

Banff, 29 October 2013

# Anderson's orthogonality catastrophe

Peter Müller (LMU München)

joint with Martin Gebert, Heinrich Küttler, Peter Otte

arXiv:1302.6124 and in preparation

Motivation: explanation of anomalies in X-ray absorption in metals

VOLUME 18, NUMBER 24

PHYSICAL REVIEW LETTERS

12 JUNE 1967

---

## INFRARED CATASTROPHE IN FERMI GASES WITH LOCAL SCATTERING POTENTIALS

P. W. Anderson

Bell Telephone Laboratories, Murray Hill, New Jersey

(Received 27 March 1967)

We prove that the ground state of a system of  $N$  fermions is orthogonal to the ground state in the presence of a finite range scattering potential, as  $N \rightarrow \infty$ . This implies that the response to application of such a potential involves only emission of excitations into the continuum, and that certain processes in Fermi gases may be blocked by orthogonality in a low- $T$ , low-energy limit.

- ground-state overlap  $\lesssim N^{-\text{const.}}$  in the macroscopic limit
- controversial discussion in the Physics literature in the 1970ies
- still of interest in Physics today
- **no mathematical explanation**

- Schrödinger operators on  $L^2(\mathbb{R}^d)$

$$H := -\Delta + V_0 \quad \text{and} \quad H' := H + V$$

$V_0$  Kato decomposable      perturbation  $0 \leq V \in L_c^\infty(\mathbb{R}^d)$

- Finite-volume restrictions to box  $\Lambda_L := [-L, L]^d$  with Dirichlet b.c.

$$H_L = \sum_{j \in \mathbb{N}} \lambda_j^L |\varphi_j^L\rangle \langle \varphi_j^L| \quad \text{and} \quad H'_L = \sum_{j \in \mathbb{N}} \mu_j^L |\psi_j^L\rangle \langle \psi_j^L|$$

- Non-interacting system of  $N$  spinless fermions on  $\bigwedge_{j=1}^N L^2(\Lambda_L)$

$$H_L^{(N)} := \sum_{j=1}^N \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes H_L^{(j)} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

- Ground states

$$\Phi_N^L := \frac{1}{\sqrt{N!}} \varphi_1^L \wedge \cdots \wedge \varphi_N^L \quad \text{and} \quad \Psi_N^L := \frac{1}{\sqrt{N!}} \psi_1^L \wedge \cdots \wedge \psi_N^L$$

- Particle number to yield given Fermi energy  $E \in \mathbb{R}$  in the mac. limit

$$N \equiv N_L(E) := \#\{j \in \mathbb{N} : \lambda_j^L \leq E\}$$

- Ground-state overlap

$$S_L(E) := \langle \Phi_{N_L(E)}^L, \Psi_{N_L(E)}^L \rangle = \det \begin{pmatrix} \langle \varphi_1^L, \psi_1^L \rangle & \cdots & \langle \varphi_1^L, \psi_{N_L(E)}^L \rangle \\ \vdots & & \vdots \\ \langle \varphi_{N_L(E)}^L, \psi_1^L \rangle & \cdots & \langle \varphi_{N_L(E)}^L, \psi_{N_L(E)}^L \rangle \end{pmatrix}$$

**Theorem A.** [Gebert, Küttler, M. - to appear in CMP]

$\forall$  sequence of lengths  $(L_n)_{n \in \mathbb{N}}$ ,  $L_n \uparrow \infty$ ,  $\exists$  subsequence  $(L_{n_k})_{k \in \mathbb{N}}$  such that for Leb.-a.e.  $E \in \mathbb{R}$

$$\limsup_{k \rightarrow \infty} \frac{\ln |S_{L_{n_k}}(E)|}{\ln L_{n_k}} \leq -\frac{\gamma(E)}{2}$$

with

$$\gamma(E) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \operatorname{tr} \left( \sqrt{V} 1_{]E-\varepsilon, E]}(H_{ac}) V 1_{[E, E+\varepsilon[}(H'_{ac}) \sqrt{V} \right).$$

