On certain instabilities occurring in the planetary three--body problem

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New Perspectives on the N--body Problem

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Outline

(1) Survey of stability results for the planetary NBP:

- the Kolmogorov Set;
- Exponentially long--time stability of semi--axes on the full phase space.
- Polynomially long--time stability of eccentricities and inclinations excluding mean--motion resonances.

(2) A description of the effect on mean--motion resonances in the planetary 3BP

- Relation between resonances and symmetries

(3) Two planets revolving quite closely, in opposite directions:

- bifurcation to hyperbolic regime;
- Graff--normal form;
- Possible future directions.

The Kolmogorov Set for NBP

Theorem [V. I. Arnold, 1963]

‘‘In the many-body problem there exists a set of initial conditions having a positive Lebesgue measure and such that, if the initial positions and velocities belong to this set, the distances of the bodies from each other will remain perpetually bounded, provided the masses of the planets are sufficiently small.’’

proof

- Arnold 1963 (planar 3BP)
- Robutel 1995 (spatial 3BP)
- Fejoz 2004 (general NBP, checking Arnold--Piartly condition)
- Chierchia-P. 2011(general NBP, checking Kolmogorov condition, with measure of the Kolmogorov set and reduction of degeneracies)

Long--Time Stability of Actions

Theorem [N. N. Nekhorossev, 1977]

(I) Let

$$H = h(I) + \mu f(I, \varphi) \quad (I, \varphi) \in A^n \times T^n \qquad T := R/(2piZ)$$

If

(i) H is real--analytic

(ii) μ is small

(iii) h is ‘‘steep’’

then

$$|I_i(t) - I_i(0)| \le R_\star := R_0 \mu^b \qquad \text{for} \qquad |t| \le T_\star := T_0 e^{\mu^{-a}}$$

(II) The same result holds if the problem if ‘‘degenerate’’, ie, $f = f(I, \varphi; u, v)$ provided $(u(t), v(t))$ remains in its domain for $|t| \le T_\star$.

-Planetary prob.: I= semi--axes, (u,v)= eccentr., inclin.

Improvements

- [Poschel 93] For the h quasi--convex, f non--degenerate case, the stability indices may be taken to be

$$a = b = \frac{1}{2n} \ .$$

- [Niederman 96], using P. Lochak approach ‘‘without small denominators’’, generalizes the values of a, b given by [Poschel 93] to the h convex, f degenerate case.

- The variation of the semi--major axes can now be confined with the improved indices a, b.

What about the degenerate actions?

- The planetary Hamiltonian $H_{NBP} = h_{kep}(I) + \mu f_{NBP}(I, \varphi, u, v)$, with $I = (I_1, \cdots, I_n)$, $\varphi = (\varphi_1, \cdots, \varphi_n)$ is not almost--integrable w.r.t. the degenerate actions $J_i = \frac{u_i^2+v_i^2}{2}$ (related to eccentricities and inclinations).
- However [Chierchia-P. 2011], in a set of phase space of well--spaced semi--axes, the average $\bar{f}_{NBP} = \frac{1}{(2pi)^n} \int_{T^n} f_{NBP} d\varphi$ may be conjugated to

$$P_s(J; I) + O(J^{s+1}; I) \qquad (*)$$

where P_s is a polynomial of arbitrary large degree s in $J_i = \frac{u_i^2+v_i^2}{2}$ (Birkhoff normal form). This normal form is Kolmogorov non--degenerate.
- The proof of (*) requires to reduce the rotational degeneracy.

Steepness (N. N. Nekhorossev, 1977)

- (*) implies stability of eccentricities and inclinations for polynomially long times, in a subset of phase space free of mean--motion resonances. Improving this time to be exponential would be related to investigate steepness of the integrable truncation $h_{\mathrm{Kep}} + \mu P_s$ underneath.
- Nekhorossev proposes to check steepness around a given I_0 via algebraic conditions on the Taylor coefficients of the expansion of f at I_0. However, these conditions become more and more complicated as n increases.
- Niederman gives a synthetic equivalent definition of steepness[a] that does not seem easier.

