Interior and free boundary regularity for Dirac-harmonic maps, harmonic maps and related PDE

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$$0 = \frac{\mathrm{d}}{\mathrm{d}t} E(u_t)_{t=o} = \int_{\mathcal{M}} \langle \mathrm{d}u, \mathrm{d}_{u^* \mathrm{T}\mathcal{N}} \phi \rangle_{\mathrm{T}^* \mathcal{M} \otimes \mathrm{u}^* \mathrm{T}\mathcal{N}} \, \mathrm{d}V_g$$

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where $\mathrm{d}_{u^*\mathrm{T}\mathcal{N}} : u^*\mathrm{T}\mathcal{N} \to \mathrm{T}^*\mathcal{M} \otimes \mathrm{u}^*\mathrm{T}\mathcal{N}$ is the induced exterior covariant derivative.Therefore u is harmonic if
$\tau(u) = \mathrm{d}^*_{u^*\mathrm{T}\mathcal{N}}(\mathrm{d}u) = 0$.
N.B. if $\tilde{\mathcal{N}}\subset\mathcal{N}$ and $u : \mathcal{M} \to \tilde{\mathcal{N}}\subset\mathcal{N}$ is harmonic then
$\tau(u) = (\mathrm{d}^*_{u^*\mathrm{T}\mathcal{N}}(\mathrm{d}u))^{\top} = 0$ weakly.

Unless $\dim(\mathcal{M}) = 1$, $u \in W^{1,2}(\mathcal{M}, \mathcal{N})$ is not continuous in general, so in order to deal with Harmonic maps we assume that $\mathcal{N} \hookrightarrow \mathbb{R}^n$ and define:

$$W^{1,2}(\mathcal{M}, \mathcal{N}) := \{u \in W^{1,2}(\mathcal{M}, \mathbb{R}^n) | u(x) \in \mathcal{N} \ \ a.e.\}$$

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To simplify we will now assume $(\mathcal{M}, g) = (B_1, g_{Eucl})$ and letting $\{\nu_K\}$ denote an orthonormal frame for $\mathrm{N}\mathcal{N}$ we have $(\Delta u)^\top = 0$ when

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We see that this PDE is quadratic in ∇u in the RHS. Since there is no L^1-theory for the Laplacian the best we can do is to say $\nabla u \in L^p$ for all $p < 2$.

However we have the following classical:

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Theorem (Rivière '92)

When $m > 2$ there exist 'nowhere continuous' weakly harmonic maps.

Partial regularity does exist in higher dimension for weakly stationary harmonic maps - which allows one to assume $\nabla u \in M^{2,m-2}$.

Letting $m = 2$ and considering $\mathcal{N} = S^d$ with the round metric, we have that $\nu(u) = u$ for a map $u : D \to S^d$ and u is harmonic if

$$-\Delta u = u|\nabla u|^2.$$

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In this setting we may observe (Shatah '88) that

$$\mathrm{div}(\mathrm{u}^{\mathrm{i}}\nabla \mathrm{u}^{\mathrm{j}} - \mathrm{u}^{\mathrm{j}}\nabla \mathrm{u}^{\mathrm{i}}) = \mathrm{u}^{\mathrm{i}}\Delta \mathrm{u}^{\mathrm{j}} - \mathrm{u}^{\mathrm{j}}\Delta \mathrm{u}^{\mathrm{i}} = 0$$

and write (Hélein '91)

$$-\Delta u^i = \sum_j (u^i \nabla u^j - u^j \nabla u^i) \cdot \nabla u^j.$$

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and write (Hélein '91)

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Note that in general we could write

$$-\Delta u^i = \sum_{K,j}(\nu_K(u)^i\nabla\nu_K(u)^j - \nu_K(u)^j\nabla\nu_K(u)^i)\cdot\nabla u^j$$

but we do not necessarily have

$$\mathrm{div}(\nu_{\mathrm{K}}(\mathrm{u})^{\mathrm{i}}\nabla\nu_{\mathrm{K}}(\mathrm{u})^{\mathrm{j}} - \nu_{\mathrm{K}}(\mathrm{u})^{\mathrm{j}}\nabla\nu_{\mathrm{K}}(\mathrm{u})^{\mathrm{i}}) = 0$$

Theorem (Wente '69, Coiffman et al '93)