**Theorem A.** [Gebert, Küttler, M. - to appear in CMP]

$\forall$  sequence of lengths  $(L_n)_{n \in \mathbb{N}}$ ,  $L_n \uparrow \infty$ ,  $\exists$  subsequence  $(L_{n_k})_{k \in \mathbb{N}}$  such that for Leb.-a.e.  $E \in \mathbb{R}$

$$\limsup_{k \rightarrow \infty} \frac{\ln |S_{L_{n_k}}(E)|}{\ln L_{n_k}} \leq -\frac{\gamma(E)}{2}$$

with

$$\gamma(E) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \operatorname{tr} \left( \sqrt{V} 1_{]E-\varepsilon, E]}(H_{ac}) V 1_{[E, E+\varepsilon[}(H'_{ac}) \sqrt{V} \right).$$

- $\gamma(E)$  well defined for Leb.-a.e.  $E \in \mathbb{R}$
- $|S_{L_{n_k}}(E)| \leq \exp \left\{ -\frac{a\gamma(E)}{2} \ln L_{n_k} + o_a(\ln L_{n_k}) \right\} \quad \forall 0 < a < 1$
- if  $d = 1$  or  $\ell^2(\mathbb{Z}^d)$  no subsequence necessary
- $\gamma(E) = 0 \quad \forall E \notin \operatorname{spec}_{ac}(H) [= \operatorname{spec}_{ac}(H')]$
- connection with Frank, Lewin, Lieb, Seiringer (2011).

## Relation to scattering theory

$$\gamma(E) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \operatorname{tr} \left( \sqrt{V} 1_{]E-\varepsilon, E]}(H_{ac}) V 1_{[E, E+\varepsilon[}(H'_{ac}) \sqrt{V} \right).$$

**Proposition.**

For Leb.-a.e.  $E \in \operatorname{spec}_{ac}(H)$

$$\gamma(E) = (2\pi)^{-2} \|S(E) - \mathbb{1}\|_{HS}^2 = (2\pi)^{-2} \|T(E)\|_{HS}^2$$

with fixed-energy scattering matrix  $S(E) : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ .

**Corollary.**

Assume  $d = 3$ ,  $V_0 = 0$ , and  $V$  radially symmetric. Then for Leb.-a.e.  $E \geq 0$

$$\gamma(E) = \frac{1}{\pi^2} \sum_{\ell=0}^{\infty} (2\ell + 1) (\sin \delta_{\ell}(E))^2$$

with scattering phases  $\delta_{\ell}(E)$ ,  $\ell \in \mathbb{N}_0$ .

Coincides with Anderson's decay exponent (only point interaction there)!

### 3 Sketch of the proof

For simplicity:  $H = H_{ac}$  and  $H' = H'_{ac}$

$$A := \begin{pmatrix} \langle \varphi_1^L, \psi_1^L \rangle & \cdots & \langle \varphi_1^L, \psi_N^L \rangle \\ \vdots & & \vdots \\ \langle \varphi_N^L, \psi_1^L \rangle & \cdots & \langle \varphi_N^L, \psi_N^L \rangle \end{pmatrix}$$

**Lemma 1.** Let  $E \in \mathbb{R}$  and recall  $N \equiv N_L(E)$ . Then

$$|S_L(E)| = |\det A| = \exp \left\{ -\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} \left( (P \Pi P)^k \right) \right\} \leq \exp \left\{ -\frac{1}{2} \operatorname{tr} (P \Pi) \right\}$$

with  $P := 1_{]-\infty, \lambda_N^L]}(H_L)$  and  $\Pi := 1_{[H_{N+1}^L, \infty[}(H'_L)$ .

### 3 Sketch of the proof

For simplicity:  $H = H_{ac}$  and  $H' = H'_{ac}$

$$A := \begin{pmatrix} \langle \varphi_1^L, \psi_1^L \rangle & \cdots & \langle \varphi_1^L, \psi_N^L \rangle \\ \vdots & & \vdots \\ \langle \varphi_N^L, \psi_1^L \rangle & \cdots & \langle \varphi_N^L, \psi_N^L \rangle \end{pmatrix}$$

**Lemma 1.** Let  $E \in \mathbb{R}$  and recall  $N \equiv N_L(E)$ . Then

$$|S_L(E)| = |\det A| = \exp \left\{ -\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} \left( (P \Pi P)^k \right) \right\} \leq \exp \left\{ -\frac{1}{2} \operatorname{tr} (P \Pi) \right\}$$

with  $P := 1_{]-\infty, \lambda_N^L]}(H_L)$  and  $\Pi := 1_{[H_{N+1}^L, \infty[}(H'_L)$ .

**Definition.** Anderson integral

$$\mathcal{I}_L(E) := \operatorname{tr} (P \Pi)$$

Lower bound on  $\mathcal{I}_L(E)$  needed!