[a] h has no critical points and its restriction to any affine subspace has only isolated critical points

3BP. The Heliocentric Reduction (9→6 d.o.f.)

$$\mathrm{H}_{3BP} = \underbrace{\sum_{i=1}^{2}\Big(\frac{|y^{(i)}|^2}{2\bar{m}_i} - \frac{\bar{m}_i M_i}{|x^{(i)}|}\Big)}_{\substack{h_{Kep} \\ \text{(integrable)}}} + \mu\,\underbrace{\Big(\frac{y^{(1)}\cdot y^{(2)}}{m_0} - \frac{m_1 m_2}{|x^{(1)} - x^{(2)}|}\Big)}_{\substack{f_{3BP} \\ \text{(perturbing function)}}}$$

$$x^{(i)},\ y^{(i)} \in R^3\ ,\quad x^{(i)} \neq 0\ ,\quad x^{(1)} \neq x^{(2)}$$

$$M_i = m_0 + \mu m_i \quad \bar{m}_i = \frac{m_0 m_i}{M_i}$$

m_0 : star μm_i : planets $i = 1, 2$ $\mu \ll 1$.

Starting Point: an Action--Angle set of symplectic coordinates for 3BP

$$\Omega = \sum_{i=1}^{2} d\Lambda_i \wedge d\bar{l}_i + d\Gamma_i \wedge dg_i + dG \wedge dg + dC_3 \wedge dz$$

$C^{(i)} = x^{(i)} \times y^{(i)}$ angular momentum of planet

$C = C^{(1)} + C^{(2)}$ total angular momentum

E_i : ellipse through $(x^{(i)}, y^{(i)})$

Action--Angle coordinates for 3BP [Deprit 1983]; [P. 2008]

$$\begin{cases} \mathrm{C}_3 & 3^{\mathrm{rd}} \text{ component of } \mathrm{C} \\ \mathrm{G} & \text{Euclidean length } |\mathrm{C}| \text{ of } \mathrm{C} \\ \mathrm{z} & \text{longitude of } \mathrm{C} \\ \mathrm{g} & \text{angle describing the rotation} \\ & \text{of the triangle } \mathrm{C}^{(1)} + \mathrm{C}^{(2)} = \mathrm{C} \end{cases}$$

$$\begin{cases} \Gamma_i & \text{Euclidean length } |\mathrm{C}^{(i)}| \text{ of } \mathrm{C}^{(i)} \\ \mathrm{g}_i & \text{perihelion of } \mathrm{E}_i \text{ w.r.t. } \mathrm{n} = \mathrm{C}^{(1)} \times \mathrm{C}^{(2)} \\ \Lambda_i = \bar{\mathrm{m}}_i \sqrt{\mathrm{M}_i \mathrm{a}_i} & (\mathrm{a}_i = \text{semi--major axis of } \mathrm{E}_i) \\ \bar{\mathrm{l}}_i & \text{mean anomaly of } x^{(i)} \text{ on } \mathrm{E}_i \qquad i = 1,\ 2 \end{cases}$$

A generalization to arbitrary n is available

Reduction of the Rotational Degeneracy
(6→5 d.o.f.)

