Spectral Theory of Laplace and Schrödinger Operators (13w5059)

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1 Overview of the Field

The 5-day workshop "Spectral Theory of Laplace and Schrödinger Operators (13w5059)" gathered at BIRS a group of specialists working on spectral analysis.The meeting focused on the study of eigenvalues of Laplace and Schrödinger type operators. In particular, we considered the following three topics:

- Isoperimetric inequalities for eigenvalues and shape optimization
- Geometric properties of eigenfunctions, in particular, nodal sets and nodal domains
- Spectral Problems in Mathematical Physics, in particular, uniform semi-classical spectral estimates

This reflects very well the planned outline of the workshop, where a short and readable overview of the field can be found. Below we report on a selection of recent developments and open problems presented at the conference.

2 Recent Developments and Open Problems

2.1 Isoperimetric inequalities for eigenvalues

Geometrically sharp "isoperimetric" estimates on eigenvalues of the Laplacian and the Schrödinger operator help us understand how frequencies and energies of physical systems are constrained by the shape and size of the domain in which the physical process takes place. For example, the Faber–Krahn inequality (Rayleigh's conjecture) tells us precisely how low the frequency of a drum can be, if the area of the drumhead is given but we are free to choose its shape. In three dimensions, the same inequality tells us the minimal energy a quantum particle can possess when it is confined to a region of given volume. The minimum is attained in each case by a domain with symmetry — the disk in two dimensions, and the ball in three dimensions.

Let us describe some recent progress and notable open problems that were presented both formally and informally during the Workshop.

Existence of minimizing domains

The first step of any "direct" method for identifying an optimal domain for an eigenvalue problem is to prove that an optimal domain exists. We focus here on eigenvalues of the Laplacian, $-\Delta u = \lambda u$, under Dirichlet boundary condition ($u = 0$ on the boundary). In trying to go beyond the basic Faber–Krahn result to the consideration of higher eigenvalues, for a long time researchers in the field were stuck: They could prove the existence of an optimal domain under a uniform boundedness assumption (that is, assuming the candidate domains all lie within a fixed disk), but could not show in general that a minimizing domain must exist. This barrier was breached in the last couple of years, in the works by Bucur [11] and Mazzoleni–Pratelli [28], who considered the minimization of the kth eigenvalue of the Dirichlet Laplacian among domains of given volume. More generally, their methods can handle eigenvalue functionals that depend in an increasing fashion on each eigenvalue and are lower semi-continuous.

Properties of minimizing domains?

Many natural questions about optimal domains remain open: What regularity do the minimizers have? Are minimizers unique? What symmetries, if any, do the minimizers possess? What can one say about the shape?

For the first eigenvalue λ_1, the Faber–Krahn theorem tells us that the unique optimal domain is a disk, among plane domains of given area. The minimizer for the second Dirichlet eigenvalue is a disjoint union of two disks. The minimizer is unknown for $\lambda_k, k \geq 3$. It is not even known whether the minimizer is connected, when $k \geq 3$. Numerical evidence supports a connectivity conjecture for $k \geq 5$, but one has no proof. Results of van den Berg and Iversen [5] give bounds on the number of connected components of minimizing domains. For connectivity of minimizing domains for linear combinations of eigenvalues, we have learned a lot recently from numerical work by Kao and Osting [25].

These problems are difficult already for very low eigenvalues. A long-standing conjecture claims that the third Dirichlet eigenvalue λ_3 is minimal for the disk among plane domains of given area. The disk is not the answer in general, though, because it is not even locally minimal for λ_k when $k \neq 1, 3$, as observed this year by A. Berger (unpublished).

Numerical work on minimizing domains

A great deal of numerical work in the last few years has been stimulated by eigenvalue minimization problems. The challenge is that one must not only compute eigenvalues to high accuracy for a single domain (which can be challenging by itself, if the domain has features such as re-entrant corners), but one must compute them to high accuracy for whole families of domains. The basic idea is to consider a high-dimensional space of trial domains with flexible geometry, and follow some kind of steepest–descent method to evolve the domain towards a (local) minimizer of the desired eigenvalue. Earlier work in this area was by Oudet [30] using finite element methods. A new method exploited by Antunes and Freitas [2] in the past few years relies on exact Bessel-type eigenfunctions that are constructed to approximately satisfy the boundary conditions. This new method gives remarkable accuracy, and can be feasibly applied up to dimension 4 at least, whereas shape optimization using finite elements in dimension 4 seems much less computationally tractable.