### 3 Sketch of the proof

For simplicity:  $H = H_{ac}$  and  $H' = H'_{ac}$

$$A := \begin{pmatrix} \langle \varphi_1^L, \psi_1^L \rangle & \cdots & \langle \varphi_1^L, \psi_N^L \rangle \\ \vdots & & \vdots \\ \langle \varphi_N^L, \psi_1^L \rangle & \cdots & \langle \varphi_N^L, \psi_N^L \rangle \end{pmatrix}$$

**Lemma 1.** Let  $E \in \mathbb{R}$  and recall  $N \equiv N_L(E)$ . Then

$$|S_L(E)| = |\det A| = \exp \left\{ -\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} \left( (P \Pi P)^k \right) \right\} \leq \exp \left\{ -\frac{1}{2} \operatorname{tr} (P \Pi) \right\}$$

with  $P := 1_{]-\infty, \lambda_N^L]}(H_L)$  and  $\Pi := 1_{[H_{N+1}^L, \infty[}(H'_L)$ .

**Definition.** Anderson integral

$$\mathcal{I}_L(E) := \operatorname{tr} (P \Pi)$$

Lower bound on  $\mathcal{I}_L(E)$  needed!

[Küttler, Otte, Spitzer - AHP to appear]

$$\mathcal{I}_L(E) = \nu(E) \ln L + o(\ln L)$$

( $d = 1, V_0 = 0, V$  more general)

### 3 Sketch of the proof

$$A := \begin{pmatrix} \langle \varphi_1^L, \psi_1^L \rangle & \cdots & \langle \varphi_1^L, \psi_N^L \rangle \\ \vdots & & \vdots \\ \langle \varphi_N^L, \psi_1^L \rangle & \cdots & \langle \varphi_N^L, \psi_N^L \rangle \end{pmatrix}$$

For simplicity:  $H = H_{ac}$  and  $H' = H'_{ac}$

**Lemma 1.** Let  $E \in \mathbb{R}$  and recall  $N \equiv N_L(E)$ . Then

$$|S_L(E)| = |\det A| = \exp \left\{ -\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} \left( (P \Pi P)^k \right) \right\} \leq \exp \left\{ -\frac{1}{2} \operatorname{tr} (P \Pi) \right\}$$

with  $P := 1_{]-\infty, \lambda_N^L]}(H_L)$  and  $\Pi := 1_{[\lambda_{N+1}^L, \infty[}(H'_L)$ .

**Definition.** Anderson integral

$$\mathcal{I}_L(E) := \operatorname{tr} (P \Pi)$$

Lower bound on  $\mathcal{I}_L(E)$  needed!

[Küttler, Otte, Spitzer - AHP to appear]

$$\mathcal{I}_L(E) = \gamma(E) \ln L + o(\ln L)$$

( $d = 1, V_0 = 0, V$  more general)

**Lemma 2.**  $\forall$  sequence of lengths  $(L_n)_{n \in \mathbb{N}}, L_n \uparrow \infty, \exists$  subsequence  $(L_{n_k})_{k \in \mathbb{N}}$  such that for Leb.-a.e.  $E \in \mathbb{R}$

$$\left| \mathcal{I}_{L_{n_k}}(E) - \operatorname{tr} \left( 1_{]-\infty, E]}(H_{L_{n_k}}) 1_{[E, \infty[}(H'_{L_{n_k}}) \right) \right| = o(\ln L_{n_k})$$

Lemma 3. Let  $E \in \mathbb{R}$ . Then

$$\text{tr} \left( 1_{]-\infty, E]}(H_L) 1_{]E, \infty[}(H'_L) \right) = \int_{]-\infty, E] \times ]E, \infty[} \frac{d\mu_L(x, y)}{(y - x)^2}$$

with  $\mu_L(B \times B') := \text{tr} \left( \sqrt{V} 1_B(H_L) V 1_{B'}(H'_L) \sqrt{V} \right)$

Lemma 3. Let  $E \in \mathbb{R}$ . Then  $\forall a > 0$

$$\operatorname{tr} \left( 1_{]-\infty, E]}(H_L) 1_{]E, \infty[}(H'_L) \right) \geq \int_{]-\infty, E] \times ]E, \infty[} \frac{d\mu_L(x, y)}{(y - x + L^{-a})^2}$$

with  $\mu_L(B \times B') := \operatorname{tr} \left( \sqrt{V} 1_B(H_L) V 1_{B'}(H'_L) \sqrt{V} \right)$