We need to regularize $e_1 = 0$ or $e_2 = 0$ or i=pi (e_1, e_2=eccentricities; i= mutual inclination)

We introduce a new set of symplectic variables

$$(\Lambda, l, v, v^\star, w, w^\star) \in R^2 \times T^2 \times R^4 \times R^4$$

with

$$\Omega = \underbrace{\sum_{i=1}^{2} d\Lambda_i \wedge dl_i + \sum_{i=1}^{2} dv_i \wedge dv_i^\star}_{\text{planar}} + \underbrace{\sum_{i=1}^{2} dw_i \wedge dw_i^\star}_{\text{spatial}}$$

Symplectic Variables for Reversed Planetary 3BP (5 d.o.f.)

$$\begin{cases} \Lambda_1 = \Lambda_1 \\ \Lambda_2 = \Lambda_2 \\ \\ \mathrm{l}_1 = \bar{\mathrm{l}}_1 + \mathrm{g}_1 + \mathrm{g} + \mathrm{z} \\ \mathrm{l}_2 = \bar{\mathrm{l}}_2 + \mathrm{g}_2 - \mathrm{g} - \mathrm{z} \end{cases}$$

$$\begin{cases} \mathrm{v}_1 = \sqrt{\Lambda_1 - \Gamma_1}\, e^{i(g_1+g+z)} \\ \mathrm{v}_2 = \sqrt{\Lambda_2 - \Gamma_2}\, e^{i(g_2-g-z)} \end{cases}$$

$$\begin{cases} \mathrm{v}_1^\star = -i\sqrt{\Lambda_1 - \Gamma_1}\, e^{-i(g_1+g+z)} \\ \mathrm{v}_2^\star = -i\sqrt{\Lambda_2 - \Gamma_2}\, e^{-i(g_2-g-z)} \end{cases}$$

$$\begin{cases} \mathrm{w}_1 = \sqrt{G - \Gamma_1 + \Gamma_2}\, e^{-i(g+z)} \\ \mathrm{w}_2 = \sqrt{G - C_3}\, e^{iz} \end{cases} \qquad \begin{cases} \mathrm{w}_1^\star = -i\sqrt{G - \Gamma_1 + \Gamma_2}\, e^{i(g+z)} \\ \mathrm{w}_2^\star = -i\sqrt{G - C_3}\, e^{-iz} \end{cases}$$

$\downarrow$ cyclic $\qquad\qquad$ $\downarrow$ cyclic

- These coordinates are available for any n [P. 2009, PhD Thesis]

Symmetries

- H_{3BP} is independent of $(w_2, w_2^\star)$:

$$H_{3BP} = h_{kep}(\Lambda) + \mu f_{3BP}(\Lambda, l, u, u^\star)$$

where $u = (v_1, v_2, w_1)$, $u^\star = (v_1^\star, v_2^\star, w_1^\star)$

- G=|C| is an integral of motion and its expression is

$$G = \Lambda_1 - \Lambda_2 - i\sum_{i=1}^{3} u_i u_i^\star$$

- H_{3BP} is invariant by
  - Reflections w.r.t. $\{x^{(3)} = 0\}$
  - Reflections w.r.t. $\{x^{(1)} = x^{(2)}\}$
  - Rotations w.r.t. the C--axis

The Secular Perturbation

The secular perturbation

$$\bar{f}_{3BP} = \frac{1}{(2pi)^2}\int_{T^2} f_{3BP} dl_1 dl_2$$

- is even in $(u, u^\star)$

- has the form $\bar{f}_{3BP} = \bar{f}_0(\Lambda) + iv \cdot Q_h(\Lambda)v^\star + iw_1 Q_v(\Lambda)w_1^\star + O_4(u, u^\star)$
  where

  - Q_h is a real, non symmetric, 2×2, and has real eigenvalues s_1, s_2 for all Λ;
  - $Q_v = (s_3)$ is 1×1, real, for all Λ;
  - $(u, u^\star) = 0$ is an elliptic fixed point for $\bar{f}_{3BP}$, for all Λ;
  - The eigenvalues s_1, s_2, s_3 verify identically the Herman Resonance

$$s_1 + s_2 + s_3 \equiv 0$$

Dynamics away from Resonances

Theorem

There exists a positive measure set of quasi--periodic tori (Kolmogorov tori) with 5 frequencies.