This numerical work provides a catalog of minimizing domains for λ_k up to $k = 15$ (and of course, one could go higher if desired). Armed with this evidence, one returns to some of the questions raised above: What symmetries, if any, do the minimizers possess? The minimizing domains appear to have a line of symmetry for $k \leq 12$, but not for λ_{13}. Perhaps the most one can hope for is that "most" minimizing domains possess a line of symmetry.

Limiting shape of minimizing domains?

Next, what can one say about the shape? The evidence is far from conclusive, but the optimal shape for minimizing λ_k among domains of given area might be converging to a disk as $k \to \infty$. The two-term Weyl asymptotic due to Ivrii lends some support to this conjecture, because together with the isoperimetric inequality it implies that for any fixed domain λ_k grows faster as $k \to \infty$ than it does for the disk of the same area. The trouble is that we do *not* consider a fixed domain: For each k we consider the minimizing domain (which naturally depends on k), and only afterwards let k grow towards infinity.

The analogous problem in the class of rectangles was solved recently by Antunes and Freitas [3], who proved that indeed the optimal rectangle for minimizing λ_k converges to a square as $k \to \infty$.

Diameter constraints

For domains of given diameter, the minimization of λ_1 for the ball follows from the Faber–Krahn theorem and the isodiametric theorem. The problem of minimizing λ_2 is already interesting and open: Does the ball provide the minimizer, as suggested by Bucur *et al.* [12]? Note that one may assume the domain to be convex, since expanding a domain to its convex hull will lower the eigenvalues (by domain monotonicity) while leaving the diameter unchanged. For each k, it is known that a minimizing domain exists for λ_k (by Blaschke selection, that is, by compactness of the class of convex bodies), and further that the minimizing domain has constant width.

Spectral gaps

The study of the spectral gap or "excitation energy" $\lambda_2 - \lambda_1$ has seen spectacular progress in the years leading up to this Workshop. The Fundamental Gap Conjecture of van den Berg and Yau asserts that the difference between the first two eigenvalues of a Schrödinger operator with convex potential on a convex domain should be minimized by the degenerate rectangular box (line segment) with zero potential. Impressive parabolic comparison techniques involving the modulus of convexity enabled Andrews and Clutterbuck [1] to prove this Gap Conjecture. Their methods are now finding their way into geometric settings, and other boundary conditions (work in progress).

Neumann boundary conditions

For Neumann boundary conditions a natural question is to maximize the eigenvalues among domains of given volume. Girouard, Nadirashvili, and Polterovich [20] showed in 2009 that among all simply connected plane domains, the second nonzero Neumann eigenvalue is maximized in the limit by the disjoint union of two identical disks. It remains a challenging open problem to remove the hypothesis of simply connectedness, and to extend this result to higher dimensions.

The Laplacian on forms, and a unifying Rayleigh conjecture

A new approach proposed recently by Savo aims to unify the Faber–Krahn lower bound on the first Dirichlet eigenvalue (minimizer is a ball) with the Payne–Weinberger lower bound on the first nonzero Neumann eigenvalue (minimizer is a line segment). He does so by identifying these results as the endpoints of a family of conjectured bounds.

Specifically, Savo examines the Laplacian on p-forms on a convex domain, under "absolute" boundary conditions. The case $p = 1$ corresponds to Payne–Weinberger with a diameter normalization on the domain, and $p = n$ to Faber–Krahn with a volume normalization. For each p, Savo [34] proves non-sharp bounds under a "p-volume" normalization on the domain. In his talk at the Workshop, he raised the fascinating open problem of obtaining sharp bounds for intermediate p-values.

Magnetic Laplacian

The magnetic Laplacian $(i\nabla + \beta(-x_2, x_1))^2$ is the Schrödinger operator for a charged particle in the plane subject to a transverse uniform magnetic field, with no electric field. Obviously the operator reduces to the usual Laplacian when the magnetic field strength β is zero.

Only two isoperimetric type results are known for the magnetic Laplacian. First, Erdös [16] proved the analogue of the Faber–Krahn result in 1996, for the first eigenvalue under Dirichlet boundary conditions on a domain in two dimensions. Second, Laugesen and Siudeja [27] last year established certain sharp upper bounds on eigenvalue sums and products, including the partition function.

