Weak convergence of empirical copula processes

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The empirical copula process

Situation: $\mathbf{X}_1, \ldots, \mathbf{X}_n$ $\mathbb{R}^d$-valued random vectors, $\mathbf{X}_i \sim F$, marginal distribution functions $F_1, \ldots, F_d$ continuous.

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Copula corresponding to F [F_j^-: generalized inverse]

$$C(\mathbf{u}) = F(F_1^-(u_1), \ldots, F_d^-(u_d)).$$

Plug-in estimation

$$F_n(\mathbf{u}) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{X_{i1} \leq u_1, \ldots, X_{id} \leq u_d\}, \quad F_{nj}(v) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{X_{ij} \leq v\}.$$

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Empirical Copula

$$C_n(\mathbf{u}) = F_n(F_{n1}^-(u_1), \ldots, F_{nd}^-(u_d)) = n^{-1} \sum_{i=1}^{n} \mathbb{I}\{X_{i1} \leq F_{n1}^-(u_1), \ldots, X_{id} \leq F_{nd}^-(u_d)\}.$$

Empirical copula process

$$\mathbb{C}_n := \sqrt{n}(C_n - C).$$

Applications of empirical copulae

- Testing for structural assumptions. For example: symmetry [Genest, Nešlehová, Quessy (2012)]. Null hypothesis: $C(u, v) = C(v, u)$ for all u, v.

$$T_n = n \int (C_n(u, v) - C_n(v, u))^2 dudv$$
$$\stackrel{H_0}{=} \int (\mathbb{C}_n(u, v) - \mathbb{C}_n(v, u))^2 dudv$$

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- Minimum-distance estimators of parametric copulas [Tsukahara 2005]. $\{C_\theta | \theta \in \Theta\}$ class of parametric candidate models. Estimator

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Derivation of asymptotic distributions:

Process convergence of $\mathbb{C}_n$ under general assumptions?

The empirical copula process in $(\ell^\infty([0,1]^d), \|\cdot\|_\infty)$

(W) Basic probabilistic assumption: defining

$$G_n(\mathbf{u}) := n^{-1} \sum_{i=1}^{n} \mathbb{I}\{F_1(X_{i1}) \leq u_1, ..., F_d(X_{id}) \leq u_d\}$$

we have

$$\sqrt{n}(G_n - C) \rightsquigarrow \mathbb{B}_C \text{ in } (\ell^\infty([0,1]^d), \|\cdot\|_\infty), \quad \mathbb{B}_C \in \mathcal{C}([0,1]^d) \text{ } a.s.$$

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Theorem [Segers (2012), Bücher and V. (2013)]
Assume that (W) holds and that additionally

$$(S1) \quad \dot{C}_j = \frac{\partial C}{\partial u_j} \text{ exists and is continuous for } \mathbf{u} \in \{\mathbf{v} \in [0,1]^d | v_j \in (0,1)\} \quad j = 1, ..., d.$$

Then for $\mathbf{u}^{(j)} := (1, ..., 1, u_j, 1..., 1)$ [u_j at j'th position]

$$\sqrt{n}(C_n - C)(\mathbf{u}) \rightsquigarrow \mathbb{G}_C(\mathbf{u}) := \mathbb{B}_C(\mathbf{u}) - \sum_{j=1}^d \dot{C}_j(\mathbf{u})\mathbb{B}_C(\mathbf{u}^{(j)}) \quad \text{in } (\ell^\infty([0,1]^d), \|\cdot\|_\infty).$$

Discussion

- Long history, e.g. Rüschendorf 1976, Gaenssler and Stute, 1987, Fermanian, Radulović, Wegkamp 2004.
- Segers (2012): i.i.d. data.
- Bücher and V. (2013): compact differentiability of copula map under (S1). In particular: extension to dependent data.
- Assumption (S1) almost minimal for weak convergence in $(\ell^{\infty}([0,1]^{d}), \|\cdot\|_{\infty})$ [Fermanian, Radulović, Wegkamp 2004]
- Contribution of Segers (2012) and Bücher, V. (2013): relax previous assumption that $\dot{C}_j$ exist and are continuous on $[0,1]^d$. This extends theory to many important examples!
- Assumption (W) on weak convergence of $\sqrt{n}(G_n - C)$ holds for most kinds of weakly dependent data.