(The proof is analogue to our proof of Arnold's Planetary Theorem)

Mean--Motion Resonances

- By Normal Form Theory [Nekhorossev, Poschel...], relations of the form

$$n_1 w^{(1)} - n_2 w^{(2)} = 0 \qquad \text{with} \quad w^{(i)} := \frac{d}{d\Lambda_i} h_{Kep}$$

with n_1, n_2 positive and co--prime, transform the effective perturbing function $\bar{f}_{NBP}$ into the projection

$$f_L = \Pi_L f := \sum_{\substack{k \in L \\ (a,a^\star) \in N^3}} f_{k,a,a^\star}(\Lambda) e^{i(k_1 l_1 - k_2 l_2)} u^a u^{\star a^\star} \qquad i = \sqrt{-1}$$

over the resonant module

$$L = \left\{ k = (k_1, k_2) = j(n_1, n_2) \ , \quad j \in Z \right\} \ .$$

The invariance by Rotations around C

- Invariance by rotations around C imposes that the coefficients of the Taylor--Fourier expansion

$$f_{3BP} = \sum_{\substack{k \in Z^2 \\ (a,a^\star) \in N^3}} f_{k,a,a^\star}(\Lambda) e^{i(k_1 l_1 - k_2 l_2)} u^a u^{\star a^\star}$$

verify

$$f_{k,a,a^\star}(\Lambda) \neq 0 \quad \Rightarrow \quad k_1 \pm k_2 + |a| - |a^\star| = 0$$

where

$$\text{sign} = \begin{cases} \uparrow & \text{for the reversed problem} \\ \downarrow & \text{for the parallel problem} \end{cases}$$

- This follows writing the Hamiltonian flow of

$$G = \Lambda_1 \mp \Lambda_2 - i \sum_{j=1}^{3} u_j u_j^\star \ .$$

Consequencies

- Fixed $p \in N$ and a resonant lattice $L = \{(n_1, n_2)\}$, a monomial of degree p

$$\sum_{\substack{k \in Z^2 \\ |a|+|a^\star|=p}} f_{k,a,a^\star}(\Lambda) e^{i(k_1 l_1 - k_2 l_2)} u^a u^{\star a^\star}$$

contains at most p+1 wave vectors belonging to L, precisely all those such that

$$j(n_1 \pm n_2) = |a^\star| - |a| \in \{-p, -p+2, \cdots, p\} \quad \text{for some } j \in Z\ .$$

- This is except for $(n_1, n_2) = (1, 1)$ in the parallel problem. In the following discussion, we shall always exclude this case.

- In particular, the only wave vectors proportional to resonances appearing into quadratic terms are, respectively

$$\begin{cases} (1,1) & \text{(reversed problem)} \\ (h+2,h)\ , \quad h \in \mathbb{N} & \text{(parallel problem)} \end{cases}$$

- The resonance $(1,1)$ causes hyperbolic effects in the quadratic part of the reversed problem.

- Féjoz, Guardia, Kaloshin and Roldan consider the resonance $(7,1)$ in the ER3BP and detect a displacement of the eccentricity.

Projection on the resonance $(1,1)$ (reversed problem)

The projection f_{3BP} on the resonance $(1,1)$ depends on the angles (l_1, l_2) only via the combinations

$$\hat{u} = u e^{-i(l_1 - l_2)/2} \qquad \hat{u}^\star = u^\star e^{i(l_1 - l_2)/2} \quad .$$

It is even in $(\hat{u}, \hat{u}^\star)$ and has the form

$$f_{(1,1)}(\Lambda, \hat{u}, \hat{u}^\star) = \bar{f}_0(\Lambda) + \underbrace{\hat{Q}_h(\Lambda) \cdot (\hat{v}, \hat{v}^\star)^2 + \hat{Q}_v(\Lambda) \cdot (\hat{w}_1, \hat{w}_1^\star)^2}_{\text{partially hyperbolic}} + O_4(\hat{u}, \hat{u}^\star; \Lambda)$$

where

- $\hat{Q}_h(\Lambda)$ is 4×4 and $J_4 \hat{Q}_h(\Lambda)$ has a couple of real opposite eigenvalues and one of purely imaginary ones.
- $\hat{Q}_v(\Lambda)$ is 2×2 and $J_2 \hat{Q}_v(\Lambda)$ has a couple of real opposite eigenvalues;

Reduction to 4 d.o.f.

