Dynamics of some symmetric n-body problems

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New Perspectives on the N-body Problem (13w5055)
January 13–18, 2013
Banff International Research Station

We will try to study n-body problems which are symmetric with respect to the action of suitable extensions of finite rotation groups[1]. The space of symmetric configurations is the complement of an arrangement of linear subspaces in a Euclidean space, and blow-up, McGehee coordinates and variational methods can –in some cases– be used to understand local dynamics (around the space of collisions) and some properties of periodic orbits.

$$\begin{aligned} \text{Masses: } & m_1, m_2, \ldots, m_n > 0 \\ \text{Positions: } & \boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n \in \mathbb{R}^d \\ \text{Homogeneity: } & -\alpha < 0 \\ \text{Potential: } & \sum_{i<j} \frac{m_i m_j}{\|\boldsymbol{q}_i - \boldsymbol{q}_j\|^\alpha} \end{aligned}$$

(1) Davide L. Ferrario/Alessandro Portaluri: On the dihedral n-body problem. In: Nonlinearity 21.6 (2008), pp. 1307–1321; idem: Dynamics of the the dihedral four-body problem. In: Discrete and Continuous Dynamical Systems - Series S (DCDS-S) 6.4 (2012), pp. 925 –974.

Symmetries

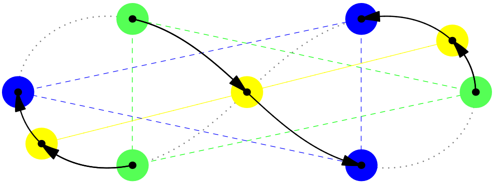
Two basic types of symmetries:

- Involving time
  - $t \mapsto t + \delta$: $x(t+\delta) = gx(t)$; [▷]
  - $t \mapsto -t$: $x(-t) = gx(t)$;
- Not involving time $\forall t, x(t) \in X^G$.

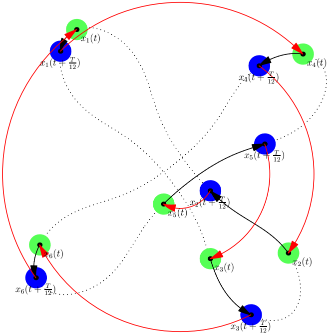
Examples:

- Antipodal symmetry $x(t+\delta) = -x(t)$.
- Devaney isosceles[(2)].
- Sitnikov.
- Chenciner Montgomery figure-eight and choreographies.
- Delgado, Vidal, Venturelli, Ferrario, Terracini, Simò, Martinez, Chen, Salomone, Xia, Gronchi, Negrini, Fusco, …

(2) Robert L. Devaney: Triple collision in the planar isosceles three-body problem. In: Invent. Math. 60.3 (1980), pp. 249–267.



Chenciner–Montgomery Eight Choreography



Two symmetric 3-choregraphies

Consider now finite subgroups of $O(2)$ (planar case) and $SO(3)$ (spatial case). Recall the classification of such groups (*point groups*):

- Plane:
  - Cyclic groups $C_n \subset SO(2)$ (of order n);
  - Dihedral groups $D_n \subset O(2)$ (of order $2n$).
- Space:
  - Cyclic C_n (of order n);
  - Dihedral D_n (of order $2n$);
  - Tetrahedral $T \cong A_4$ (of order 12);
  - Octahedral $O \cong S_4$ (of order 24);
  - Icosahedral $Y \cong A_5$ (of order 60).

For subgroups of $O(3)$, one obtains full groups adding to the above the *inversion* $a\colon x \mapsto -x$, (which is in the center of $SO(3)$) and yields full groups $I \times C_n$, $I \times D_n$, with $I = \{1, a\}$ …or the groups of *mixed type* (those without the inversion a).