Many isoperimetric questions stand open for the magnetic Laplacian. One could take most of the results known for the Laplacian in two dimensions (normalizing by area, diameter, inradius, conformal mapping radius, etc.) and ask whether an analogous result holds in the presence of a magnetic field. In higher dimensions the picture is less clear, since the magnetic field direction breaks the symmetry.

Hot spots conjecture for Neumann eigenfunctions

The Hot Spots Conjecture of Jeffrey Rauch says that the first non-constant eigenfunction of the Neumann Laplacian on a convex domain achieves its maximum and minimum values on the boundary. The problem has seen intense activity in the past year on PolyMath [33], particularly due to Nigam, Siudeja, and Tao. The problem remains open, although there is hope for a numerical proof for triangular domains using interval arithmetic.

The oval conjecture

Let γ be a closed curve of length 2π parametrized by arclength, and let κ be its curvature. Consider the Schrödinger operator on γ with the potential κ^2. The oval conjecture states that the principal eigenvalue of this operator attains its minimum on a family of ovals connecting the unit circle to a segment of length π traversed in both directions. This problem has attracted considerable interest, particularly due to its connection to the Lieb–Thirring conjecture, as well as to the simplicity and elegance of its formulation. It has been studied by many mathematicians, including several workshop participants, such as R. Benguria, M. Loss and A. Burchard. Some results on this problem were discussed at the previous meetings in 2009 at Oberwolfach, and in 2006 at AIM in Palo Alto. Significant progress on the oval conjecture was reported at this workshop: Jochen Denzler showed that the problem has a solution given by a planar convex analytic curve with strictly positive curvature.

2.2 Nodal domains and nodal sets of eigenfunctions

Nodal geometry on manifolds

Several talks at the meeting focused on recent developments in the study of nodal domains and nodal sets of the eigenfunctions of the Laplacian and other differential operators, such as the conformally covariant GJMS operators. The latter were the subject of the talk of D. Jakobson, who reported on his work with several collaborators on the study of conformal invariants arising from nodal sets [13]. These results may be viewed as the first steps in a promising new direction of research on the interface of conformal geometry and spectral theory.

Y. Canzani spoke about her joint work with J. Toth on the number of intersections between an analytic curve on a compact real analytic Riemannian surface without boundary, and the nodal set of a Laplace eigenfunction. They have shown that under a certain technical assumption on the curve, the number of its intersections with the nodal set of an eigenfunction corresponding to an eigenvalue λ is bounded above by $O(\sqrt{\lambda})$ (see [36] for some related earlier results). This assumption is not easy to verify, but it is believed to be satisfied generically. In particular, as was shown by Bourgain–Rudnick [10], it holds for any curve of strictly positive curvature on a torus.

Improvements of Pleijel's nodal domain theorem

The talk of S. Steinerberger was concerned with the celebrated Pleijel's theorem which gives an asymptotic upper bound on the number of nodal domains of Laplace eigenfunctions. Pleijel's theorem is proved using Weyl's law and the Faber-Krahn inequality, and it is easy to see from the proof that the constant in Pleijel's estimate is not sharp. It was conjectured in [31] that the optimal constant in Pleijel's inequality is attained on a rectangle. In 2013, J. Bourgain obtained a slight improvement of Pleijel's constant, combining the stability of the Faber-Krahn inequality with a packing argument [9]. S. Steinerberger reported on a related, but somewhat different approach which can be viewed as an "uncertainty principle" for partitions of the plane [35]. As a consequence, he showed that the methods of Bourgain and his own cannot yield a better Pleijel's constant than that of a hexagon, which is bigger than the conjectured optimal constant. This observation is closely related to the "hexagonal conjecture" of B. Helffer, T. Hoffmann-Ostenhof, and their collaborators in the theory of spectral minimal partitions (see [22]).

Nodal count on graphs

The topic of G. Berkolaiko's talk was the nodal count on graphs. It is known that the nth eigenfunction of the Schrödinger opeator on a quantum graph has at least $n-1$ zeros. The difference between the actual number of zeros and $n-1$ is called the nodal surplus. An analogous quantity, called the nodal deficiency, can also be defined for Laplace eigenfunctions on manifolds and domains. Recently, the speaker and his collaborators have obtained very interesting results interpreting these quantities as Morse indices of certain functionals [6, 7]. An example of such a functional is the eigenvalue itself considered as a function of the magnetic flux introduced into the system. As a consequence of this interpretation, in the case of quantum graphs one expects to have eigenfunctions of arbitrary high energy with the nodal surplus equal to zero. In joint work with R. Band and T. Weyand, G. Berkolaiko constructed a family of graphs with a nontrivial nodal surplus for all $n > 1$. It turns out that this anomaly has a physical meaning: In particular, it could be linked to the existence of Dirac points in the graphene honeycomb lattice.

