{F_0} \to F_0$ is finite and unramified of degree $m > 1$. This contradiction proves the lemma. $\square$

In the last step of the above proof, we made use of the following result.

Lemma 6.4. *Let a finite group G act algebraically on a simply-connected complete reduced curve Γ. Then Γ/G is simply-connected.*

Proof. Using Van kampen's theorem we see easily that each component of Γ is a cuspidal rational curve and the dual graph of Γ is a tree. We will prove the result by induction of $|G|$ and the number of components of Γ. We call a component C_1 of Γ a tip of Γ if C_1 meets the union of the remaining components of Γ in a single point.

First assume that all the components of Γ meet in a single point p. Then every component of Γ is a tip of Γ. For any component C_1 of Γ the union of its translates by G, say Γ_1 is stable under G. It is easy to see that Γ_1/G is an irreducible rational curve. Since any two components of Γ meet only at p, we see that the various components of Γ/G meet only at the image of p in Γ/G. This implies, again by Van kampen's Theorem, that Γ/G is simply-connected.

Now we consider the case when not every component of Γ is a tip of Γ. Since the dual graph of Γ is a tree, there is an irreducible component C_1 of Γ which is a tip of Γ. Then each translate of C_1 by G is also a tip of Γ. Let Γ_1 be the union of translates of C_1. The union of the components of Γ which are not translates of C_1, say Γ_2, is also G-stable and easily seen to be connected. It is also simply-connected, being a sub-curve of Γ. By induction, both Γ_1/G and Γ_2/G are simply-connected. They meet in a single point in Γ/G, which is the image of $\Gamma_1 \cap \Gamma_2$. This shows that Γ/G is simply-connected. □

Next we will prove the following closely related result.

Lemma 6.5. *Let $f : X \to Y$ be a proper morphism from a smooth 3-fold X onto a normal affine surface such that a general fiber of f is $\mathbb{P}^1$. If a fiber F of f (taken with reduced structure) is simply-connected, then the exceptional divisor in any resolution of singularities of Y at $p := f(F)$ is simply-connected. Conversely, if the exceptional divisor in the resolution of a singular point p of Y is simply-connected, then the corresponding fiber $f^{-1}(p)$ is simply-connected.*

Proof. We will give a proof of only one implication. The other implication is similar.

So assume that F_{red} is simply-connected. We can assume that Y is a small Stein neighborhood of p. By the above arguments it follows that X is simply-connected (since F_{red} is a strong deformation retract of X). Let $\widetilde{Y} \to Y$ be a resolution of singularities. We can find a sequence of blowing ups with smooth centers of X, say $\widetilde{X} \to X$, such that f extends to a proper morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$. Since $\widetilde{X} \to X$ is a proper birational morphism and X is smooth, we know that $\widetilde{X}$ is also simply-connected.

A general fiber of the morphism $\widetilde{X} \to \widetilde{Y}$ is $\mathbb{P}^1$. Hence we have a surjection $\pi_1(\widetilde{X}) \to \pi_1(\widetilde{Y})$. This implies that $\widetilde{Y}$ is simply-connected. Since the exceptional divisor E is a strong deformation retract of $\widetilde{Y}$, it follows that E is simply-connected. □

Lemma 6.6. *Let $f : X \to Y$ be a proper morphism from a smooth 3-fold onto a normal surface Y. Let F be a 1-dimensional fiber of f. Then Y has at worst a quotient singularity at $f(F)$.*

Proof. Let $p := f(F)$. Let C be a component of F and let $q \in C$ be a general point. For a general transverse hyperplane section S of X at q, the point q is isolated in the inverse image of p for the morphism $f|_S$. It follows that that the completion of the local ring of X at q is integral over the completion of the local ring of Y at p. This implies (using Mumford's result on the topology of normal surface singularities) that Y has at worst a quotient singularity at p. □

Combining Lemmas 6.5 and 6.6, we get the following:

Theorem 6.7. *Let $f : X \to Y$ be as in the statement of Lemma 6.5. Then the fiber F is simply-connected.*

Proof. This follows since the exceptional divisor of a resolution of singularity of Y at $f(F)$ is a tree of non-singular rational curves. □

Next we use the following result from [33, p. 107, (2.8.6.3)] to deduce another property of a singular fiber of a $\mathbb{P}^1$-fibration.