'Non-smooth' copulas: examples

Problem: The assumption

$$\dot{C}_j \text{ exists and is continuous for } \mathbf{u} \in \{\mathbf{v} \in [0,1]^d | v_j \in (0,1)\}$$

is satisfied by many, but not by all interesting copulas. Example

$$C(\mathbf{u}) := \lambda u_1 u_2 + (1-\lambda) min(u_1, u_2)$$

Here

$$\dot{C}_1(\mathbf{u}) = \lambda u_2 + (1-\lambda) 1_{\{u_1 < u_2\}},$$
$$\dot{C}_2(\mathbf{u}) = \lambda u_1 + (1-\lambda) 1_{\{u_1 > u_2\}},$$

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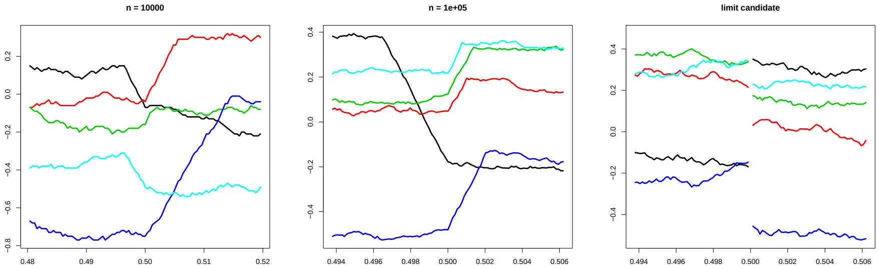
Other examples

- extreme value copulas with non-differentiable Pickands dependence function
- Marshall-Olkin copulas
- Archimedean copulas with non-smooth generators
- ...

Lack of uniform convergence.



- Left and middle: 'typical realizations' of empirical copula process
- Right: sample paths of candidate limit process.

Observation: we need a metric for which jump functions and smoothed jump functions are 'close'. Generalize Skorohod's M_2 metric.

Introducing hypi convergence

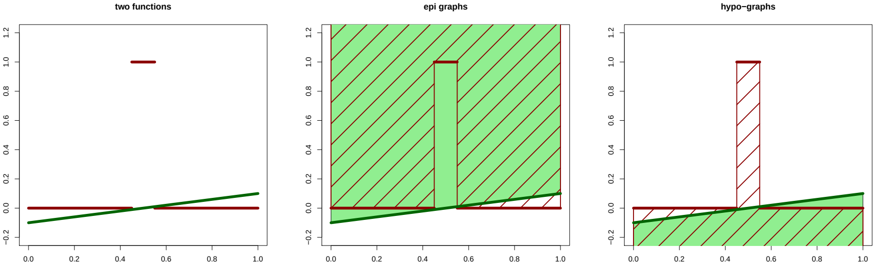
Epi- and hypograph of a function $f \in \ell^\infty([0,1]^d)$:

$$\operatorname{epi} f := \{(\mathbf{u}, t) \in [0,1]^d \times \mathbb{R} \mid f(\mathbf{u}) \leq t\}$$
$$\operatorname{hypo} f := \{(\mathbf{u}, t) \in [0,1]^d \times \mathbb{R} \mid f(\mathbf{u}) \geq t\}$$

The hypi semi-metric is defined as

$$d_{\mathrm{hypi}}(f,g) = \max\{d_{\mathcal{F}}(\mathrm{cl}(\operatorname{epi} f), \mathrm{cl}(\operatorname{epi} g)), d_{\mathcal{F}}(\mathrm{cl}(\operatorname{hypo} f), \mathrm{cl}(\operatorname{hypo} g))\}.$$

where $d_{\mathcal{F}}$ is metric on closed sets inducing Painlevé–Kuratowski convergence.