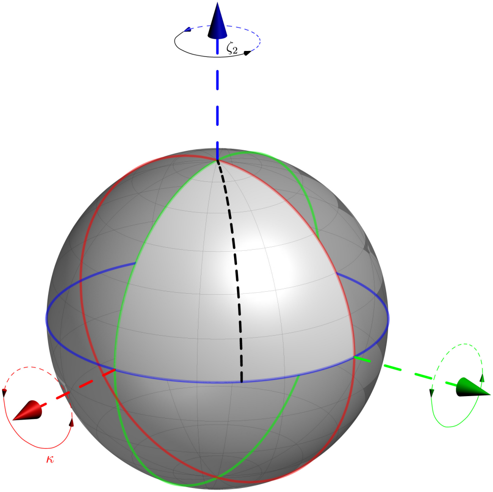
Now consider a rotation group $K \subset SO(3)$ of order n, and n bodies with equal masses "naturally" symmetric with respect to K. Here "naturally" means that the permutation action on $\{1,\ldots,n\}$ is the (natural) Cayley left action of K on $K \approx \{1,\ldots,n\} \approx K$ by assigning indices to the elements of K. For each g, there exists a corresponding permutation $\sigma \in S_n$ defined by $gg_i = g_{\sigma i}$. In other words, if $K = \{g_1,\ldots,g_n\}$, we consider configurations of n points (with equal masses) $q_1,\ldots,q_n \in \mathbb{R}^3$. If X is the $3n$-dimensional configuration space, then the induced symmetry $g\colon X \to X$ is defined by

$$g \cdot (\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n) = \left(g\boldsymbol{q}_{\sigma^{-1}(1)}, g\boldsymbol{q}_{\sigma^{-1}(2)},\ldots,g\boldsymbol{q}_{\sigma^{-1}(n)}\right).$$

The space of symmetric configurations hence is

$$\begin{aligned} X^K &= \{x \in X : Kx = x\} \\ &= \{x = (\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n) : \boldsymbol{q}_i = g_i g_j^{-1}\boldsymbol{q}_j\} \cong \{\boldsymbol{q}_1\} = \mathbb{R}^3 \end{aligned}$$

(Configuration spaces)



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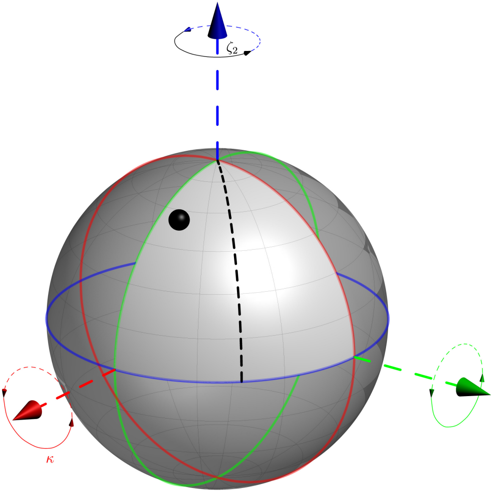

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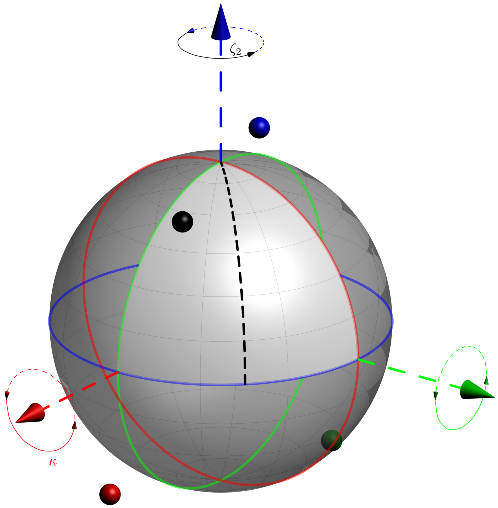

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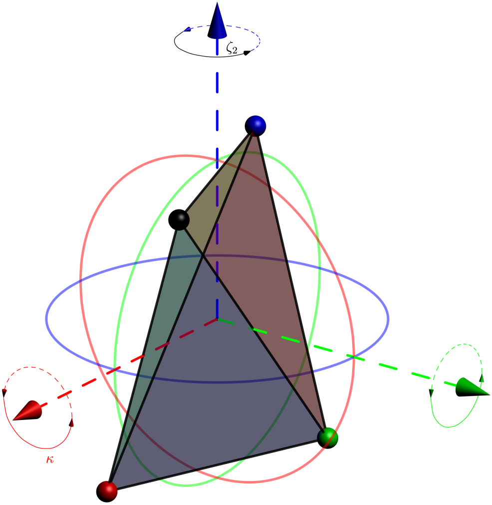