Proposition 6.8. *Let $f : X \to Y$ be a proper morphism with X normal, Y smooth and a general fiber $\mathbb{P}^1$. Let F be a 1-dimensional fiber of f. Then $H^1(F_{\mathrm{red}}, \mathcal{O}) = (0)$.*

If C is an irreducible component of F_{red} then we have a surjection $H^1(F_{\mathrm{red}}, \mathcal{O}) \to H^1(C, \mathcal{O})$. Hence $H^1(C, \mathcal{O}) = (0)$. Clearly, $H^0(C, \mathcal{O}) = \mathbb{C}$. Hence the arithmetic genus of C is 0. This implies that C is a smooth rational curve. This shows that every component of F_{red} is a smooth rational curve, and Lemma 6.5 implies that F_{red} is simply-connected.

7. Twisted additive group schemes and their actions

Let $p : Y \to X$ be an $\mathbb{A}^1$-fibration such that Y is affine. If X is affine and A is factorially closed in B where $A = \Gamma(X, \mathcal{O}_X)$ and $B =$

$\Gamma(Y, \mathcal{O}_Y)$, then there exists a G_a-action on Y such that the morphism p coincides with the quotient morphism $q : Y \to Y//G_a$ (see [23, Lemma 1.2]). However, if X is not affine, such a G_a-action does not exist for otherwise X must be isomorphic to Spec Ker δ, where δ is the locally nilpotent derivation (lnd for short) associated to the G_a-action. In order to treat X as the quotient space of a certain group scheme acting on the variety Y, we have to consider the line bundle over X equipped with the group structure which is locally isomorphic over X to the additive group scheme G_a. This idea was already exploited in Dubouloz [11] and applied to the case where $p : Y \to X$ is an $\mathbb{A}^1$-bundle. We are interested in extending the result to the case where p is an $\mathbb{A}^1$-fibration. We begin with redefining the line bundle $\mathbb{L}$ with additive group structure over X.

As in the previous sections, we consider all varieties and schemes defined over k. But the most of the theory can be developed over an affine scheme $S = \operatorname{Spec} R$ over $\mathbb{Q}$. Let X be a scheme over k and let $\mathcal{L}$ be an invertible sheaf. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open covering of X such that $\mathcal{L}|_{U_i} = \mathcal{O}_{U_i} e_i$, where $e_j = s_{ji} e_i$ for $s_{ji} \in \Gamma(U_{ij}, \mathcal{O}_X^*)$, where $U_{ij} = U_i \cap U_j$. The line bundle $\mathbb{L}$ over X associated to $\mathcal{L}$ is the affine scheme $\operatorname{Spec} \mathcal{O}_X[\mathcal{L}]$, where $\mathcal{O}_X[\mathcal{L}]|_{U_i} = \mathcal{O}_{U_i}[e_i]$ is a polynomial ring over $\mathcal{O}_{U_i}$ in the variable e_i. Define the comultiplication Δ, the coinverse ι and the counit ε locally by

$$\Delta|_{U_i}(e_i) = e_i \otimes 1 + 1 \otimes e_i, \quad \iota|_{U_i}(e_i) = -e_i, \quad \varepsilon|_{U_i}(e_i) = 0 \ ,$$

where the tensor products are taken over $\mathcal{O}_{U_i}$. Then $\mathbb{L}$ is a group scheme over X, which is locally over U_i isomorphic to the additive group scheme G_{a,U_i}. Hence we denote $\mathbb{L}$ also by $G_{a,\mathcal{L}}$ and call it the $\mathcal{L}$-twisted additive group scheme over X.