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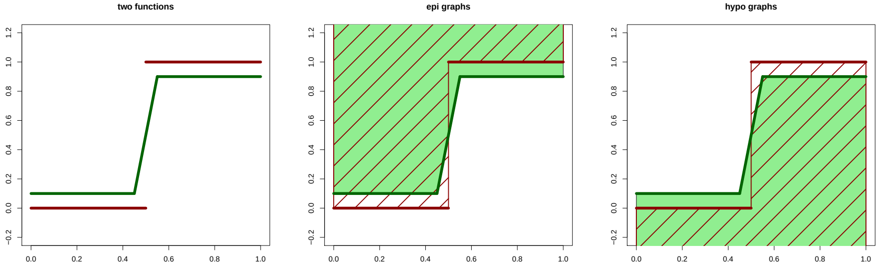

The empirical copula process on $(\ell^{\infty}_{d_{\mathrm{hypi}}}([0,1]^d), d_{\mathrm{hypi}})$

$\mathcal{D}(C) := \{\mathbf{u} \in [0,1]^d \mid \dot{C}_j(\mathbf{u}) \text{ does not exist or is not continuous for some } 1 \leq j \leq d\}$

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Theorem [Bücher, Segers, V. 2013] Under the assumptions

a) $\sqrt{n}(G_n - C) \rightsquigarrow \mathbb{B}_C$ in $(\ell^\infty([0,1]^d), \|\cdot\|_\infty)$, $\quad \mathbb{B}_C \in \mathcal{C}([0,1]^d)$ *a.s.*

b) $\mathcal{D}(C)$ is a Lebesgue-null set

we have

$$[\mathbb{C}_n]_{d_{\mathrm{hypi}}} = [\sqrt{n}(C_n - C)]_{d_{\mathrm{hypi}}} \rightsquigarrow [\mathbb{C}_C]_{d_{\mathrm{hypi}}}$$

in $(\ell^\infty_{d_{\mathrm{hypi}}}([0,1]^d), d_{\mathrm{hypi}})$, where

$$\mathbb{C}_C(\mathbf{u}) = \mathbb{B}_C(\mathbf{u}) + dC_{(-\alpha_1,\ldots,-\alpha_d)},$$

where, for $a = (a_1, \ldots, a_d) \in \{\mathcal{C}([0,1])\}^d$,

$$dC_a(\mathbf{u}) = \sup_{\varepsilon>0} \inf \left\{ \sum_{j=1}^d \dot{C}_j(\mathbf{v})\, a_j(v_j) : \mathbf{v} \in [0,1]^d \setminus \mathcal{D}(C),\ |\mathbf{v} - \mathbf{u}| < \varepsilon \right\}.$$

Some properties of d_{hypi}

Theorem [Bücher, Segers, V. 2013] For functions $f_n, f \in \ell^\infty([0,1]^d)$

- Let μ be a finite measure on $[0,1]^d$ and f with $\mu(\text{discontinuity points } f) = 0$. Then $d_{\mathrm{hypi}}(f_n, f) \to 0$ implies $\|f_n - f\|_p \to 0$.
- Convergence $d_{\mathrm{hypi}}(f_n, f) \to 0$ implies $\sup f_n \to \sup f, \inf f_n \to \inf f$.
- If f continuous in x $d_{\mathrm{hypi}}(f_n, f) \to 0$ implies $f_n(x) \to f(x)$, also uniformly over compact sets.

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Some of many possible applications

- Minimum distance estimators for irregular copulas.
- L^2 tests for structural assumptions (tests for symmetry, copulas being extreme-value, copulas belonging to parametric classes, ...)
- Analysis of local power curves of L^2-distance and Kolmogorov-Smirnov type tests.

Thank you!