Self-similar solutions for curvature flows

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Part I

1. Intro

Mean curvature flow $X_t = H = \triangle_g X$ or effective part
$(X_t)^N = H = \triangle_g X$
consider three symmetries: shrinking, translating, rotating
○ shrinker.
$X(x,t) = \sqrt{-t} F(x)$

$$-\frac{1}{2\sqrt{-t}} F^N = \frac{1}{\sqrt{-t}} \triangle_g F$$

or

$$\triangle_g F = -\frac{1}{2} F^N$$

In particular $F = (x, f(x)) \in R^n \times R^1$

$$\operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) = -\frac{1}{2}\frac{-x \cdot Df + f}{\sqrt{1+|Df|^2}}$$

or

$$\triangle f - \frac{f_i f_j}{1+|Df|^2} f_{ij} = \frac{1}{2}(x \cdot Df - f)$$

when f radial

$$\frac{f_{rr}}{1+f_r^2} + \frac{n-1}{r} f_r = \frac{1}{2}(x \cdot Df - f)$$

examples: sphere, cylinder, shamrock ...

○ Translator

$X(x,t) = F(x) + te$

$$e^N = \triangle_g F$$

In particular $F = (x, f(x))$, $e = (0, \cdots, 0, 1)$

$$\operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) = \frac{1}{\sqrt{1+|Df|^2}}$$

eg. grim reaper $\frac{f_{xx}}{1+f_x^2} = 1$, $f = -\ln\cos x$

◦ Rotator

$X(x,t) = F(x) + t\partial_\theta$

1d $X(x,t) = F(x) + tJF(x)$

$$(JF)^N = \triangle_g F$$

In particular $F = (x, f(x))$

$$\operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) = \left\langle (-f, x), \frac{(-f_x, 1)}{\sqrt{1+f_x^2}} \right\rangle = \frac{ff_x + x}{\sqrt{1+f_x^2}}$$

In polar coordinates $(r(\theta, t), \theta)$
1st mean curvature flow
$(r_t \partial_r)^N = H$
$N = JT / |T| = J [r_\theta (\cos\theta, \sin\theta) + r(-\sin\theta, \cos\theta)] / |T| =$
$[r_\theta (-\sin\theta, \cos\theta) + r(-\cos\theta, -\sin\theta)] / |T|$
$\beta = \measuredangle (T, \partial_x)$

$$-\frac{r_t r}{|T|} = \frac{d\beta}{ds} = \frac{d\beta}{d\theta}\frac{d\theta}{ds}$$

or

$$-r_t r = \frac{d\beta}{d\theta}$$

2nd rotator

$r\left(\theta, t\right) = r\left(\theta + t\right)$

$$-r_{\theta} r = \frac{d\beta}{d\theta}$$

then

$$\frac{d\beta}{dr} = -r$$

and $\beta = -\frac{1}{2} r^2$ or

$$\arctan f_x = -\frac{1}{2}\left(x^2 + f^2\right).$$

By symmetry, $r = \sqrt{-2\beta}$ & $r = \sqrt{2\left(\beta - \pi\right)}$ form the Yin-Yang rotator.

2. **Theorem** (Lu WANG, 09) Any ancient self-similar solution $(x, f(x,t)) = \left(x, \sqrt{-t}\, u\left(\frac{x}{\sqrt{-t}}\right)\right)$ to

$$f_t = g^{ij}(Df)\,\partial_{ij} f \text{ in } R^n \times (-\infty, 0) \Leftrightarrow g^{ij}(Du)\,\partial_{ij} u = \frac{1}{2} x \cdot Du - \frac{1}{2} u \text{ in } R^n$$

is linear, $f(x,t) = Du(0) \cdot x = u(x)$.

Superharmonic Way (2012).

Step1. Superharmonic inner product $w = \langle N, e_{n+1} \rangle = 1/V > 0$ satisfies

$$\left(g^{ij}\partial_{ij} - \partial_t\right) w = -|A|^2 w \leq 0.$$

By self-similarity,

$$g^{ij}\partial_{ij} w - \frac{1}{2} x \cdot Dw = -|A|^2 w \leq 0.$$

Heuristic: As $g^{ij}\partial_{ij} w \leq \frac{1}{2} x \cdot Dw$, the amplifying force in the right forces w up near ∞. Otherwise, bounded w becomes unboundedly negative near ∞. Hence super solution w attains its min at a finite point, then constant.