Lemma 7.1. (1) *Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on X. Then $\mathcal{L} \cong \mathcal{M}$ if and only if $\mathbb{L}$ and $\mathbb{M}$ are isomorphic group schemes over X.*

(2) *Let $g : X' \to X$ be a morphism of schemes. Then $\mathbb{L} \times_X X' \cong G_{a,g^*\mathcal{L}}$ as groups schemes over X'.*

(3) *Let $f : Y \to X$ be a morphism of schemes with Y a normal affine scheme and let $\sigma : G_{a,\mathcal{L}} \times Y \to Y$ be an action of X-group schemes on Y. Then the action σ corresponds bijectively to a section δ of $\Gamma(Y, \mathcal{T}_{Y/X} \otimes f^*(\mathcal{L})^{-1})$ such that $\delta|_{U_i}$ is an lnd on $\Gamma(f^{-1}(U_i), \mathcal{O}_Y)$, where $\mathcal{U} = \{U_i\}_{i \in I}$ is an affine open covering of X.*

Proof. (1) Suppose that $\mathcal{L} \cong \mathcal{M}$. Then there exists an affine open covering $\mathcal{U} = \{U_i\}_{i \in I}$ such that $\mathcal{L}|_{U_i} = \mathcal{O}_{U_i} e_i$ and $\mathcal{M}|_{U_i} = \mathcal{O}|_{U_i} z_i$ for every $i \in I$, where $e_j = s_{ji} e_i$ and $z_j = t_{ji} z_i$ on U_{ij} with $s_{ji}, t_{ji} \in \Gamma(U_{ji}, \mathcal{O}_X^*)$ and such that $t_{ji} = u_j s_{ji} u_i^{-1}$ for $u_i \in \Gamma(U_i, \mathcal{O}_X^*)$. Then the mapping $z_i \mapsto \widetilde{e}_i := u_i e_i$ induces an X-group scheme isomorphism between $\mathbb{M}$ and $\mathbb{L}$. Conversely, suppose that there exists an X-group scheme isomorphism ${}^a\varphi : \mathbb{M} \to \mathbb{L}$, where $\varphi : \mathcal{O}_X[\mathcal{L}] \to \mathcal{O}_X[\mathcal{M}]$ is an $\mathcal{O}_X$-algebra isomorphism such that $\Delta\varphi = (\varphi \otimes \varphi)\Delta$, where Δ is the comultiplication. We take an affine open covering $\mathcal{U}$ of $X, \{s_{ji}\}$ and $\{t_{ji}\}$ as above. Then $\varphi_i := \varphi|_{U_i} : \mathcal{O}_{U_i}[e_i] \to \mathcal{O}_i[z_i]$ is an $\mathcal{O}_{U_i}$-isomorphism induced by $\varphi(e_i) = a_i z_i + b_i$, where $a_i \in \Gamma(U_i, \mathcal{O}_X^*)$ and $b_i \in \Gamma(U_i, \mathcal{O}_X)$. Since $\varphi_i = \varphi_j$ on U_{ji}, it follows that $t_{ji} = a_j^{-1} s_{ji} a_i$ and $b_j = s_{ji} b_i$. Since we have $\Delta\varphi_i = (\varphi_i \otimes \varphi_i)\Delta$, it follows that $b_i = 0$. Hence $\varphi(e_i) = a_i z_i$ for every $i \in I$. Thus, by the change of base $z_i \mapsto a_i z_i$, φ induces an isomorphism $\mathcal{L} \cong \mathcal{M}$.

(2) Straightforward by the definition.

