Limiting Distributions of the Error Terms

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B^p-almost periodic functions

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- We say that the real function $\phi(y)$ is a B^2-*almost periodic* function if for any $\epsilon > 0$ there exists a real-valued trigonometric polynomial

$$P_{N(\epsilon)}(y) = \sum_{n=1}^{N(\epsilon)} r_n(\epsilon) e^{i\lambda_n(\epsilon)y}$$

such that

$$\limsup_{Y\to\infty} \frac{1}{Y} \int_0^Y |\phi(y) - P_{N(\epsilon)}(y)|^2 dy < \epsilon^2.$$

Riemann's Explicit Formula

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- $$\psi(x) = x - \sum_{\substack{\zeta(\rho)=0 \\ |\Im(\rho)| \leq T}} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2(xT)}{T} + \log x\right),$$

  valid for $x \geq 2$ and $T > 1$

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- On the Riemann hypothesis, it follows that

$$\frac{\psi(e^y) - e^y}{e^{y/2}} = \Re\left(\sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ 0<\gamma\leq T}} \frac{-2e^{iy\gamma}}{\rho} \right) + O\left(\frac{e^{\frac{y}{2}} \log^2(e^y T)}{T} + ye^{-\frac{y}{2}} \right).$$

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$$\phi(y) = \text{Constant}+\text{Real Trigonometric Polynomial}+\text{Error}.$$

Wintner's Theorem (1935)

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- Under the assumption of the Riemann hypothesis

$$\frac{\psi(e^y) - e^y}{e^{y/2}}$$

is a B^2-almost periodic function and so it has a limiting distribution.

Applications

Oscillation Theorems

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- **Question:** Does $P_\pi = \{x \geq 2;\ \pi(x) < \mathrm{Li}(x)\}$ has a density?

Logarithmic Density

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- For $P \subset \mathbb{R}^+$ if

$$\delta(P) = \lim_{x \to \infty} \frac{1}{\log x} \int_{t \in P \cap [2,X]} \frac{dt}{t}$$

exists we say that P has logarithmic density $\delta(P)$.

Linear Independence Conjecture (LI)

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- The multiset of the positive ordinates of the zeros of the Riemann zeta function is linearly independent over $\mathbb{Q}$.

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- **Theorem** Under the RH

$$\frac{\pi(e^y) - \mathrm{Li}(e^y)}{ye^{y/2}}$$

has a limiting distribution ν_π. Moreover under the LI $\hat{\nu}_\pi$ (the Fourier transform of ν_π) can be calculated in terms of Bessel functions, and in addition

$$\delta(P_\pi) = 0.99999973\cdots .$$

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- **Conjecture**
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- Mertens' Conjecture implies the Riemann hypothesis.
- **Oldyzko-te Riel (1985)** Mertens' Conjecture is false.

Explicit Formula for $M(x)$

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- Under the assumptions of RH and the simplicity of zeros of $\zeta(s)$ for $x \geq 2$ and $T \in \mathcal{T}$ we have

$$M(x) = \sum_{\substack{|\gamma| \leq T \\ \rho = 1/2 + i\gamma}} \frac{x^\rho}{\rho \zeta'(\rho)} + E(x, T).$$

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- **Conjecture (Gonek)** As $T \to \infty$ we have

$$J_{-1}(T) \sim \frac{3}{\pi^3} T.$$

Ng, 2004

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- **Theorem** Assume RH and $J_{-1}(T) \ll T$. Then

$$\frac{M(e^y)}{e^{y/2}}$$

has a limiting distribution ν_M. Moreover under LI the Fourier transform $\hat{\nu}_M$ can be calculated.

Mazur-Stein's Problem

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- **Problem** What can we say about $\delta(\{x; S(x) > 0\})$?
- It seems that if $\mathrm{rank}_{\mathbb{Q}}(E)$ is large then $a_E(p) < 0$ more often and so

$$\delta(\{x; S(x) > 0\}) < \frac{1}{2}.$$

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- It can be shown (under GRH and LI) that $\delta(\{x; S(x) > 0\})$ exists.
- Under the assumptions of some standard conjectures Sarnak has shown that

$$\delta(\{x; S(x) > 0\}) = \frac{1}{2}.$$

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Joint Work with Ng and Shahabi