Step2. The self-similar term rw_r with barrier like $-\varepsilon\left(|x|^2 - 100^2\right) + \min_{B_{100}} w$ forces superharmonic w to attain its global minimum at a finite point.
Strong max principle then implies that $w \equiv const. > 0$.
Step3. By the equation for w, one concludes $|A| = 0$.

RMK. Subharmonic way (2010) little longer $\beta = \arctan|Df|$

$$g^{ij}\partial_{ij}\beta - \frac{1}{2}x \cdot D\beta = \cot\beta\ \left(|A|^2 - |\nabla_g \beta|^2\right)$$

Drugan (2010) had a high co-dimension version.

RMK. Integral way (Lu WANG) even longer, but first 09, after Ecker-Huisken 90s w/ polynomial growth condition.

3. Rigidity of simple (or embedded) curve shrinker: it must be a circle w/ radius $\sqrt{2}$.

Th'm. (Gage, Hamilton, Abresch-Langer, Epstein-Weinstein, Huisken, X.P. Zhu, Andrews, ...) Any embedded solution to

$$\triangle_g X = -\frac{1}{2}X^N, \quad \text{locally } H = \left(\frac{f_x}{\sqrt{1+f_x^2}}\right)_x = \frac{1}{2}\frac{xf_x - f}{\sqrt{1+f_x^2}}$$

is the circle with radius $\sqrt{2}$ centered at the origin.

The following is another proof 2012 (A first integral and extrinsic approach).

Step1. (Colding-Minicozzi) A conservative quantity

$$\ln H - \frac{1}{4}|X|^2 = const. \quad \text{or } H = ce^{|X^2|/4}.$$

Just integrate

$$H_x = \left[\frac{1}{2}\frac{xf_x - f}{\sqrt{1+f_x^2}}\right]_x = \frac{1}{2}\frac{f_{xx}}{\left(\sqrt{1+f_x^2}\right)^3}(x + f\ f_x) = \frac{1}{4}H\ \left(|X|^2\right)_x.$$

RMK. This conservation law is already in the first integral of the support equ. $u_{\theta\theta} + u = 2/u$

$$u_\theta^2 + u^2 - 4 \ln u = const,$$

once u_θ is recognized as the tangent part of $|X|^2 = |X^T|^2 + |X^N|^2 = u_\theta^2 + u^2$.
From the conservative quantity, one directly sees that either i) $H \equiv 0$ (the shrinker is a line through the origin) or ii) $H \neq 0$ and $|X|$ is bounded.

The boundedness is from $|X|/2 \geq |X^N|/2 = |H| = |c| e^{|X|^2/2}$. All these facts (except the boundedness [C-M]) are known.

Step2. An extrinsic way to the embedded convex S^1-shrinker.
Obs1. Symmetry w.r.t. the line through furthest or nearest points, in general critical pts of distance $|X|$, by uniqueness of ode.
Obs2. At (&only at) the critical pts of H or equivalently $|X|$, $\frac{1}{2}|X| = |c|\, e^{|X|^2/4}$. In turn, it is either the furthest or nearest pt. Indeed, at critical pts, $\langle X, X_x \rangle = \langle X, (1, f_x) \rangle = 0$, then $X^T = 0$, and the identity; (vice versa).
By the analyticity of the shrinker, those critical points of distance are discrete, unless the curve is already the circle. By symmetry, any two consecutive critical pts of distance must be distinct ones: furthest and nearest. Let the angle between the corresponding rays (through A f.&B n. pts) be θ. As the shrinker is simple closed one, this angle θ can only be $2\pi/k : \pi, 2\pi/3, 2\pi/4, \cdots$.
* $\theta = 2\pi/3$ is not allowed, as the critical pts are alternating, unless the curve is already is circle. (Remember shamrocks.)

* $\theta = \pi$. Let's line up the furthest pt A & nearest pt B along vertical y-axis. By convexity, there is one and only one vertical tangent point on the right side of the curve. The v. tangent pt separates the right side of curve into two graphs over an interval $[0, \xi]$ on x-axis.