(3) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open covering of X such that $\mathcal{L}|_{U_i} = \mathcal{O}_{U_i} e_i$ for all i and $e_j = s_{ji} e_i$ on U_{ji}, and let $\mathcal{V} = \{f^{-1}(U_i)\}_{i \in I}$. Write $U_i = \text{Spec } A_i$ and $f^{-1}(U_i) = \text{Spec } B_i$. Then the coaction σ^* restricted onto $f^{-1}(U_i)$ is given by

$$\sigma^*|_{U_i} : B_i \longrightarrow B_i \otimes A_i[e_i], \quad b \mapsto \sum_{n \geq 0} \frac{1}{n!} (\delta_i)^n (b) e_i^n$$

such that we have on $f^{-1}(U_{ji})$

$$\sum_{n \geq 0} \frac{1}{n!} (\delta_i|_{U_{ji}})^n (b) e_i^n = \sum_{n \geq 0} \frac{1}{n!} (\delta_j|_{U_{ji}})^n (b) e_j^n$$

for $b \in \Gamma(f^{-1}(U_{ji}, \mathcal{O}_Y)$. Here δ_i is an A_i-trivial lnd on B_i. We need some extra explanations on the notations $\delta_i|_{U_{ji}}$ and $\delta_j|_{U_{ji}}$. The lnd δ_i as viewed an element of $\text{Der}_{A_i}(B_i)$ corresponds to the vector field associated to the G_a-action on $f^{-1}(U_i)$ defined by δ_i. Then $\delta_i|_{U_{ji}}$ is the restriction of the vector field to $V_{ji} := f^{-1}(U_{ji})$. Since this vector field corresponds to the G_a-orbit directions of the points in V_{ji}, $\delta_i|_{V_{ji}}$ is an lnd of $\Gamma(V_{ji}, \mathcal{O}_Y)$. Similarly, $\delta_j|_{V_{ji}}$ is the lnd of $\Gamma(V_{ji}, \mathcal{O}_Y)$ induced by δ_j. Then $\delta_i = s_{ji}\delta_j$. In fact, it suffices that the coincidence occurs on the open set of V_{ji} with a closed set of codimension ≥ 2 removed. Hence we can restrict $\delta_i|_{U_{ji}}$ and $\delta_j|_{U_{ji}}$ on the smooth locus of V_{ji} by the normality hypothesis of Y. Let $\mathcal{T}_{Y/X}$ be the relative tangent sheaf $\mathcal{T}_{Y/X} := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{Y/X}, \mathcal{O}_X)$. Since $\delta_j = s_{ji}^{-1}\delta_i$ for each pair (i, j) and $\delta_i \in \Gamma(f^{-1}(U_i), \mathcal{T}_{Y/X})$, we know that $\{\delta_i\}$ defines a section of $\Gamma(Y, \mathcal{T}_{Y/X} \otimes_{\mathcal{O}_X} f^*(\mathcal{L})^{-1})$ such that δ_i is locally nilpotent over $f^{-1}(U_i)$. Conversely, given such a section δ of $\Gamma(Y, \mathcal{T}_{Y/X} \otimes_{\mathcal{O}_X} f^*(\mathcal{L})^{-1})$, it is clear that δ_i defines a G_a-action on $f^{-1}(U_i)$ and that these locally-defined G_a-actions patch together to define an action of an X-group scheme $\mathbb{L} = G_{a,\mathcal{L}}$ on Y so that $Y /\!/ \mathbb{L} = X$. □

Corollary 7.2. *If $\mathcal{L} \cong \mathcal{O}_X$, then $G_{a,\mathcal{L}} \cong G_a$ and $G_{a,\mathcal{L}}$-action on X is equivalent to a G_a-action on X.*

The following result is also known in [11].