2.3 Spectral problems in mathematical physics

In mathematical physics, the spectral analysis of eigenvalues often treats two classes of problems: The asymptotic behavior for large eigenvalue numbers (e.g., semi-classical results), or the study of particular outstanding eigenvalues or combinations of eigenvalues (e.g., isoperimetric problems, fundamental gaps, etc.)

Uniform semi-classical spectral estimates

The talk by A. Laptev presented a review of recent improvements on a class of well-known semi-classical estimates. Given the Dirichlet Laplacian in a bounded open domain, it is known that the partial sums of the first n eigenvalues can be estimated from below exactly by the corresponding quantity computed originally in the semi-classical limit. These so-called Berezin-Li-Yau inequalities are a standard reference for uniform spectral bounds. Inspired by an observation by Melas [29], in recent years much attention has been paid to the improvement of such uniform bounds by suitable lower order terms. Progress has been made both for higher Riesz means of eigenvalues, where (in contrast to Melas's result) one obtains remainder terms of the expected second Weyl order [37], [19], but also in the case of sums of eigenvalues, the limiting case for the original inequalities to be known with sharp constants [26].

Despite these efforts, various tantalizing classical conjectures on uniform bounds for counting functions of eigenvalues remain open, in particular, the famous Pólya conjecture [32].

Stability in isoperimetric problems

Qualified remainder terms are also of interest in the context of isoperimetric inequalities. The classical isoperimetric problem considers in the first stage an optimal constant for the ratio of some quantity (e.g., the ground state energy) by a geometric characteristic (e.g., a power of the volume of a domain or of a phase space volume). In the second stage one studies the existence and uniqueness of an optimizer. In the third stage one is interested in the stability of the isoperimetric ratio with respect to perturbations of the optimal configuration. Classical results in this area are the quantitative isoperimetric inequality of Fusco, Maggi, and Pratelli [18] and the quantitative Sobolev inequality of Bianchi and Egnell [8]. In his talk R. Frank presented his work with Carlen and Lieb proving the stability of Keller's isoperimetric problem, namely the problem of minimizing the ground state of a Schrödinger operator among all potentials with a given L^p-norm. Their result [14] is based on quantified Gagliardo-Nirenberg-Sobolev and Hölder bounds. In particular, their new version of the well-known Hölder inequality shows a deep and intriguing link between modern and "classical" mathematics.

Spectral analysis for magnetic Hamiltonians

The inclusion of magnetic fields leads to spectral problems that are mathematically challenging and relevant for applications. This played a role in several presentations.

Already the description of the structure of the bottom of the spectrum is non-trivial and involves a substantial amount of subtle semi-classical analysis. This was shown by B. Helffer in a broad review on magnetic wells in the semi-classical limit [21, 23, 17]. He focused on spectral gaps, typically the gap between the first and second eigenvalues, and on the description of effective Hamiltonians, which can explain via standard semi-classical analysis the complete spectral picture including tunneling.

2.4 Other topics

Spectral analysis of linear operator pencils

M. Levitin presented the results of his joint work with E. B. Davies, as well as with D. Elton and I. Polterovich on the spectra of linear operator pencils. A pencil is an operator family of the form $A - \lambda B$. If A and B are self-adjoint operators, but not sign–definite, the pencil may exhibit non-self adjoint features, such as the existence of complex eigenvalues and irregular spectral asymptotics. One of the examples used to illustrate these phenomena was a one-dimensional Dirac operator pencil arising in the study of graphene waveguides [15].

Numerical methods and spectral analysis

N. Nigam gave a beautiful survey talk on applications of numerical analysis to spectral theory. She discussed different aspects of finite element methods for eigenvalue problems, and emphasized the conditions under which the numerical calculations may (or may not) be considered reliable. Of special interest to spectral theorists was an overview of recent developments in validated numerics, opening up a possibility to prove eigenvalue inequalities and other spectral results rigorously using computer-assisted methods. Several examples, in particular, related to isoperimetric inequalities for eigenvalues on surfaces [24], were presented.