Interaction potential

Consider the binary collision subspace $\Delta_{ij} = \{\boldsymbol{q}_i = \boldsymbol{q}_j\} \subset X$. The *projection* π_{ij} onto Δ_{ij} given by

$$\begin{aligned}\pi_{ij}(x) &= \pi_{ij}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_i, \ldots, \boldsymbol{q}_j, \ldots, \boldsymbol{q}_n) \\ &= (\boldsymbol{q}_1, \ldots, \frac{m_i\boldsymbol{q}_i + m_j\boldsymbol{q}_j}{m_i + m_j}, \ldots, \frac{m_i\boldsymbol{q}_i + m_j\boldsymbol{q}_j}{m_i + m_j}, \ldots, \boldsymbol{q}_n)\end{aligned}$$

is well-defined, and orthogonal with respect to the mass-metric on X. Now, observe that if $\|x\|_M$ denotes the mass-metric on X

$$\begin{aligned}\|x - \pi_{ij}(x)\|_M^2 &= m_i\|\boldsymbol{q}_i - \frac{m_i\boldsymbol{q}_i + m_j\boldsymbol{q}_j}{m_i + m_j}\|^2 + m_j\|\boldsymbol{q}_j - \frac{m_i\boldsymbol{q}_i + m_j\boldsymbol{q}_j}{m_i + m_j}\|^2 \\ &= \ldots = \frac{m_i m_j}{m_i + m_j}\|\boldsymbol{q}_i - \boldsymbol{q}_j\|^2\end{aligned}$$

The potential

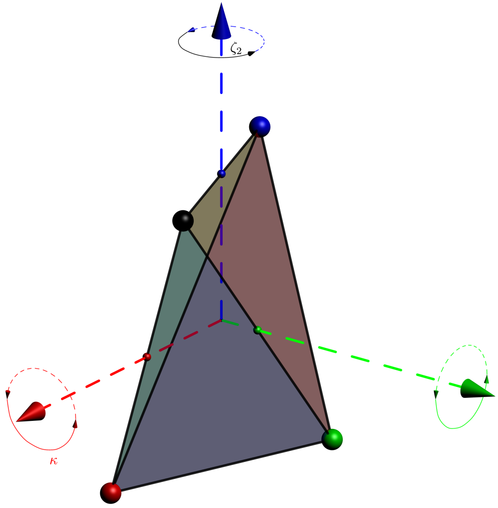
$$\sum_{i<j} \frac{m_i m_j}{\|\boldsymbol{q}_i - \boldsymbol{q}_j\|^\alpha}$$

can be therefore written as

$$\sum_{i<j} \frac{(m_i + m_j)^{-\alpha/2} (m_i m_j)^{1+\alpha/2}}{\|x - \pi_{ij}(x)\|_M} .$$

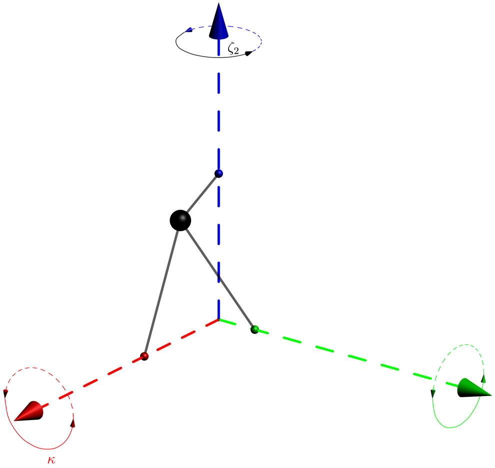
It is a weighted sum of powers of distances from x to binary collision subspaces Δ_{ij}.
Its restriction to symmetric configurations $X^K \subset X$ (all equal masses at the moment, but it can be easily generalized, e.g. isosceles or Sitnikov or multiple choreographies or ...)? If $x \in X^K$, in general it is not true that $\pi_{ij}(x) \in X^K$, but it happens that again it is a weighted sum of powers of distances from subspaces.