Lemma 7.3. *Let X be a smooth variety and let $\overline{f} : \overline{Y} \to X$ be a $\mathbb{P}^1$-bundle. Let S be a section of $\overline{f}$ such that the complement $Y := \overline{Y} \setminus S$ is an affine variety, and let $f := \overline{f}|_Y$. Then there exists an invertible sheaf $\mathcal{L}$ on X such that the $\mathcal{L}$-twisted additive group $G_{a,\mathcal{L}}$ acts on Y and $Y//G_{a,\mathcal{L}} \cong X$. Hence Y is a $G_{a,\mathcal{L}}$-torsor over X.*

Proof. We have an exact sequence

$$0 \to \mathcal{O}_{\overline{Y}} \to \mathcal{O}_{\overline{Y}}(S) \to \mathcal{O}_S(S) \to 0 \ .$$

Since $\overline{f} : \overline{Y} \to X$ is a $\mathbb{P}^1$-bundle, we have $R^1\overline{f}_*\mathcal{O}_{\overline{Y}} = 0$ and hence an exact sequence

$$0 \to \overline{f}_*\mathcal{O}_{\overline{Y}} \to \overline{f}_*\mathcal{O}_{\overline{Y}}(S) \xrightarrow{\varphi} \mathcal{L} \to 0,$$

where $\mathcal{O}_S(S)$ is viewed as an invertible sheaf $\mathcal{L}$ on X, $\overline{f}_*\mathcal{O}_{\overline{Y}}(S)$ is a rank 2 vector bundle $\mathcal{E}$ on X and $\overline{f}_*\mathcal{O}_{\overline{Y}} = \mathcal{O}_X$. Furthermore $\overline{Y} = \text{Proj}\,_Y S^\bullet(\mathcal{E})$, where $S^\bullet(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ over $\mathcal{O}_X$. Choose an affine open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X such that $\mathcal{L}|_{U_i} = \mathcal{O}_{U_i}e_i$ and

$$\mathcal{E}|_{U_i} = \mathcal{O}_{U_i}e + \mathcal{O}_{U_i}\widetilde{e}_i \quad \text{with} \quad \varphi(\widetilde{e}_i) = e_i \ .$$

Then we have

$$\begin{pmatrix} \widetilde{e}_j \\ e \end{pmatrix} = \begin{pmatrix} s_{ji} & t_{ji} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{e}_i \\ e \end{pmatrix} ,$$

where $s_{ji} \in \Gamma(U_{ji}, \mathcal{O}_X^*)$ and $t_{ji} \in \Gamma(U_{ji}, \mathcal{O}_X)$. Since the section S is defined by the surjection φ, we have

$$Y = \bigcup_i \text{Spec}\ A_i[\widetilde{e}_i/e]$$

with $U_i = \text{Spec}\ A_i$. Let $x_i = \widetilde{e}_i/e$ and define an A_i-trivial lnd δ_i on $\Gamma(f^{-1}(U_i), \mathcal{O}_Y)$ by $\delta = \partial/\partial x_i$. Then we have $x_j = s_{ji}x_i + t_{ji}$ and

$$s_{ji}\frac{\partial}{\partial x_j}(x_j) = s_{ji} = \frac{\partial}{\partial x_i}(s_{ji}x_i + t_{ji})$$

on $U_{ji} = U_j \cap U_i$. This implies that $\delta_j = s_{ji}^{-1}\delta_i$. Let $\mathcal{L}$ be an invertible sheaf on X with transition functions $\{s_{ji}\}$ with respect to $\mathcal{U}$. Then the Y-group scheme $G_{a,\mathcal{L}}$ acts on Y via $\{\delta_i\}_{i\in I}$, and $Y//G_{a,\mathcal{L}}$ is identified with X. $\square$

Example 7.4. Let $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and let Δ be the diagonal of Σ_0. Let $[x_0 : x_1]$ (resp. $[y_0 : y_1]$) be the homogeneous coordinates of the first factor (resp. the second factor) of Σ_0. The complement $X := \Sigma_0 \setminus \Delta$ is the Danielewski surface. By the Segre embedding, we introduce regular functions on X as

$$x = \frac{x_0y_0}{x_1y_0 - x_0y_1},\ y = \frac{x_1y_1}{x_1y_0 - x_0y_1},\ z = \frac{x_0y_1}{x_1y_0 - x_0y_1},\ u = \frac{x_1y_0}{x_1y_0 - x_0y_1}.$$