Let $E, D \in L^2(B_1 \subset \mathbb{R}^2, \wedge^1\mathbb{R}^2)$ *and* $\phi \in W_0^{1,2}$ *weakly solve*

$$\Delta\phi = E \cdot D$$

and

$$\mathrm{d}E = \mathrm{d}^* D = 0$$

then ϕ *is continuous and*

$$\|\phi\|_{L^\infty} + \|\nabla\phi\|_{L^2} + \|\nabla^2\phi\|_{L^1} + \|\nabla\phi\|_{L^{2,1}} \leq C\|E\|_{L^2}\|D\|_{L^2}.$$

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Therefore we can at least conclude the full regularity for harmonic maps from a disc into a round sphere. Actually we have in all dimensions $E, D \in L^2(\mathbb{R}^m, \wedge^1 \mathbb{R}^m)$, $\mathrm{d}E = \mathrm{d}^* D = 0$ then

$$E \cdot D \in \mathcal{H}^1 \subset L^1.$$

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Then find an appropriate frame in which to express the equation. This frame will turn any 'bad' non-linearities into 'good' Wente-type non-linearities.
The ideas introduced by Hélein have been improved on and generalised since, in particular the partial regularity for harmonic maps in higher dimensions was proved using the same methods by Evans '91 (into spherical targets) and Bethuel '93 (general targets).

Rivière generalised Hélein's methods and proved

Theorem (Rivière '07)

Let $\Omega \in L^2(B_1, so(n) \otimes \mathbb{R}^2)$ *and* $u \in W^{1,2}(B_1, \mathbb{R}^n)$ *solve*

$$\mathrm{d}^*_\Omega(\mathrm{d}u) = -\Delta u - \Omega \cdot \nabla u = 0 \tag{1}$$

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For higher dimensions Rivière-Struwe proved:

Theorem (Rivière-Struwe '08)

Let $\Omega \in M^{2,m-2}(B_1, so(n) \otimes \mathbb{R}^m)$ *and* $u \in W^{1,2}(B_1, \mathbb{R}^n)$,
$\nabla u \in M^{2,m-2}(B_1)$ *solve*

$$\mathrm{d}^*_\Omega(\mathrm{d}u) = -\Delta u - \Omega \cdot \nabla u = 0. \tag{2}$$

Then there exists $\epsilon > 0$ *such that if* $\|\Omega\|_{M^{2,m-2}(B_1)} \leq \epsilon$ *then* u *is Hölder continuous.*

The key to this theorem lies in the existence of a Coulomb gauge in the Morrey space setting:

Theorem (Rivière-Struwe)

Given any such Ω there exist $\epsilon = \epsilon(m, n) > 0$, $P \in W^{1,2}(B_1, SO(n))$ and $\xi \in W_0^{1,2}(B_1, so(n) \otimes \wedge^{m-2}\mathbb{R}^m)$ such that whenever $\|\Omega\|_{M^{2,m-2}(B_1)} \leq \epsilon$ then

$$P^{-1}\mathrm{d}P + P^{-1}\Omega P = *\mathrm{d}\xi, \qquad \mathrm{d} * \xi = 0$$

and

$$\|\nabla\xi\|_{M^{2,m-2}} + \|\nabla P\|_{M^{2,m-2}} \leq C\|\Omega\|_{M^{2,m-2}}.$$

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Re-writing (1) with respect to P we have

$$\begin{aligned} \mathrm{d}^*(P^{-1}\mathrm{d}u) &= -*(\mathrm{d}(P^{-1}) \wedge *\mathrm{d}u) - P^{-1}\Delta u \\ &= *(P^{-1}\mathrm{d}P \wedge *P^{-1}\mathrm{d}u) + *(P^{-1}\Omega P \wedge *P^{-1}\mathrm{d}u) \\ &= *(*\mathrm{d}\xi \wedge *P^{-1}\mathrm{d}u) = *\mathrm{d}\xi \cdot P^{-1}\mathrm{d}u \end{aligned}$$

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- Applications to regularity of poly/fractional harmonic maps (Da Lio - Rivière, Schikorra, Struwe)
- Interior regularity (Wang-Xu '09) and free boundary regularity (S-Zhu '13) for Dirac-harmonic maps
- New global estimates for harmonic maps in two dimensions (S-Lamm '13)

We provide the following addition to the results of Rivière and Rivière-Struwe:

Theorem ($m = 2$:Rivière, S-Topping 2010. $m \geq 2$: S 2011)