Potential on symmetric configurations



$$U = \sum_{H \subset K} \frac{C_H}{\|\boldsymbol{q} - \pi_H(\boldsymbol{q})\|^{\alpha}}$$

The subgroup $H \subset K$ ranges over all the isotropy subgroups of K. The orthogonal projection $\pi_H \colon E \to E^H$ project the configuration space E onto the subspace E^H fixed by H, and C_H is a corresponding positive coefficient.



Let $\boldsymbol{q}, \boldsymbol{p}$ be the canonical coordinates, $(\boldsymbol{q}, \boldsymbol{p}) \in$ phase space. Since U is $-\alpha$-homogeneous, in *McGehee coordinates* (with mass-metric $\|\cdot\| = \|\cdot\|_M$) $\rho = \|\boldsymbol{q}\|, \boldsymbol{s} = \rho^{-1}\boldsymbol{q}, \boldsymbol{z} = \rho^{\alpha/2}\boldsymbol{p}$ after rescaling time and defining

$$v = \langle \boldsymbol{z}, \boldsymbol{s} \rangle, \boldsymbol{w} = \boldsymbol{z} - \langle \boldsymbol{a}, \boldsymbol{s} \rangle \boldsymbol{s}$$

(where $\boldsymbol{w}$ is tangent to the sphere) Newton equations become :

$$\begin{cases} \rho' = \rho v \\ v' = \|\boldsymbol{w}\|^2 + \frac{\alpha}{2} v^2 - \alpha U(\boldsymbol{s}) \\ \boldsymbol{s}' = \boldsymbol{w} \\ \boldsymbol{w}' = -\|\boldsymbol{w}\|^2 \boldsymbol{s} + (\frac{\alpha}{2} - 1) v \boldsymbol{w} + \nabla_{\boldsymbol{s}} U(\boldsymbol{s}) \ , \end{cases}$$

where $\nabla_{\boldsymbol{s}} U$ is the componente of the gradient of U tangent to the inertia ellipsoid $S = \{\|\boldsymbol{q}\| = 1\}$.

McGehee coordinates (cont.)

The coordinates $\rho, v, \boldsymbol{s}, \boldsymbol{w}$ yield a map (homeomorphism outside $\{\rho = 0\}$) defined on the phase space

$$(\boldsymbol{q}, \boldsymbol{p}) \mapsto (\rho, v, \boldsymbol{s}, \boldsymbol{w}) \in [0, +\infty) \times \mathbb{R} \times TS,$$

where TS is the tangent bundle of S. The energy H can be written as

$$2\rho^{\alpha} H = v^2 + \|\boldsymbol{w}\|^2 - 2U(\boldsymbol{s}) \, .$$

All trajectories going to a *total collisions* touch a submanifold of the boundary $\{\rho = 0\}$, termed the MgGehee *total collision manifold* M_0, defined by the equation

$$v^2 + \|\boldsymbol{w}\|^2 = 2U(\boldsymbol{s}) \, .$$

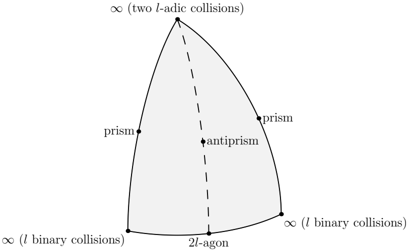
This equation defines also the projection of all parabolic trajectories as a subset of $\mathbb{R} \times TS$, where one eliminates ρ. (Hence, given a solution in M_0, one can integrate ρ and obtain the full parabolic motion)

Partial collisions are a cone of a subset $\Delta \subset S$.
M_0 is a sphere bundle on $S \setminus \Delta$, with fibers $\approx S$. The flow on M_0 is gradient-like (due to v), and stops at *singular* points in $\Delta \subset S$, or at *equilibrium* points, i.e., points satisfying the equations

