BIRS Workshop “Non-Gaussian Multivariate Statistical Models and their Applications” May 23, 2013

Skew *t*-copula and its estimation:

For application to risk aggregation

Toshinao Yoshiba
(Bank of Japan / The Institute of Statistical Mathematics)
E-mail: toshinao.yoshiba@boj.or.jp

Views expressed in this presentation are those of the author and do not necessarily reflect the official views of the Bank of Japan.

Outline

I. Introduction

II. Construction of skew *t*-copula

III. Estimation of skew *t*-copula

IV. Tail dependence of skew *t*-copula

V. Conclusions and open problems

I. Introduction

- Practical needs for risk aggregation
  1. Overall dependence is captured by linear/rank correlation.
  2. Lower tail dependence is captured.
  3. Asymmetry of tail dependence is captured.

Which copula?

- 1 : Gaussian copula
- 1 & 2 : Student's *t*-copula
- 1 & 2 & 3 : Skew *t*-copula
  - HAC or vine copula may be another way.

I. Introduction (cont.)

- Variety of skew *t*-copula
  - Wide variety for multivariate skew *t*-distribution

1. Demarta and McNeil (2005) copula (GH skew *t*)

$$\mathbf{X} = \boldsymbol{\gamma} V^{-1} + \frac{\mathbf{Z}}{\sqrt{V}} \qquad V \sim G(\nu/2, \nu/2),\ \mathbf{Z} \sim N_d(0, \Sigma)$$

skewness vector

Student's *t* random vector

2. Smith, Gan, and Kohn (2012) copula

- Implied in Sahu, Dey, and Branco (2003) skew *t*-distribution

$$\mathbf{X} = \frac{\Gamma |\mathbf{W}| + \mathbf{Z}}{\sqrt{V}} \qquad V \sim G(\nu/2, \nu/2),\ \mathbf{Z} \sim N_d(0, \Sigma)$$

$$\mathbf{W} \sim N_d(0, I),\ \Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_d)$$

skewness matrix

skewed random vector

I. Introduction (cont.)

- Variety of skew *t* copula (cont.)
  3. Azzalini and Capitanio (2003) distribution
  – Kollo and Pettere (2010) tried to construct its copula

$$\mathbf{X} = \frac{\mathbf{Y}}{\sqrt{V}} \qquad V \sim G(\nu/2, \nu/2),\ \mathbf{Y} \sim SN_d(\underline{\boldsymbol{\lambda}}, \Psi)$$

skewness vector

$$Y_j = \frac{\underline{\lambda_j} |Z_0| + Z_j}{\sqrt{1+\lambda_j^2}} \qquad Z_0 \sim N(0,1),\ \mathbf{Z} \sim N_d(0, \Psi)$$

skewness

– Z_0 is scalar which is different from Sahu, Dey, and Branco (2003) .

II. Construction of skew *t*-copula

- Azzalini and Capitanio (2003) *d*-variate skew *t*-distribution $\mathrm{St}_d(0,\Omega,\boldsymbol{\alpha},\nu)$ has density

$$g(\mathbf{x}) = 2t_{d,\nu}(\mathbf{x};\Omega)T_{1,\nu+d}\left(\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x}\sqrt{\frac{\nu+d}{\mathbf{x}^{\mathrm{T}}\Omega^{-1}\mathbf{x}+\nu}}\right),$$

where $\Omega = \Lambda^{-1}(\Psi+\boldsymbol{\lambda}\boldsymbol{\lambda}^{\mathrm{T}})\Lambda^{-1}, \boldsymbol{\alpha} = \dfrac{\Lambda\Psi^{-1}\boldsymbol{\lambda}}{\sqrt{1+\boldsymbol{\lambda}^{\mathrm{T}}\Psi^{-1}\boldsymbol{\lambda}}},$

$$\Lambda = \mathrm{diag}(\sqrt{1+\lambda_1^2},\ldots,\sqrt{1+\lambda_d^2}).$$

  - $t_{d,\nu}(\cdot;\Omega)$: *d*-variate Student's *t* density with d.f.$=\nu$ and correlation matrix Ω
  - $T_{1,\nu+d}(\cdot)$: univariate Student's *t* distribution function with d.f.$=\nu+d$