Let $u \in W^{1,2}$, $\nabla u \in M^{2,m-2}$ weakly solve (for $B_1 \subset \mathbb{R}^m$)

$$-\Delta u = \Omega . \nabla u + f$$

for $\Omega \in M^{2,m-2}(B_1, so(n) \otimes \bigwedge^1 \mathbb{R}^m)$ and $f \in L^p$, $\frac{m}{2} < p < m$. Then there exist $\epsilon = \epsilon(n, m, p)$ and $C = C(n, m, p)$ such that whenever $\|\Omega\|_{M^{2,m-2}(B_1)} \leq \epsilon$ we have

$$\|\nabla^2 u\|_{M^{\frac{2p}{m},m-2}(B_{\frac{1}{2}})} + \|\nabla u\|_{M^{\frac{2p}{m-p},m-2}(B_{\frac{1}{2}})} \leq C(\|u\|_{L^1(B_1)} + \|f\|_{L^p(B_1)}).$$

In particular if $f \equiv 0$ then $\nabla u \in L^q$ for any $q < \infty$ and $\epsilon = \epsilon(n, m, q)$.

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In particular if $f \equiv 0$ then $\nabla u \in L^q$ for any $q < \infty$ and $\epsilon = \epsilon(n,m,q)$.

The results are sharp when $m = 2$.

Corollaries of this theorem:

- recover the ϵ-regularity and estimates for harmonic maps (S) and Dirac-harmonic maps (S-Zhu) in all dimensions

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There have been at least two generalisations of this theorem, in particular

- A. Schikorra (2012) has proved a similar 'higher - integrability' result for more general ('non-local') systems, using different methods.
- R. Moser (2012) has proved a similar estimate for harmonic maps but with $1 < p < \infty$ - with applications to the regularity of harmonic map flow for dimension $m \geq 3$.

Inspired by the supersymmetric nonlinear sigma model from quantum field theory Chen-Jost-Li-Wang '06 introduced the notion of a Dirac harmonic map

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$$L(\phi,\psi) = \int_{\mathcal{M}} |\nabla\phi|^2 + \langle \not{D}\psi, \psi\rangle \mathrm{d}V_{\mathcal{M}}.$$

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A Dirac-harmonic map is a critical point (ϕ,ψ) of L.Similarly for harmonic maps, letting $\mathcal{N}\subset\mathbb{R}^n$ there exists some $\Omega = \Omega(\phi,\nabla\phi,\psi) \in L^2$ such that ϕ solves

$$-\Delta\phi = \Omega \cdot \nabla\phi$$

with the spinor solving

$$\not{\partial}\psi = \Theta \cdot \psi$$

for some $\Theta = \Theta(\nabla\phi) \in L^2(B_1, gl(n) \otimes \wedge^1\mathbb{R}^m)$

Chen-Jost-Wang-Zhu '12 introduced the appropriate free boundary problem for Dirac-harmonic maps from spin surfaces with boundary, which provides a mathematical interpretation of the D-branes in superstring theory. Now $\partial\mathcal{M} \neq \emptyset$ and for some smooth submanifold $\mathcal{S}\subset\mathcal{N}$ we consider constrained maps $\phi : (\mathcal{M}, \partial\mathcal{M}) \to (\mathcal{N}, \mathcal{S})$ and spinors that satisfy a Chirality-type boundary condition compatible with the supporting manifold $\mathcal{S}$.

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Theorem (Chen et al '12)

Let (ϕ, ψ) be a weakly Dirac-harmonic map from $\mathcal{M}$ to $\mathcal{N}$ with free boundary on $\mathcal{S}$. If in addition we assume that $\mathcal{S}$ is totally geodesic, then (ϕ, ψ) is smooth.

Theorem (S-Zhu '13)

Let (ϕ, ψ) be a weakly Dirac-harmonic map from $\mathcal{M}$ to $\mathcal{N}$ with free boundary on $\mathcal{S}$. Then (ϕ, ψ) is smooth up to the boundary. Moreover given a local chart about $\partial\mathcal{M}$: for any $k \in \mathbb{N}$ there exist $\epsilon = \epsilon(\mathcal{N}, \mathcal{S})$ and $C = C(k, \mathcal{N}, \mathcal{S})$ such that if (ϕ, ψ) is a weakly Dirac-harmonic map from B_1^+ to $\mathcal{N}$ with free boundary $\phi(I)$ on $\mathcal{S}$ satisfying $\|\nabla\phi\|_{L^2(B_1^+)} \leq \epsilon$, then

$$\|\nabla^k\phi\|_{L^\infty(B^+_{\frac{1}{2}})} + \|\nabla^k\psi\|_{L^\infty(B^+_{\frac{1}{2}})} \leq C(\|\nabla\phi\|_{L^2(B_1^+)} + \|\psi\|_{L^4(B_1^+)}).$$