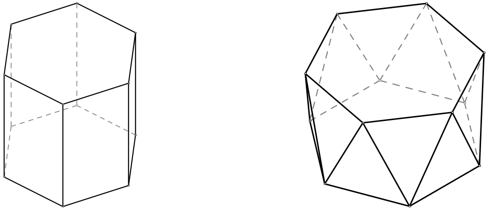
$$v^2 = U(\boldsymbol{s}),\ \nabla_{\boldsymbol{s}} U(\boldsymbol{s}) = \mathbf{0},\ \boldsymbol{w} = \mathbf{0},$$

which correpond to *central configurations*: stationary points for the restricted potential U ($\boldsymbol{s} \in S : \nabla_{\boldsymbol{s}} U(\boldsymbol{s})$). Other equilibrium points in the phase space do not exist.
Equilibrium points must be *found*, singular points must be *regularized*...

C.C. FOR DIHEDRAL CONFIGURATIONS



Central configurations for D_l-symmetric configurations of $2l$ bodies



C.C. FOR DIHEDRAL CONFIGURATIONS (CONT.)

(1) *If $G = D_l$ is the dihedral group with $2l$ elements, then central configurations for D_l-symmetric configurations are only those of the previous slide ($2l$-agon, l-prism and l-antiprism).*

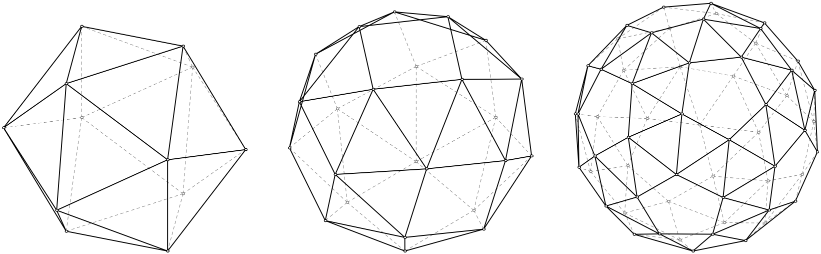
(2) *Moreover, all the corresponding equilibrium points in the M_0 flow are hyperbolic*[(3)]*.*

(3) *For the 4-body Klein group, and any $\alpha \in (0, 2)$, there are 12 square central configurations (4 for each coordinate plane), and 8 tetrahedra, which are minima for U.*
Dimensions of the stable and unstable manifolds in M_0: 2 and 2 for the tetrahedral CC's, 3 and 1 ($v > 0$) or 1 and 3 ($v < 0$) for the squares.

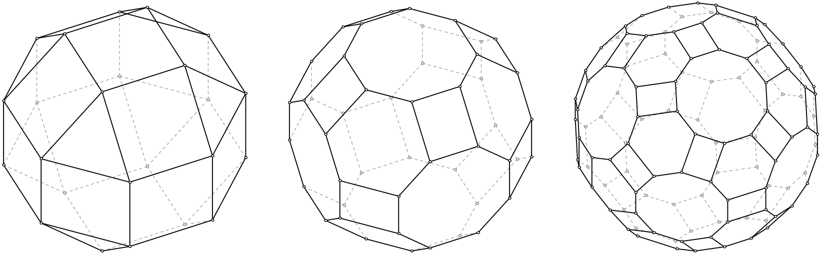
(4) *For the l-dihedral $2l$-body problem and $\alpha \in (0, 2)$, the three families of central configurgations have dimensions of the stable and unstable manifolds in M_0 equal to: prism and planar the same as square CC for the 4-body, all antiprisms the same as tetrahedral CC.*

[(3)] Ferrario/Portaluri: On the dihedral *n*-body problem (see n. (1)).

Other groups?