Then X is identified with a hypersurface

$$xy = zu = z(z+1),$$

where $u = z+1$. The first projection $\overline{f} : \Sigma_0 \to \mathbb{P}^1$ induces an $\mathbb{A}^1$-bundle morphism $f : X \to \mathbb{P}^1$ defined by

$$f(x, y, z) = [x : u] = [z : y].$$

With the notations of Lemma 7.1, $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(2)$. Let $\mathcal{U} = \{U_0, U_1\}$, where $U_0 = \{x_0 \neq 0\}$ and $U_1 = \{x_1 \neq 0\}$. Then $f^{-1}(U_0) = \mathrm{Spec}\, k[t, x]$ with $U_0 = \mathrm{Spec}\, k[t], y = t(tx-1), z = tx - 1, u = tx$ and $f^{-1}(U_1) = \mathrm{Spec}\, k[s, y]$ with $U_1 = \mathrm{Spec}\, k[s], x = s(sy+1), z = sy, u = sy + 1$. Let $\delta_0 = \partial/\partial x$ and $\delta_1 = \partial/\partial y$. Then it follows that $t^2(\partial/\partial y) = \partial/\partial x$, i.e., $t^2\delta_1 = \delta_0$. Hence, with $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(2)$, $\mathbb{L}$ acts on X in such a way that $X/\!/\mathbb{L} = \mathbb{P}^1$.

A similar argument to Example 7.4 gives the following example.

Example 7.5. Let Σ_d be the Hirzebruch surface of degree d and let S be an ample section with $n := (S^2)$. Let $X := \Sigma_d \setminus S$ which is called the Danilov-Gizatullin surface (see [18]). Let $f : X \to \mathbb{P}^1$ be the restriction of the canonical $\mathbb{P}^1$-fibration of Σ_d. Then $G_{a,\mathcal{O}_{\mathbb{P}^1}(n)}$ acts on X in such a way that $X/\!/G_{a,\mathcal{O}_{\mathbb{P}^1}(n)} = \mathbb{P}^1$.

Next we consider the case where $f : Y \to X$ is simply an $\mathbb{A}^1$-fibration. We need the following result.

Lemma 7.6. (1) *Let $X = \mathrm{Spec}\, A, Y = \mathrm{Spec}\, B$ and $f : Y \to X$ be an $\mathbb{A}^1$-fibration, where B is an affine domain and A is an affine subdomain of B. Then there exists an A-trivial lnd δ of B.*

(2) *Let δ, δ' be A-trivial lnds on B. Then there exist nonzero element $a, a' \in A$ such that $a'\delta = a\delta'$.*

(3) *Suppose that B is factorial and A is factorially closed in B. Further assume that both δ and δ' are reduced, i.e., there are no non-invertible elements of A which divides δ (or δ'). Then $\delta = u\delta'$, where $u \in A^*$.*

Proof. (1) There exists an affine open set $D(a)$ of X such that $f^{-1}(D(a)) \cong D(a) \times \mathbb{A}^1$. Hence $B[a^{-1}] = A[a^{-1}][x]$. Let $\delta = a^N(\partial/\partial x)$, where N is chosen in such a way that $\delta(b_i) \in B$ with $B = A[b_1, \ldots, b_r]$. The δ is an A-trivial lnd of B.

(2) Let $K = Q(A)$. Then δ, δ' incuce K-trivial lnds of $B_K := B \otimes_A K = K[x]$. Then $\delta = \lambda(\partial/\partial x)$ and $\delta' = \mu(\partial/\partial x)$, where $\lambda, \mu \in K$. Then there exist $a, a' \in A \setminus \{0\}$ such that $a'\delta = a\delta'$.