II. Construction of skew *t*-copula (cont.)

- Marginal distribution for $\mathrm{St}_d(0,\Omega,\boldsymbol{\alpha},\nu)$ is $\mathrm{St}_1(0,1,\lambda_j,\nu)$
  - j-th marginal density

$$g_j(x) = 2t_{1,\nu}(x)T_{1,\nu+1}\left(\lambda_j x\sqrt{\frac{\nu+1}{x^2+\nu}}\right)$$

Kollo and Pettere (2010) misspecify as α_j

$$X_j = \frac{\lambda_j|Z_0|+Z_j}{\sqrt{V}\sqrt{1+\lambda_j^2}}$$

skewness

$$Z_0 \sim N(0,1),\ \mathbf{Z} \sim N_d(0,\Psi)$$

  - If $\mathbf{Z}$ is correlated, then $\boldsymbol{\alpha} \neq \boldsymbol{\lambda}$

II. Construction of skew *t*-copula (cont.)

- Skew *t*-copula based on Azzalini and Capitanio (2003)

$$C_{st}(u_1,\ldots,u_d;\Omega,\boldsymbol{\lambda},\nu)$$

$$=\mathrm{St}_d(\mathrm{St}_1^{-1}(u_1;0,1,\lambda_1,\nu),\ldots,\mathrm{St}_1^{-1}(u_d;0,1,\lambda_d,\nu);0,\Omega,\boldsymbol{\alpha},\nu)$$