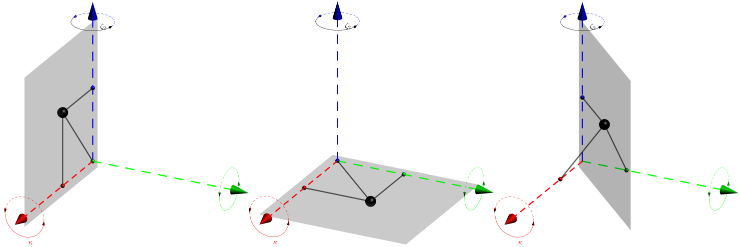
Minimal CC's for T (of order 12), the O (of order 24) and Y (of order 60) and their 2*-covers.*



Recall that for a rotation group, $S \approx S^2$ and M_0 is a four-dimensional S^2-bundle over $S \smallsetminus \Delta$.

For each rotation in the symmetry group G, there is a collision axis, and two antipodal collision points in S. Coxeter planes contain pairs of rotation axes, and are invariant in the flow.

That is, each of the symmetry planes gives rise to an invariant surface in M_0 containing l-agon collisions, with a rectangular flow analogous to the square flow.



(Regularized flow)

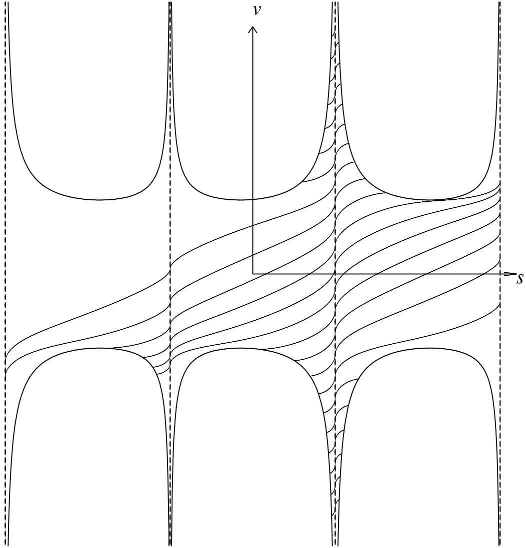
(5) *For any α, a bouncing regularization is possible, but only locally within the plane, by setting for the horizontal plane*

$$u = \frac{\sin^{\alpha}(2\theta)}{\sqrt{W(\theta)}} w$$

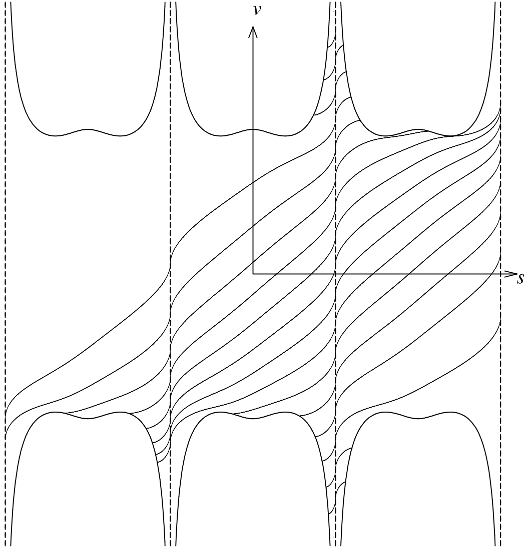
with $W(\theta) = \sin^{\alpha}(2\theta) U(\theta)$ *and changing time accordingly. Here* $\theta \approx \boldsymbol{s}$ *and* $w \approx \boldsymbol{w}$*. Similar formulas hold for the prism and tetrahedral case.*

For $\alpha = 1$ a Levi-Civita double covering map can be defined, which gives the “bouncing” regularization on invariant planes. But, as far as we know, not explicitely for any symmetry group (cfr. Lemaitre-Moeckel-Montgomery).

Covering of the prism section



Covering of the tetrahedral section



In the negative energy region, one can expect to find (many?) periodic collisionless orbits.
A few can be proven to exist by applying previous reults[(4)(5)], minimizing the Lagrangean action on the Sobolev space of G-equivariant loops, for suitable G. Let σ, τ and ρ be the permutation, time and space representation of G, and X the configuration space.

