Specializations of nonsymmetric Macdonald polynomials at infinity

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Notation

$R \subset \mathbb{R}^n$	reduced irreducible root system
$\alpha_i \in R$	simple roots
$\alpha_i^\vee \in R^\vee$	simple coroots
s_i	simple reflections
$W = \langle s_i \rangle$	Weyl group
$\ell(w)$	length function
w_0	long element
$Q = \bigoplus \mathbb{Z}\alpha_i$	root lattice
P	weight lattice

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Untwisted affinization of $R^\vee$ (results hold more generally)

$R^\vee + \mathbb{Z}\delta$	affine coroots
$\alpha_0^\vee = -\theta^\vee + \delta$	$\theta^\vee =$ highest coroot
$W_{\mathrm{aff}} = Q \rtimes W$	affine Weyl group
$W_{\mathrm{ext}} = P \rtimes W$	extended affine Weyl group
$\Pi = W_{\mathrm{ext}}/W_{\mathrm{aff}} \cong P/Q$	length zero elements
$w =: t_{\mathrm{wt}(w)}\mathrm{dir}(w)$	where $w \in W_{\mathrm{ext}}$, $\mathrm{wt}(w) \in P$, $\mathrm{dir}(w) \in W$

Nonsymmetric Macdonald polynomials

The *nonsymmetric Macdonald polynomials* $E_\lambda(X; q, v)$ lie in the group algebra $\mathbb{Q}(q,v)[P] = \mathbb{Q}(q,v)[X^\lambda : \lambda \in P]$; they form a basis.

They are variants of the symmetric Macdonald polynomials $P_\lambda(X; q, v)$, which form a basis of $\mathbb{Q}(q,v)[P]^W$ and generalize the Weyl characters ($q = v^2$), Hall-Littlewood polynomials ($q = 0$), Jack polynomials ($v^2 = q^k$), and other important families of symmetric polynomials.

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Demazure-Lusztig operators

$$T_i = vs_i + \frac{v - v^{-1}}{X^{\alpha_i} - 1}(s_i - 1) \qquad \text{where} \qquad w(X^\lambda) = X^{w(\lambda)}$$

These operators are an important ingredient in the construction of the E_λ.

$T_w := T_{i_1} \cdots T_{i_\ell}$ is independent of the reduced expression $w = s_{i_1} \cdots s_{i_\ell}$.

Example for $R = B_2$

$$
\begin{aligned}
E_{(-1,0)}(X;q,v) = X^{(-1,0)} &+ \frac{(1-v)(1+v)}{1-qv^2}\left(X^{(0,1)} + X^{(0,-1)}\right) \\
&+ \frac{(1-v)(1+v)(1-qv^6)}{(1-qv^2)(1-qv^3)(1+qv^3)}\ X^{(1,0)} \\
&+ \frac{(1-v)(1+v)(1+qv^2)(1-qv^4)}{(1-qv^2)(1-qv^3)(1+qv^3)}
\end{aligned}
$$

Remarks

- Sage calculates $E_\lambda(X;q,v)$ for any (affine) type.
- $E_\lambda(X;q,v)$ is well-defined at $q^{\pm 1} = 0$ or $v^{\pm 1} = 0$.
- Let m_λ denote the minimal coset representative of t_λ for W_{ext}/W. Then X^μ appears in $E_\lambda(X;q,v)$ iff $m_\mu \leq m_\lambda$ in Bruhat order.

Some specializations of $E_\lambda(X; q, v)$

$q = 0$ p-adic Iwahori-Spherical functions (Ion)	$q = \infty$ p-adic Iwahori-Whittaker functions (Brubaker-Bump-Licata)
$v = 0$ level-one affine Demazure characters (Ion)	$v = \infty$???

Alcove paths

Let $u, w \in W_{\text{ext}}$ and fix a reduced expression $w = \pi s_{i_1} \cdots s_{i_\ell}$.

Definition

An *alcove path* of type $(i_1, \ldots, i_\ell)$ starting at u is a sequence of elements $u_0, u_1, \ldots, u_\ell \in W_{\text{ext}}$ satisfying

$$u_0 = u\pi \quad \text{and} \quad u_k \in \{u_{k-1}, u_{k-1}s_{i_k}\} \text{ for } k \geq 1.$$

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By abuse of notation, we write $\mathcal{B}(u, w)$ for the set of alcove paths of type $(i_1, \ldots, i_\ell)$ starting at u.

Say that p has a $\pm$-*fold* at step k if $u_k = u_{k-1}$ and

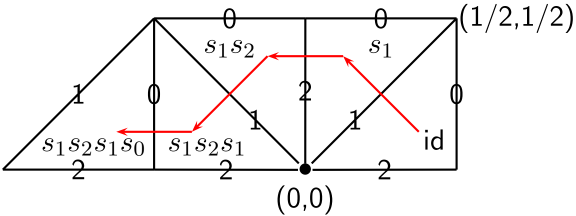
$$u_{k-1}(\alpha_{i_k}^\vee) \in \mathbb{Z}\delta \pm R_+^\vee.$$

Visualizing alcove paths

Alcoves are connected components of $\mathbb{R}^n \setminus \bigcup_{\alpha^\vee+m\delta}\{x : \langle \alpha^\vee, x\rangle + m = 0\}$.

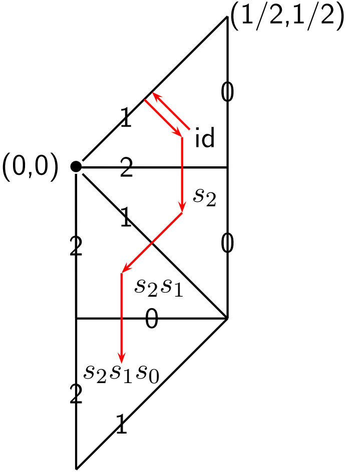
W_{aff} acts simply-transitively on the set of alcoves.