Theorem (S-Zhu '13)

Let (ϕ, ψ) be a weakly Dirac-harmonic map from $\mathcal{M}$ to $\mathcal{N}$ with free boundary on $\mathcal{S}$. Then (ϕ, ψ) is smooth up to the boundary. Moreover given a local chart about $\partial\mathcal{M}$: for any $k \in \mathbb{N}$ there exist $\epsilon = \epsilon(\mathcal{N}, \mathcal{S})$ and $C = C(k, \mathcal{N}, \mathcal{S})$ such that if (ϕ, ψ) is a weakly Dirac-harmonic map from B_1^+ to $\mathcal{N}$ with free boundary $\phi(I)$ on $\mathcal{S}$ satisfying $\|\nabla\phi\|_{L^2(B_1^+)} \leq \epsilon$, then

$$\|\nabla^k\phi\|_{L^\infty(B^+_{\frac{1}{2}})} + \|\nabla^k\psi\|_{L^\infty(B^+_{\frac{1}{2}})} \leq C(\|\nabla\phi\|_{L^2(B_1^+)} + \|\psi\|_{L^4(B_1^+)}).$$

The theory we develop recovers the interior ϵ-regularity for Dirac-harmonic maps (Wang-Xu) along with the partial interior regularity (Hélein, Bethuel, Evans, Rivière-Struwe) and free boundary regularity (Jost-Gulliver, Scheven) for harmonic maps in all dimensions with smooth estimates.

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- A classical lemma coupled with the interior regularity estimates allow us to assume that ϕ maps a neighbourhood of any point $p \in \partial\mathcal{M}$ (B_1^+, say) into a (Fermi-)coordinate neighbourhood of some $q \in \mathcal{S} \subset \mathcal{N}$ (under a smallness assumption).

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- We show that the Chirality-type boundary condition for the spinor is essentially Riemann-Hilbert condition for $\overline{\partial}$.
- This enables us to prove that $\psi \in W^{1,p}$ for any $p < 2$ up to the boundary. The equation for the map ϕ in these coordinates becomes (for some $\Omega \in L^2(B_1^+)$ and $A \in L^\infty \cap W^{1,2}(B_1^+, GL(d))$):

$$\mathrm{d}^*(A\mathrm{d}\phi) = \langle \Omega, A\mathrm{d}\phi \rangle + f \quad f \in L^p$$

$$\frac{\partial \phi^\top}{\partial \overrightarrow{n}} = \Re(P_{\mathcal{S}}(\overrightarrow{n} \cdot \psi^\perp; \psi^\top)) \qquad \phi^\perp = 0$$

We can now reduce to:

Theorem (S-Zhu '13)

Let $0 \leq s \leq d$, $\Lambda < \infty$ and $1 < p < 2$. Consider A, Ω, f as above but with $(g, k) \in (W^{1,p}_\partial(I, \mathbb{R}^s), W^{2,p}_\partial(I, \mathbb{R}^{d-s}))$, and $u \in W^{1,2}$ weakly solving

$$\mathrm{d}^*(A\mathrm{d}u) = \Omega \cdot A\mathrm{d}u + f \quad \text{in} \quad B_1^+,$$

$$\frac{\partial u^i}{\partial \vec{n}} = g^i \;\; 1 \leq i \leq s, \qquad u^j = k^j \;\; s+1 \leq j \leq d \quad \text{on } I$$

$\Lambda^{-1}|\xi| \leq |A(x)\xi| \leq \Lambda|\xi|$ moreover we require that $A(x)$ commutes with $R := \begin{pmatrix} Id_s & 0 \\ 0 & -Id_{d-s} \end{pmatrix}$. Then $u \in W^{2,p}_{loc}(B_1^+ \cup I)$, in particular $\nabla u \in L^q$ up to the boundary for some $q > 2$.