Obs. (Extrinsic)

$$1 = \left| \int_0^{\xi} \left(\frac{f_x}{\sqrt{1+f_x^2}} \right)_x dx \right| = \left| \int_0^{\xi} c e^{|X|^2/4} dx \right|.$$

Since $|X|$ is monotonic on the whole right curve, this $\theta = \pi$ configuration is not allowed.

* $\theta = 2\pi/4, 2\pi/5, \cdots$
First line up the furthest pt. A (w/ $|A| = a$ and $|B| = b$) along the vertical y-axis, then
Obs. (Extrinsic)

$$\sin\theta = \left| \int_0^{b\sin\theta} \left(\frac{f_x}{\sqrt{1+f_x^2}} \right)_x dx \right| = \left| \int_0^{b\sin\theta} ce^{|X^2|/4} dx \right| \le |c|\, e^{a^2/4}\ b$$

Thus $1 \le |c|\, e^{a^2/4}\ b$.
On the other hand, line up the nearest pt B along the vertical y-axis, then
Obs. (Extrinsic)

$$\sin\theta = \left| \int_0^{a\sin\theta} \left(\frac{f_x}{\sqrt{1+f_x^2}} \right)_x dx \right| = \left| \int_0^{a\sin\theta} ce^{|X^2|/4} dx \right| \ge |c|\, e^{b^2/4}\ a$$

Thus $1 \ge |c|\, e^{b^2/4}\ a$.

So far we have

$$a\ e^{b^2/4} \leq \frac{1}{|c|} \leq b\ e^{a^2/4}.$$

Recall

$$\frac{1}{2} b e^{-b^2/4} = |c| = \frac{1}{2} a e^{-a^2/4}.$$

We then get $ab/2 \leq 1 \leq ab/2$. Thus $ab = 2$.
Finally the sol. to the system

$$\begin{cases} ab = 2 \\ a e^{-a^2/4} = b e^{-b^2/4} \end{cases}$$

is $a = b = \sqrt{2}$. Therefore, these angles $0 < \theta \leq \pi/2$ are not allowed, unless the shrinker is already a circle.

4. Immersed S^2 shrinker (Drugan, 12)
Equ over r-axis

$$\frac{f_{rr}}{1+f_r^2} + \frac{n-1}{r} f_r = \frac{1}{2}\left(x \cdot Df - f\right)$$

over rotating-axis

$$\frac{h_{yy}}{1+h_y^2} - \frac{(n-1)}{h} = \frac{1}{2}\left(yh_y - h\right)$$

Step 1. Power series for “singular” equation $\frac{f_{rr}}{1+f_r^2} + \frac{1}{r} f_r = \frac{1}{2}\left(rf_r - f\right)$ near $r = 0$
Step2. Small height top branch cross r-axis, then blow-up
Step3. Bottom branch convex up
Step4. Bottom branch cross back radial axis, then blow-up
Step5. Lift small height until left blow-up point is the cross one (everything below sphere height 2)

5. Question

Is embedded S^2 type shrinker in R^3 the standard S^2?

RMK.

- Mean convex case, $H \geq 0$, Yes. Huisken 90s.
- star shaped, yes.

* Angenent 80s, embedded torus shrinker.
* Moller 2011, embedded high genus shrinkers.
* Drugan-Kleene 2013, ∞-many immersed rotational shrinkers of topological types: sphere, torus, plane, cylinder.

Part II curvature flows w/ potential

1. Intro:

- Lagrangian mean curvature flow in R^{2n}

$$\partial_t U = g^{ij}\partial_{ij} U \quad \text{w/ } U = Dv$$

Euclidean $\left(R^{2n}, dx^2 + dy^2\right)$, $g = I + D^2 v D^2 v \Leftrightarrow \partial_t v = \arctan D^2 v$
Pseudo-Euclidean $\left(R^{2n}, dxdy\right)$, $g = D^2 v \quad \Leftrightarrow \quad \partial_t v = \ln\det D^2 v$

- Kahler Ricci flow

$$\left.\begin{aligned} \partial_t g_{i\bar{k}} &= -R_{i\bar{k}} \\ g_{i\bar{k}} &= v_{i\bar{k}} \\ Ric &= -\partial\bar{\partial}\ln\det\partial\bar{\partial} v \end{aligned}\right\} \quad \Leftrightarrow \quad \partial_t v = \ln\det\partial\bar{\partial} v$$