(3) Note that the assumption implies that A is factorial. Since $a'\delta = a\delta'$, one can cancel out the $\gcd(a, a')$ from a and a', and assume that $\gcd(a, a') = 1$. Let z be an arbitrary element of B. Then $a\delta'(z) = a'\delta(z)$. Since B is factorial and A is factorially closed in B, any prime element of A remains prime in B. Hence a divides $\delta(b)$. Namely a divides δ. Since δ is reduced by the assumption, $a \in A^*$. Similarly, $a' \in A^*$. Hence $\delta = (aa'^{-1})\delta'$ with $u := aa'^{-1} \in A^*$. □

Theorem 7.7. *Let $f : Y \to X$ be an $\mathbb{A}^1$-fibration such that*

(1) *Y is a factorial affine variety of dimension less than or equal to three* [6] *and the image of f contains all codimension one points of X.*
(2) *Every fiber of f either is the empty set or has equi-dimension one.*
(3) *X is normal.*

Then there exists an invertible sheaf $\mathcal{L}$ on X such that the $\mathcal{L}$-twisted X-group scheme $G_{a,\mathcal{L}}$ acts on the X-scheme Y and $Y//G_{a,\mathcal{L}} = X$.

Proof. (1) Note that f is an affine morphism. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open covering of X. For every $i \in I$, let $V_i = f^{-1}(U_i)$. Since V_i is affine, we write $V_i = \operatorname{Spec} B_i$ and $U_i = \operatorname{Spec} A_i$, where we can assume that A_i and B_i are affine domains. Then $f_i := f|_{V_i} : V_i \to U_i$ is an $\mathbb{A}^1$-fibration. Then there exists an A_i-trivial lnd δ_i such that $A_i \subseteq \operatorname{Ker} \delta_i \subset B_i$, where $\operatorname{Ker} \delta_i$ is an affine domain by the assumption (1). Then f_i splits as

$$f_i : V_i \xrightarrow{\pi_i} Z_i \xrightarrow{\tau_i} U_i,$$

where $Z_i = \operatorname{Ker} \delta_i$. Then τ_i is a birational morphism whose fibers are either the empty set or a finite set by the assumption (2). Since X is normal, τ_i is then an open immersion by Zariski's Main Theorem. Since the image of τ_i contains all codimension one points of U_i, it follows that τ_i is biregular.

[6] This assumption is used only for the finite generation of $\operatorname{Ker}\delta_i$ in the subsequent proof.

(2) Let $U_{ji} = U_i \cap U_j$ and $V_{ji} = V_i \cap V_j = f^{-1}(U_{ji})$. Write $U_{ji} = \mathrm{Spec} A$ and $V_{ji} = \mathrm{Spec}\, B$. Since $\pi_i = f|_{V_i} : V_i \to U_i$ is the quotient morphism by the G_a-action σ_i associated to the lnd δ_i. Hence V_{ji} is a G_a-stable open set of V_i, whence the restriction $\sigma_i|_{V_{ji}}$ corresponds to the lnd δ of B. Since B is affine domain, the argument in the step (1) shows that $A = \mathrm{Ker}\, \delta$. Similarly, $\sigma_j|_{V_{ji}}$ corresponds to the lnd δ' of B such that $\mathrm{Ker}\, \delta' = A$. Note that B is factorial. In fact, if D is an irreducible subvariety of codimension 1 of V_{ji}, let $\overline{D}$ be the closure of D in Y. Since Y is factorial, there exists a regular function g on Y such that $\overline{D} = V(g)$. Then D is defined by $g|_{V_{ji}} = 0$. Since A is factorially closed in B, Lemma 7.6, (3) implies that $\delta = u\delta'$ with $u \in A^*$. Namely we have $\delta_i|_{V_{ji}} = s_{ji}\delta_j|_{V_{ji}}$. Since it follows that $s_{ki} = s_{kj}s_{ji}$, let $\mathcal{L}$ be the invertible sheaf on X with the transition functions $\{s_{ji}\}$ with respective to $\mathcal{U}$. Then $G_{a,\mathcal{L}}$ acts on the X-scheme Y locally via $\{\delta_i\}_{i \in I}$ so that $Y/\!/G_{a,\mathcal{L}} = X$. $\square$

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