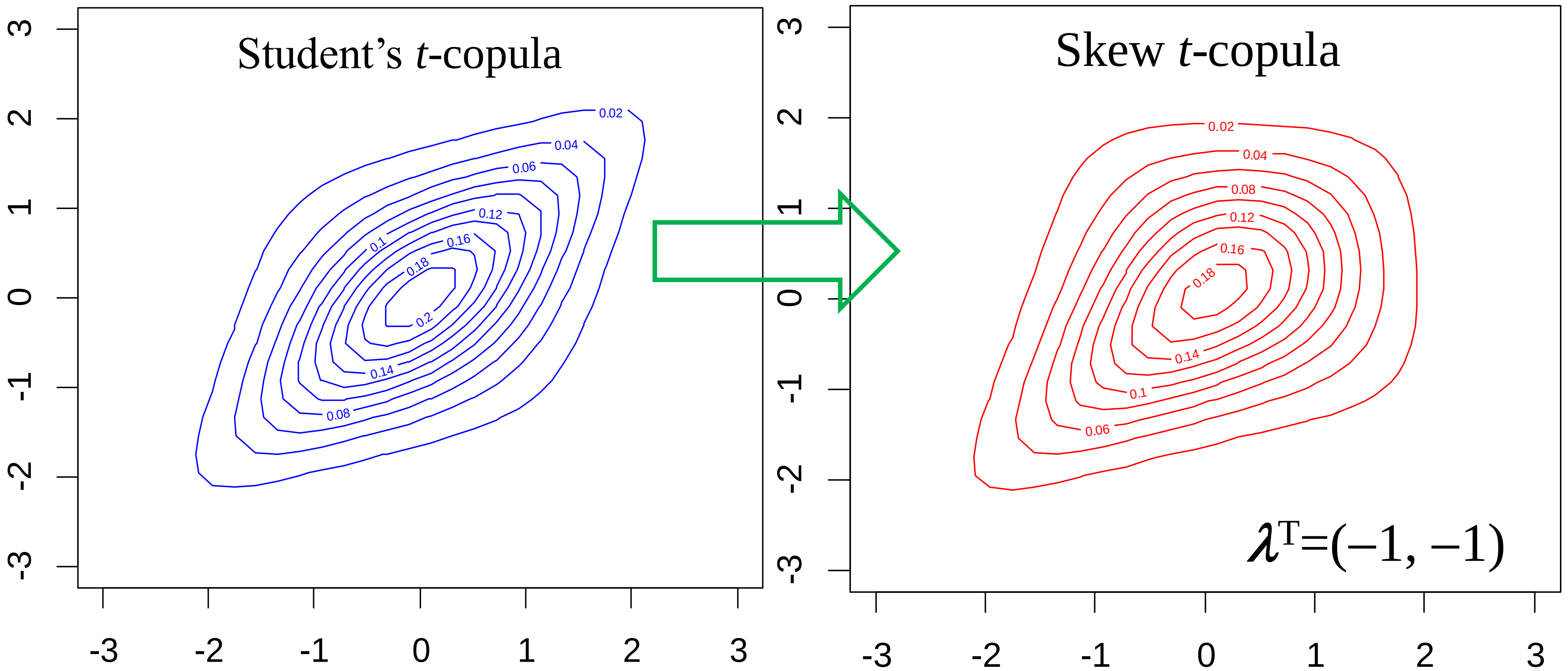
where

$$\boldsymbol{\alpha}=\frac{\Lambda\Psi^{-1}\boldsymbol{\lambda}}{\sqrt{1+\boldsymbol{\lambda}^{\mathrm{T}}\Psi^{-1}\boldsymbol{\lambda}}}=\frac{\Lambda(\Lambda\Omega\Lambda-\boldsymbol{\lambda}\boldsymbol{\lambda}^{\mathrm{T}})^{-1}\boldsymbol{\lambda}}{\sqrt{1+\boldsymbol{\lambda}^{\mathrm{T}}(\Lambda\Omega\Lambda-\boldsymbol{\lambda}\boldsymbol{\lambda}^{\mathrm{T}})^{-1}\boldsymbol{\lambda}}},$$

$$\Lambda=\mathrm{diag}(\sqrt{1+\lambda_1^2},\ldots,\sqrt{1+\lambda_d^2}).$$

II. Construction of skew *t*-copula (cont.)

- Student's *t*-copula $\Rightarrow$ Skew *t*-copula



  - $\rho = 0.5,\ \nu = 3$
  - Marginal distribution: Standard Normal

III. Estimation of skew *t*-copula

- Maximum likelihood estimation (MLE)
  - For n independent observations $\mathbf{u}_1, \ldots, \mathbf{u}_n$ unif. distributed on $[0,1]^d$, log-likelihood for parameter $\theta = (\Omega, \boldsymbol{\lambda}, \nu)$ is

$$l(\theta) = \sum_{i=1}^{N} l_i(\theta),$$

$$l_i(\theta) = \ln\left(\frac{2\Gamma((\nu+d)/2)}{(\pi\nu)^{d/2}\Gamma(\nu/2)}\right) - \frac{1}{2}\ln|\Omega| - \frac{\nu+d}{2}\ln\left(1+\frac{\mathbf{x}_i^{\mathrm{T}}\Omega^{-1}\mathbf{x}_i}{\nu}\right)$$

$$+\ln\left[T_{1,\nu+d}\left(\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{x}_i\sqrt{\frac{\nu+d}{\mathbf{x}_i^{\mathrm{T}}\Omega^{-1}\mathbf{x}_i+\nu}}\right)\right] - \sum_{j=1}^{d}\ln g_j(x_{ij};\lambda_j,\nu),$$

where

$$\mathbf{x}_i = (x_{i1},\ldots,x_{id}),\ \mathbf{u}_i = (u_{i1},\ldots,u_{id}),\ x_{ij} = \mathrm{St}_1^{-1}(u_{ij};0,1,\lambda_j,\nu)$$

  - Accurate quantile function is time consuming.
    - e. g. `qst` function in `sn` library for R

III. Estimation of skew *t*-copula (cont.)

- MLE (cont.)
  - Log-likelihood calculation time for n=1,000 bivariate data, using accurate quantile function (`qst` in `sn` ver. 0.4-17) takes 4.4 seconds on Windows Vista with Intel Core 2 Duo CPU 2.4GHz.
  - Using empirical quantile with 100,000 random numbers 20 times faster.
  - However, because empirical quantile may have large error in the tail, it is an issue <u>whether the empirical quantile function is enough accurate or not</u>.

III. Estimation of skew *t*-copula (cont.)

- Method of Moments
  - Using some moments which does not depend on marginal distribution is alternative estimation to MLE and intuitive.
  - Correlation matrix Ω (or Ψ) is estimated from sample rank correlation. Degree of freedom parameter ν is estimated from average tail dependence. Skewness parameter λ is estimated from difference in upper and lower tail dependence.
  - To estimate correlation matrix, <u>closed-form of rank correlation is needed.</u>

IV. Tail dependence of skew *t*-copula

- Tail dependence of bivariate copula
  - Tail conditional probabilities with some threshold *u*

$$\eta_L(u) = \Pr[F_2(X_2) < u \mid F_1(X_1) < u] = \frac{C(u,u)}{u} \qquad \text{lower}$$

$$\eta_U(u) = \Pr[F_2(X_2) > u \mid F_1(X_1) > u] = \frac{1-2u+C(u,u)}{1-u} \qquad \text{upper}$$