(6) *Let* $K = \ker \tau$. *If* $\rho(K) \subset SO(3)$ *is a finite group of rotations acting transitively on the index set* $\{1, \ldots, n\}$*, and if* $X^G = \{0\}$*, then there exists a G-equivariant collisionless minimizer.*

How to define group actions satisfying this condition?

(7) Corollary. *Given $K \subset SO(3)$ a subgroup of order n, with permutation regular representation $\hat{\sigma} \colon K \to \Sigma_n$, if $g \in N_{O(3)}K$ is such that $(\mathbb{R}^3)^g = 0$, and $s \in \Sigma_n$ is the permutation on K defined by conjugation with g, then the subgroup G of $SO(3) \times \Sigma_n$ generated by the graph of $\hat{\sigma}$ and the element (g, s) satisfies the hypotheses of (6), with ρ, σ natural projections and τ defined as $\tau(K) = 0$, $\tau((g, \sigma)) = 1$.*

(8) Corollary. *Let $K \subset SO(3)$ be a subgroup of order n as above. Then the antipodal map $g = -I \in O(3)$ normalizes K and induces the trivial conjugation permutation s.*

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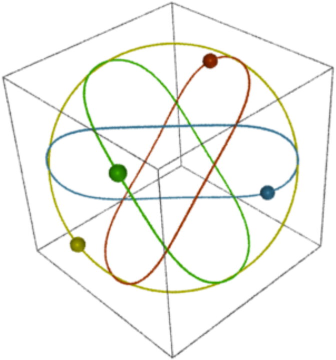
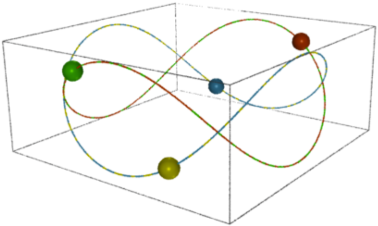
(4) Davide L. Ferrario/Susanna Terracini: On the Existence of Collisionless Equivariant Minimizers for the Classical n-body Problem. In: Invent. Math. 155.2 (2004), pp. 305–362.

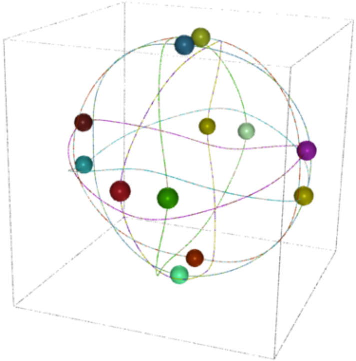
(5) Davide L. Ferrario: Transitive decomposition of symmetry groups for the n-body problem. In: Adv. Math. 213.2 (2007), pp. 763–784, URL: `http://dx.doi.org/10.1016/j.aim.2007.01.009`.



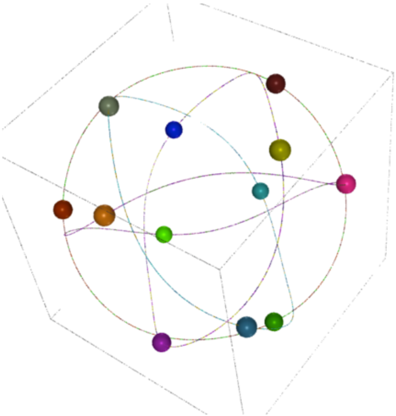
Klein group, $g = -I$ (but the minimizer is also $\mathbb{Z}_3$-symmetric): [▷]



Klein group, $g =$ Hip-Hop rotation: [▷]



$K =$ tetrahedral group, $g =$ Hip-Hop 4-rotation: [▷]



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Remarks

- It is possible to consider multiple copies of the same symmetric minimizing orbit, and a minimizer will exist (eight 3-choreographies + 21 singletons: [▷], a 3-choreography + a 5-choreography + a 7-choreography + a 9-choreography and 3 singletons - $|G| = 630$ [▷]).

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- Take two subspaces, fixed by involutions, with a single intersection. Minimize in the space of all paths going from one component of a subspace to a component of the other $\implies$ there exists a collisionless minimizer, yielding a symmetric minimizer (periodic or quasi-periodic ...).

References

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Thank you