Lemma 3. Let  $E \in \mathbb{R}$ . Then  $\forall a > 0$

$$\operatorname{tr} \left( 1_{]-\infty, E]}(H_L) 1_{]E, \infty[}(H'_L) \right) \geq \int_{]-\infty, E] \times ]E, \infty[} \frac{d\mu_L(x, y)}{(y - x + L^{-a})^2}$$

with  $\mu_L(B \times B') := \operatorname{tr} \left( \sqrt{V} 1_B(H_L) V 1_{B'}(H'_L) \sqrt{V} \right)$

Lemma 4.  $\forall 0 < a < 1$  for Leb.-a.e.  $E \in \mathbb{R}$

$$\int_{]-\infty, E] \times ]E, \infty[} \frac{d\mu_L(x, y)}{(y - x + L^{-a})^2} \geq \int_{]-\infty, E] \times ]E, \infty[} \frac{d\mu(x, y)}{(y - x + L^{-a})^2} + O_a(1)$$

with  $\mu(B \times B') := \operatorname{tr} \left( \sqrt{V} 1_B(H) V 1_{B'}(H') \sqrt{V} \right)$

Idea of the proof: regularise, Helffer-Sjöstrand, remove regularisation!

Lemma 3. Let  $E \in \mathbb{R}$ . Then  $\forall a > 0$

$$\operatorname{tr} \left( 1_{]-\infty, E]}(H_L) 1_{]E, \infty[}(H'_L) \right) \geq \int_{]-\infty, E] \times ]E, \infty[} \frac{d\mu_L(x, y)}{(y - x + L^{-a})^2}$$

with  $\mu_L(B \times B') := \operatorname{tr} \left( \sqrt{V} 1_B(H_L) V 1_{B'}(H'_L) \sqrt{V} \right)$

Lemma 4.  $\forall 0 < a < 1$  for Leb.-a.e.  $E \in \mathbb{R}$

$$\int_{]-\infty, E] \times ]E, \infty[} \frac{d\mu_L(x, y)}{(y - x + L^{-a})^2} \geq \int_{]-\infty, E] \times ]E, \infty[} \frac{d\mu(x, y)}{(y - x + L^{-a})^2} + O_a(1)$$

with  $\mu(B \times B') := \operatorname{tr} \left( \sqrt{V} 1_B(H) V 1_{B'}(H') \sqrt{V} \right)$

Idea of the proof: regularise, Helffer-Sjöstrand, remove regularisation!

Lemma 5.  $\forall a > 0$  for Leb.-a.e.  $E \in \mathbb{R}$

$$\int_{]-\infty, E] \times ]E, \infty[} \frac{d\mu(x, y)}{(y - x + L^{-a})^2} = a \gamma(E) \ln L + o_a(\ln L)$$

## Ground State of a Magnetic Impurity in a Metal

PHILIP W. ANDERSON

*Bell Telephone Laboratories, Murray Hill, New Jersey*

(Received 21 July 1967)

... This is the exact version of the orthogonality theorem proved as an inequality in Ref. 15. It is interesting that the main difference from the previous result is to replace  $\sin^2\delta$  by  $\delta^2$ . ...

## Ground State of a Magnetic Impurity in a Metal

PHILIP W. ANDERSON

*Bell Telephone Laboratories, Murray Hill, New Jersey*

(Received 21 July 1967)

... This is the exact version of the orthogonality theorem proved as an inequality in Ref. 15. It is interesting that the main difference from the previous result is to replace  $\sin^2 \delta$  by  $\delta^2$ . ...

**Theorem B.** [Gebert, Küttler, M., Otte - in prep.]

$\forall$  sequence of lengths  $(L_n)_{n \in \mathbb{N}}$ ,  $L_n \uparrow \infty$ ,  $\exists$  subsequence  $(L_{n_k})_{k \in \mathbb{N}}$  such that for Leb.-a.e.  $E \in \mathbb{R}$

$$\limsup_{k \rightarrow \infty} \frac{\ln |S_{L_{n_k}}(E)|}{\ln L_{n_k}} \leq -\frac{\tilde{\nu}(E)}{2}$$

with

$$\tilde{\nu}(E) := \frac{\|\arcsin |T(E)/2|\|_{HS}^2}{\pi^2}.$$

Compare Theorem A:

$$\nu(E) := \frac{\|T(E)/2\|_{HS}^2}{\pi^2}.$$