  - Tail dependence (definition)

$$\eta_L = \lim_{u\to 0+} \eta_L(u), \eta_U = \lim_{u\to 1-} \eta_U(u)$$

  - Lower and upper tail dependence of Student's *t*-copula

$$\eta_L = \eta_U = 2T_{1,\nu+1}\left(-\sqrt{\frac{(1-\rho)(\nu+1)}{(1+\rho)}} \right) \equiv \eta_t$$

IV. Tail dependence of skew *t*-copula (cont.)

- Tail dependence of bivariate skew *t*-copula
  - Bortot (2010) derives tail dependences as

$$\eta_L = \frac{1 - T_{1,\nu+2}\left(2\alpha\sqrt{(\nu+2)(1+\rho)/2}\right)}{1 - T_{1,\nu+1}\left(\lambda\sqrt{\nu+1}\right)}\eta_t$$

$$\eta_U = \boxed{\frac{T_{1,\nu+2}\left(2\alpha\sqrt{(\nu+2)(1+\rho)/2}\right)}{T_{1,\nu+1}\left(\lambda\sqrt{\nu+1}\right)}}\eta_t \qquad \lambda_1 = \lambda_2 = \lambda$$

$$\alpha = \frac{\lambda}{\sqrt{|(1+\rho)^2 - \lambda^2(1-\rho^2)|}}$$

  - Fung and Seneta (2010) derive some integral forms.

IV. Tail dependence of skew *t*-copula (cont.)

- Lower and upper tail dependence depend on skewness parameter λ

$$\lambda > (<) 0 \Leftrightarrow \boxed{\frac{T_{1,\nu+2}(2\alpha\sqrt{(\nu+2)(1+\rho)/2})}{T_{1,\nu+1}(\lambda\sqrt{\nu+1})}} > (<) 1$$

– $\lambda < 0 \Rightarrow \eta_L > \eta_t > \eta_U$

– $\lambda > 0 \Rightarrow \eta_L < \eta_t < \eta_U$

- Bortot (2010) uses different approximation from Fung and Seneta (2010) . <u>It is also an issue whether both tail dependence coincide.</u>

$$\int_{-\infty}^{-\sqrt{(1-\rho)(\nu+1)/(1+\rho)}} T_{1,\nu+1}(z) \frac{\partial}{\partial z} T_{1,\nu+2}\left(\frac{\alpha\sqrt{\nu+2}\{\sqrt{(1-\rho^2)/(\nu+1)}z-(1+\rho)\}}{\sqrt{1+z^2/(\nu+1)}} \right) dz = 0?$$

V. Conclusions and open problems

- We construct skew *t*-copula implied in Azzalini and Capitanio (2003) multivariate skew *t*-distribution
  – Comparative analysis with Demarta and McNeil (2005) or Smith, Gan, and Kohn (2012) skew *t*-copula is an issue.

- Estimation of skew *t*-copula have several open problems
  – For MLE, accurate and fast quantile function for univariate skew *t*-distribution is needed.
  – For method of moments, closed-form of rank correlation is needed.
  – Which way should practitioners go including some other robust methods than MLE, method of moments?

References

Azzalini, A. and A. Capitanio (2003) "Distributions generated by perturbation of symmetry with emphasis on a multivariate skew *t*-distribution," *Journal of the Royal Statistical Society Series B*, **65**(2), 367–389.

Bortot, P. (2010) "Tail dependence in bivariate skew-Normal and skew-*t* distributions," Working paper.

Demarta, S. and A. J. McNeil (2005) "The *t* Copula and Related Copulas," *International Statistical Review*, **73**(1), 111–129.

Fung, T. and E. Seneta (2010) "Tail dependence for two skew *t* distributions," *Statistics & Probability Letters*, **80**(9-10), 784–791.

Kollo, T. and G. Pettere,(2010) "Parameter estimation and application of the multivariate skew *t*-copula," in Jaworski, P. *et al*. eds. *Copula Theory and Its Applications*, Chap.15, 289–298, Springer.

Sahu, S. K., D. K. Dey, and M. D. Branco (2003) "A New Class of Multivariate Skew Distributions with Applications to Bayesian Regression Models," *The Canadian Journal of Statistics*, **31**(2), 129–150.

Smith, M. S., Q. Gan, and R. J. Kohn (2012) "Modelling dependence using skew *t* copulas: Bayesian inference and applications," *Journal of Applied Econometrics*, **27**(3), 500–522.