Consider self similar shrinking sols in $R^{2n}\times(-\infty, 0)$:
$v(x,t) = -tu\left(x/\sqrt{-t}\right)$

$$\arctan D^2 / \ln\det D^2 / \ln\det\partial\bar{\partial}\ \ u = \frac{1}{2}x\cdot Du(x) - u(x).$$

2. **Th'm** (Chau-Chen-Y. 10)

- Let u be an entire smooth sol to $\arctan\lambda_1 + \cdots + \arctan\lambda_n = \frac{1}{2}x \cdot Du(x) - u$ in R^n. Then $u = u(0) + \frac{1}{2}\left\langle D^2u(0)\,x, x\right\rangle$.
- Let u be an entire smooth convex sol to $\ln\det D^2u = \frac{1}{2}x \cdot Du(x) - u$ in R^n satisfying $D^2u(x) \geq \frac{2(n-1)}{|x|^2}$ for large $|x|$. Then u is quadratic.
- Let u be an entire smooth pluri-subharmic sol to $\ln\det\partial\bar\partial u = \frac{1}{2}x \cdot Du(x) - u$ in C^m satisfying $\partial\bar\partial u \geq \frac{2m-1}{2|x|^2}$ for large $|x|$. Then u is quadratic.

RMK. For arctan case: Y. 09 bounded Hessian, Chau-Chen-He (09) $\left|D^2u\right| \leq 1-\delta$, R. Huang-Z. Wang (10), $\left|D^2u\right| \leq 1$, rigidity was derived. For $\ln\det D^2$ case w/ similar lower bound, R. Huang-Z. Wang (10) derived the rigidity.

RMK. Q. Ding-Xin (12), for $\ln\det D^2$ without any lower bound, derived the rigidity.

Proof. The argument is similar to the co-dim 1 case. (In fact the other way around.)

• Euclidean arctan case:

Step 0. Equs $v(x,t) = -tu\left(\frac{x}{\sqrt{-t}}\right)$, $v_t(x,t) = \Theta\left(\frac{x}{\sqrt{-t}}\right)$

$$v_t = \arctan D^2 v \Leftrightarrow \arctan D^2 u = \frac{1}{2} x \cdot Du(x) - u(x)$$

$$\partial_t(v_t) = tr\left[\left(I + D^2 v D^2 v\right)^{-1} D^2 v_t\right] \Leftrightarrow \; g^{ij}\partial_{ij}\Theta(x) = \frac{1}{2} x \cdot D\Theta(x)$$

Heuristic: The amplifying force in the right forces bounded Θ to be constant.

Step 1. Phase Θ attains its max at a finite point.

As $g^{-1} = \left(I + D^2 v D^2 v\right)^{-1} \leq I$, we can construct a convex upper barrier b s.t.

$$g^{ij}\partial_{ij} b \leq \delta_{ij}\partial_{ij} b = \triangle b \leq \frac{1}{2} r b_r.$$

In fact $b = \varepsilon r^{1+\delta} + \max_{\partial B_{r_0}} \Theta$ is a super sol on $R^n \backslash B_{r_0}$, and larger than Θ at ∂B_{r_0} and ∞. By the weak max principle

$$\Theta \leq \varepsilon r^{1+\delta} + \max_{\partial B_{r_0}} \Theta \text{ on } \ R^n \backslash B_{r_0} \ .$$

Let ε go to 0, we have $\max_{R^n} \Theta = \max_{B_{r_0}} \Theta$.
Step 2. Phase Θ is constant by the strong max principle.
Step 3. Potential u is quadratic by Euler's formula applied to

$$\Theta(0) = \frac{1}{2} x \cdot Du(x) - u(x) \, .$$

• Case $\ln \det D^2 u$ and $\ln \det \partial \bar{\partial}$: Review the above argument, only lower bound on Hessian is enough, matching the barrier equ, we have the inverse quadratic decay (completeness) condition on the metric to reach rigidity.

3. **Th'm** (Drugan-Lu-Y. 13) Let u be an entire smooth pluri-subharmonic sol to $\ln \det \partial\bar{\partial} u = \frac{1}{2} x \cdot Du(x) - u$ in C^m s.t. the metric $g = \partial\bar{\partial} u$ is complete. Then $u(x)$ is quadratic.