In a work in progress with Tobias Lamm we also have the following global estimate for almost harmonic maps in two dimensions:

In a work in progress with Tobias Lamm we also have the following global estimate for almost harmonic maps in two dimensions:

Theorem (S-Lamm '13)

Given a closed $(\mathcal{N}, h)$ *and any map* $u \in W^{1,2}(B_1, \mathcal{N})$ *with* $\|\nabla u\|_{L^2} + \|\tau(u)\|_{L^2} \leq \Lambda$*, there exists a* $C = C(\Lambda, \mathcal{N})$ *such that*

$$\|\nabla u\|_{L^{2,1}(B_{\frac{1}{2}})} \leq C.$$

Recall the E-L equation for harmonic maps:

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As before, we can make sense of this PDE for maps $u \in W^{1,2}$ and and where we had before: $\mathrm{d}^*_{\Omega}(\mathrm{d}u) = 0$ we have

$$\overline{\partial}_{\Omega}(\partial u) = \overline{\partial}(\partial u) + \Omega^{\overline{z}} \wedge \partial u = 0$$

for

$$\Omega^i_j := \sum_K (\nu_K(u)^i \nabla \nu_K(u)^j - \nu_K(u)^j \nabla \nu_K(u)^i).$$

Now we can think more geometrically and use the following classical result of Koszul-Malgrange '58:

Theorem

Let $U \subset \mathbb{C}^k$ be simply connected and consider $\omega : U \to gl(n, \mathbb{C}) \otimes \wedge^1 \mathbb{C}^k$ (local connection forms). Then there exists a frame $S : U \to GL(n, \mathbb{C})$ (a 'holomorphic frame') solving

$$S^{-1}\overline{\partial} S + S^{-1}\omega^{\overline{z}} S = 0$$

if and only if

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So when $k = 1$ we always have the existence of such S for smooth ω. Unfortunately this theorem is false if $\omega \in L^2$ - it would be nice for us since then

$$\overline{\partial}(S^{-1}\partial u) = 0.$$

However we have the following result of Hélein:

Lemma

Given $\Omega \in L^{2,1}(B_1, gl(n,\mathbb{C}) \otimes \wedge^1\mathbb{R}^2)$ *there exists* $\epsilon > 0$ *such that whenever* $\|\Omega\|_{L^{2,1}} \leq \epsilon$ *there exists* $S \in C^0 \cap W^{1,2}(B_1, GL(n,\mathbb{C}))$ *solving*

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with

$$\mathrm{dist(S, Id)} \leq \frac{1}{3}.$$

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The proof uses a very simple fixed point argument: In fact Hélein used this in his proof of the regularity theory, however he required a lot of machinery to put us into the position where $\Omega \in L^{2,1}$.

Using a very simple observation we prove the following:

Theorem (S '12)

Suppose $\Omega \in L^2(B_1, u(n) \otimes \wedge^1 \mathbb{R}^2)$ *has the following Hodge decomposition:*

$$\Omega = \mathrm{d}a + *\mathrm{d}b$$

for $a \in W^{1,2}(B_1, u(m))$ *and* $b \in W_0^{1,2}(B_1, u(m))$ *and* $\nabla b \in L^{2,1}$*. Now, there exists* $\epsilon > 0$ *such that whenever*

$$\|\Omega\|_{L^2} + \|\nabla b\|_{L^{2,1}} \leq \epsilon$$

there exists $S \in C^0 \cap W^{1,2}(B_1, GL(n, \mathbb{C}))$ *solving*

$$S^{-1}\overline{\partial} S + S^{-1}\Omega^{\overline{z}} S = 0$$

with

$$\mathrm{dist}(\mathrm{S}, \mathrm{U(n)}) \leq \frac{1}{3}.$$

Now for

$$\Omega^i_j := \sum_K (\nu_K(u)^i \mathrm{d}\nu_K(u)^j - \nu_K(u)^j \mathrm{d}\nu_K(u)^i)$$

for any such Hodge decomposition we have

$$\Delta b^i_j = \sum_K 2\mathrm{d}\nu_K(u)^i \wedge \mathrm{d}\nu_K(u)^j = \sum_K 2\mathrm{d}\nu_K(u)^i \cdot *\mathrm{d}\nu_K(u)^j$$

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Therefore by our Wente estimates we satisfy the conditions of the Theorem and we can conclude that, for harmonic maps there exists a frame S such that (when $\|\nabla u\|_{L^2}$ is sufficiently small)

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which recovers the full regularity, an energy convexity result (Colding-Minicozzi) and an estimate on the energy in the local Hardy space (Lamm-Lin) under the weakest assumption that $\mathcal{N}$ is C^2.

Thank you for your attention!