Examples: Alcove paths for $R = B_2$. $\alpha_1 = (1,-1),\ \alpha_2 = (0,1)$



$u = \mathrm{id},\ \ w = s_1s_2s_1s_0$

no folds



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$+$-fold at step 1

Ram-Yip formula

Recall that m_λ is the minimal coset representative of t_λ for W_{ext}/W.

Define $w_\lambda \in W$ by $t_\lambda = m_\lambda w_\lambda$.

Let $\mathrm{wt}(p) = \mathrm{wt}(u_\ell)$, $\mathrm{dir}(p) = \mathrm{dir}(u_\ell)$.

Theorem (Ram-Yip)

$$T_u E_\lambda(X; q, v) = v^{-\ell(w_\lambda)} \sum_{p \in \mathcal{B}(u, m_\lambda)} X^{\mathrm{wt}(p)} v^{\ell(\mathrm{dir}(p))} f^+(p) f^-(p)$$

Here $f^\pm(p)$ are explicit rational functions of q, v built from the $\pm$-folds.

They are products of terms of the form (where $a, b \geq 0$)

$$\frac{v^{-1} - v}{1 - q^a v^b} \quad \text{for } +$$

$$\frac{(v^{-1} - v) q^a v^b}{1 - q^a v^b} \quad \text{for } -$$

Specialization at $q = \infty$

Let $\mathcal{B}^-(u, w)$ be the set of alcove paths with all folds negative.

Let $|p|$ denote the number of folds in an alcove path p.

Proposition (O.-Shimozono)

$$E_\lambda(X; \infty, v^{-1}) = v^{\ell(w_0) - 2\ell(w_\lambda)} \sum_{p \in \mathcal{B}^-(\mathrm{id}, m_\lambda)} X^{\mathrm{wt}(p)} v^{\ell(w_0 \mathrm{dir}(p))} (v^{-1} - v)^{|p|}$$

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Proof. Use the formula

$$E_\lambda(X^{-1}; q^{-1}, v^{-1}) = v^{\ell(w_0) - 2\ell(w_\lambda)} T_{w_0} E_{-w_0(\lambda)}(X; q, v)$$

and take $q \to 0$ in the Ram-Yip formula for the right-hand side.

Quantum Bruhat graph

Our formula for at $v = \infty$ requires the *quantum Bruhat graph*, which has vertices $w \in W$ and directed labeled edges $w \xrightarrow{\alpha} ws_\alpha$ for $\alpha \in R_+$ and

$$\begin{aligned} &\ell(ws_\alpha) = \ell(w) + 1 && \text{(Bruhat edge)} \\ \text{or} \quad &\ell(ws_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho \rangle + 1 && \text{(quantum edge)} \end{aligned}$$

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We also need a projection of $p \in \mathcal{B}(u, w)$ to a sequence in $W^{|p|}$, defined by successively deleting simple reflections at fold positions from left to right and taking dir.

Example. Take $u = \mathrm{id}$ and $w = t_{(-1,0)} = s_1 s_2 s_1 s_0$ for $R = B_2$. Let p have folds at steps 1 and 3. Then the projection of p is (y_0, y_1, y_2) where

$$\begin{aligned} uw = s_1 s_2 s_1 s_0 \qquad & y_0 = \mathrm{dir}(s_1 s_2 s_1 s_0) = \mathrm{id} \\ s_2 s_1 s_0 \qquad & y_1 = \mathrm{dir}(s_2 s_1 s_0) = s_1 \\ s_2 \quad s_0 \qquad & y_2 = \mathrm{dir}(s_2 s_0) = s_2 s_1 s_2 s_1 \end{aligned}$$

Specialization at $v = \infty$

Let $\overleftarrow{\mathcal{QB}}(u, w)$ be the subset of $\mathcal{B}(u, w)$ made up of alcove paths that project to *reverse* paths in the quantum Bruhat graph.

Theorem (O.-Shimozono)

$$E_\lambda(X; q^{-1}, \infty) = \sum_{p \in \overleftarrow{\mathcal{QB}}(\mathrm{id}, m_\lambda)} X^{\mathrm{wt}(p)} q^{n(p)} \quad \text{for explicit } n(p) \in \mathbb{Z}_{\geq 0}.$$

Remarks

- An analogous result at $v = 0$ due to Lenart was our starting point.
- Proof uses the "T_{w_0}-formula" but is more subtle than at $q = \infty$.
- Corollary: $E_\lambda(X; q^{-1}, \infty)$ has coefficients in $\mathbb{Z}_{\geq 0}[q]$.
- Cherednik and E. Feigin conjecture a relation to the PBW filtration of level-one affine Demazure modules, for antidominant λ.

Example for $R = B_2$

$\lambda = (-1, 0) \qquad m_\lambda = t_\lambda = s_1 s_2 s_1 s_0$

Let $p \in \mathcal{B}(\mathrm{id}, s_1 s_2 s_1 s_0)$ with folds at steps 1 and 3.

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Then:

- $\mathrm{wt}(p) = \mathrm{wt}(s_2 s_0) = (1, 0)$
- p projects to the following reverse path in the quantum Bruhat graph

$$\mathrm{id} \xleftarrow{\alpha_1} s_1 \xleftarrow{\alpha_1 + 2\alpha_2} s_2 s_1 s_2 s_1$$

  with both edges quantum.

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  with both edges quantum.

$$E_{(-1,0)}(X; q^{-1}, \infty) = X^{(-1,0)} + q^2 X^{(1,0)} + q\left(X^{(0,-1)} + X^{(0,1)} + X^{(0,0)}\right)$$