RMK. C^m can be replaced by any domain Ω containing the origin.
Proof. The idea is still to force the volume (or phase) element attains its global max at a finite point, instead of using a barrier as in [Chau-Chen-Y], now by considering its radial derivative–which is the scalar curvature.
Step 0. Equs: $v(x,t) = -tu\left(\frac{x}{\sqrt{-t}}\right)$, $v_t(x,t) = \Phi\left(\frac{x}{\sqrt{-t}}\right)$,
$v_{tt}(x,t) = \frac{-S\left(\frac{x}{\sqrt{-t}}\right)}{-t}$

$$v_t = \ln \det \partial\bar{\partial} v \Leftrightarrow \ln \det \partial\bar{\partial} u = \frac{1}{2} x \cdot Du(x) - u$$

$$\partial_t (v_t) = tr\left(\partial\bar{\partial} v\right)^{-1} \partial\bar{\partial} v_t \Leftrightarrow -S = g^{i\bar{k}} \partial_{i\bar{k}} \Phi = \frac{1}{2} x \cdot D\Phi(x).$$

$$\partial_t (v_{tt}) = tr\left(\partial\bar{\partial}v\right)^{-1}\partial\bar{\partial}v_{tt} - tr\left(\partial\bar{\partial}v\right)^{-1}\partial\bar{\partial}v_t\left(\partial\bar{\partial}v\right)^{-1}\partial\bar{\partial}v_t$$

$$\leq tr\left(\partial\bar{\partial}v\right)^{-1}\partial\bar{\partial}v_{tt} - \frac{1}{m}\left[tr\left(\partial\bar{\partial}v\right)^{-1}\partial\bar{\partial}v_t\right]^2$$

$$\text{or} \quad R_t \geq \triangle_g R + \frac{1}{m}R^2$$

$\Leftrightarrow$

$$g^{i\bar{k}}\partial_{i\bar{k}}S \leq -\frac{1}{m}S^2 + S + \frac{1}{2}x \cdot DS(x),$$

where we used $\frac{-S\left(\frac{x}{\sqrt{-t}}\right)}{-t} = v_{tt} = tr\left(\partial\bar{\partial}v\right)^{-1}\partial\bar{\partial}v_t =$ $tr\left(\partial\bar{\partial}v\right)^{-1}\partial\bar{\partial}\ln\det\partial\bar{\partial}v = -tr\left(g^{-1}Ric\right) = -R(x,t)$.

Obs. One has $S_{\min} \in [0, m]$ if $S_{\min}$ is achieved at a finite point, as then $0 \leq -\frac{1}{m}S_{\min}^2 + S_{\min}$.

Step 1. Scalare curvature $S \geq 0$ for complete ancient sol to Ricci flow (B. L. Chen 09). We have a direct elliptic argument in the self-similar case.
Step 2. Volume element $\Phi = \ln \det \partial\bar{\partial} u$ attains its max at the origin, since

$$\frac{1}{2} r \, \Phi_r = -S \leq 0.$$

Step 3. Volume element Φ is constant $\Phi(0)$ by the strong max principle, as

$$g^{i\bar{k}} \partial_{i\bar{k}} \Phi = \frac{1}{2} x \cdot D\Phi(x).$$

Step 4. Kahler potential u is quadratic by Euler formula for homogeneous functions applied to

$$\Phi(0) = \frac{1}{2} x \cdot Du(x) - u(x).$$

RMK. The above proof works for real M-A case,
$\ln \det D^2 w = \frac{1}{2} x \cdot Dw(x) - w$ in Ω. Just complexify $w(x)$ along $iR^n : u(x + iy) = w(x)$, completeness is kept.

4. Question. Any entire solution to $\ln \det \partial\bar{\partial} u = \frac{1}{2} x \cdot Du\,(x) - u$ in C^m is quadratic?

RMK. Self-similar makes solution to the eigenvalue equation more rigid. In contrast, there exist nontrivial (non flat) entire and complete solution to complex M-A equations $\ln \det \partial\bar{\partial} u = 0$ in C^m by LeBrun, Hitchin ... 80s.