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- $(\lambda_n)_{n\in\mathbb{N}}$= A non-decreasing sequence of positive numbers which tends to infinity.
- $(r_n)_{n\in\mathbb{N}}$= A complex sequence.
- c= A real number.
- y_0 and X_0 positive reals.
- We consider the class of functions

$$\phi(y) = c + \Re\Big(\sum_{\lambda_n \le X} r_n e^{i\lambda_n y}\Big) + \mathcal{E}(y, X),$$

for any $X \geq X_0 > 0$ where $\mathcal{E}(y, X)$ satisfies

$$\lim_{Y\to\infty} \frac{1}{Y} \int_{y_0}^{Y} |\mathcal{E}(y, e^Y)|^2 dy = 0.$$

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- **Theorem** If $r_n \ll \frac{1}{\lambda_N^{\beta}}$ for $\beta > \frac{1}{2}$ and

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then $\phi(y)$ has a limiting distribution.

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- **Theorem** If
$$\sum_{T < \lambda_n \leq T+1} 1 \ll \log T,$$
and for $0 \leq \theta < 3 - \sqrt{3}$,
$$\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 \ll T^{\theta},$$
then $\phi(y)$ has a limiting distribution.

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- ν_ϕ is the limiting distribution in the previous theorems.
- μ_{ν_ϕ} =The mean of $\nu_\phi = c$.
- $\sigma^2_{\nu_\phi}$=The variance of $\nu_\phi = c^2 + \frac{1}{2}\sum_{n=1}^{\infty} |r_n|^2$.

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General Limiting Distribution Theorems

- $\phi(y) = c + \Re\Big(\sum_{\lambda_n \le X} r_n e^{i\lambda_n y} \Big) + \mathcal{E}(y, X)$
- ν_ϕ is the limiting distribution in the previous theorems.
- **Theorem** If $\{\lambda_m\}$ is linearly independent over $\mathbb{Q}$ then

$$\hat{\nu}(\xi) = \int_{\mathbb{R}} e^{-i\xi t} d\nu(t) = e^{-ic\xi} \prod_{m=1}^{\infty} J_0(|r_m|\xi),$$

where

$$J_0(z) = \int_0^1 e^{-iz\cos(2\pi t)} dt.$$

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- E an elliptic curve over $\mathbb{Q}$.
- $L(s, E)$ be the (normalized) L-function of E.
- $$\lambda_E(p) = \frac{p + 1 - N_E(p)}{\sqrt{p}}.$$
- For $\Re(s) > 1$ we have

$$-\frac{L'(s, E)}{L(s, E)} = \sum_{p^k}^{\infty} \frac{(\log p)\lambda_E(p^k)}{p^{ks}}.$$

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- Under the assumption of the GRH for $L(s, E)$ we have

$$
\begin{aligned}
e^{-y/2} \sum_{p^k \le e^y} (\log p) \lambda_E(p^k) &= -2\mathrm{ord}_{s=1/2} L(s, E) \\
&\quad + \Re\Big(\sum_{0 < \gamma \le T} \frac{-2e^{i\gamma y}}{\rho} \Big) \\
&\quad + \mathcal{E}(y, T).
\end{aligned}
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- We have

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$$\sum_{\gamma \leq T} \frac{\gamma^2}{|\rho|^2} \ll T \log T.$$

- Under the assumption of the GRH for $L(s, E)$,

$$e^{-y/2} \sum_{p^k \leq e^y} (\log p) \lambda_E(p^k)$$

has a limiting distribution ν.

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- Under the assumption of LI we have

$$\hat{\nu}(\xi) = e^{-ic\xi}\prod_{\gamma>0} J_0\left(\frac{2\xi}{\sqrt{\frac{1}{4}+\gamma^2}}\right).$$

- ν is symmetric about its mean, so under BSD if $\mathrm{rank}_{\mathbb{Q}}(E) > 0$ then

$$\delta(\{x \geq 2;\ \sum_{p^k\leq x}(\log p)\lambda_E(p^k) < 0\}) > \frac{1}{2}.$$

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- $P_M = \#\{x > 0;\ |M(x)| \leq \sqrt{x}\}$.
- Under the assumptions of RH, LI, and $J_{-1}(T) \ll T$, $\delta(P)$ exists and

$$\delta(P) \geq 1 - 2\exp\left(-\frac{1}{2\sigma_{\nu_M}^2}\right).$$

Applications

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So
$$\delta(P_M) \geq \delta(P_\pi).$$