Isospectrality and "hearing the shape of a drum"

Peter Herbrich gave a very nice overview of the subject of hearing, or not hearing, properties of drums while also giving several of his recent results. In particular, he presented results on the isospectrality of various domains with mixed Dirichlet and Neumann boundary conditions obtained via the method of transplantation. Using these ideas one can catalog the low order cases of isospectral domains under various conditions and can see how isospectrality follows from transplantation. These ideas build upon the work of Gordon, Webb, and Wolpert and of Pierre Bérard from the early 90s, with the general theme of these investigations going back to the seminal paper of Mark Kac (1966), "Can one hear the shape of a drum?" Herbrich's talk showed that this topic is very much alive, with new insights and developments occurring regularly.

3 Presentation Highlights

There have been many excellent talks at the workshop. Some of them are mentioned below.

- J. Clutterbuck (Australian National University): *Proof of the fundamental gap conjecture*

  This special talk was devoted to the solution of the longstanding gap conjecture, namely that the fundamental gap for convex domains is minimized in the limit by a line segment. This problem has been a regular topic of preceding conferences (see, for example, [4]). The presentation of the joint proof by Andrews and Clutterbuck was certainly one of the highlights of the workshop.

- Rupert Frank (Caltech): *Stability estimates for the lowest eigenvalue of a Schrödinger operator*

  The stability of isoperimetric problems is expressed in terms of qualified remainder terms in the isoperimetric bound resulting in a quantified continuity with respect to perturbations of the optimizer. This has been presented for the ground state of the Schrödinger operator. The result invokes an improvement of the Hölder inequality, providing a link between modern and classical analysis.

- Nilima Nigam (Simon Fraser University): *Numerical analysis of spectral problems, validated numerics, and proof*

  While the computation of ground states is a rather robust procedure, the numerics for higher eigenvalues needs much more attention. This highly informative talk showed the possibilities and, at the same time, the limitations of numerical computations for higher eigenvalues.

- Alessandro Savo (Università di Roma la Sapienza): *On the spectrum of the Hodge Laplacian and the John ellipsoid*

  The usual Laplacian is a standard object in spectral analysis. It is quite surprising that for the Hodge Laplacian much less is known. In his talk Savo put forward a unified view of the classical Faber-Krahn and Payne-Weinberger inequalities, presenting them as special cases of geometric inequalities for the first eigenvalue of the Hodge Laplacian on convex domains. Several challenging open problems were posed.

- Stefan Steinerberger (Universität Bonn): *A geometric uncertainty principle and Pleijel's estimate*

  The talk outlined a beautiful connection between estimates on nodal domains and a basic geometrical covering problem. This illustrates once again the fundamental role of spectral analysis as a link between various branches of mathematics.

4 Scientific Progress Made

This workshop is the third in a series on related topics (the first two took place in 2006 at AIM in Palo Alto, and in 2009 at the MFO Oberwolfach). Therefore it is natural that certain problems discussed at earlier meetings have been addressed at the workshop, and new progress has been reported. One should certainly point out the solution of the gap conjecture by Andrews and Clutterbuck (for the history of this conjecture from well prior to their proof, see Ashbaugh's write-up [4] for the 2006 AIM workshop; this was one of the two main topics of that workshop). The talks of Henrot and Girouard touched upon recent advances in the study of isoperimetric problems for Steklov eigenvalues, which is a rapidly developing direction in the field. Denzler presented new results on the fascinating oval conjecture which remains open so far. Certain tantalizing classical conjectures on universal estimates (Pólya's conjecture, the Lieb-Thirring hypothesis) remain a challenge. The talks presented an overview of a very lively subject, and the three open problems sessions held during the workshop helped to reinforce this impression.

5 Outcomes of the Meeting

The feedback from the participants has been very positive. One particular feature of this meeting (along with its predecessors, in 2006 at AIM in Palo Alto, and in 2009 at the MFO Oberwolfach) is, as A. Burchard points out, the following:

"What has made these workshops particularly interesting is that they have been very focused (on eigenvalue problems), yet have attracted researchers across several disciplines, most notably Mathematical Physics, Spectral Geometry, Functional Analysis and Convexity."

The meeting featured intensive discussions between scientists working on seemingly different subjects. Spectral analysis, however, links these branches together and translates problems from one mathematical language into another.

It is worth noting that a large number of strong young mathematicians participated in the workshop. This resulted in a fruitful balance between junior and senior scientists.

The only criticism touches upon the negative effect of the strike at Canadian embassies, which prevented several invited guests from attending the conference due to visa problems.

The organizers are very grateful to BIRS and its staff for the excellent organization of the meeting and superb working and living conditions.

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