This suggests to switch to the following symplectic variables

$$\begin{cases} \mathtt{G} := \Lambda_1 - \Lambda_2 - \mathtt{i}\sum_{\mathtt{j}=1}^{3} \mathtt{u_j} \cdot \mathtt{u_j^\star} & \rightarrow \ \text{integral} \\ \hat{\mathtt{L}} := \Lambda_1 + \Lambda_2 \end{cases}$$

$$\begin{cases} \mathtt{g} := \frac{1}{2}(\mathtt{l_1} - \mathtt{l_2}) & \rightarrow \ \text{cyclic and slow} \\ \hat{\mathtt{l}} := \frac{1}{2}(\mathtt{l_1} + \mathtt{l_2}) \end{cases}$$

$$\begin{cases} \hat{\mathtt{u}}_\mathtt{j} = \mathtt{u_j} \mathtt{e}^{-\mathtt{i}(\mathtt{l_1}-\mathtt{l_2})/2} \\ \hat{\mathtt{u}}_\mathtt{j}^\star = \mathtt{u_j^\star} \mathtt{e}^{\mathtt{i}(\mathtt{l_1}-\mathtt{l_2})/2} \end{cases} \qquad \mathtt{j} = 1,\ 2,\ 3.$$

- Neglecting g, this reduction reduces to put into the Hamiltonian

$$\begin{cases} \Lambda_1 = \frac{1}{2}(\hat{L} + G + i\sum_{j=1}^{3} \hat{u}_j \hat{u}_j^\star) \\ \Lambda_2 = \frac{1}{2}(\hat{L} - G - i\sum_{j=1}^{3} \hat{u}_j \hat{u}_j^\star) \end{cases} \quad \begin{cases} l_1 = \hat{l} \\ l_2 = \hat{l} \end{cases} \quad \begin{cases} u_i = \hat{u}_i \\ u_i^\star = \hat{u}_i^\star \end{cases}$$

- This is general: for any given resonance (n_1, n_2), the reduction

$$\begin{cases} \Lambda_1 = \frac{1}{n_1 \pm n_2}(\hat{L} + n_1(G + i\sum_{j=1}^{3} \hat{u}_j \hat{u}_j^\star) \\ \Lambda_2 = \frac{1}{n_1 \pm n_2}(\hat{L} - n_2(G + i\sum_{j=1}^{3} \hat{u}_j \hat{u}_j^\star) \end{cases} \quad \begin{cases} l_1 = n_2 \hat{l} \\ l_2 = n_1 \hat{l} \end{cases} \quad \begin{cases} u_i = \hat{u}_i \\ u_i^\star = \hat{u}_i^\star \end{cases}$$

  reduces completely rotations preserving the Hamiltonian structure of motion equations.

- This is because, by linear algebra, one can always find a new set of variables having G as a generalized momentum and $g = n_1 l_1 - n_2 l_2$ as its conjugated variable (cyclic and slow).

Advantages of this Reduction

- It is regular, in contrast to Poincaré variables after Jacobi reduction, that are singular for zero inclination;

- It is adapted to the study of dynamics around resonances: Expanding in powers of $(\hat{u}, \hat{u}^\star)$

$$h(\Lambda_1, \Lambda_2) = h_0(\hat{L}; G) + \underbrace{i(n_1 w^{(1)} - n_2 w^{(2)})}_{\text{small}} \sum_{j=1}^{3} \hat{u}_j \hat{u}_j^\star + \cdots$$

- It is available for any number of planets.

Back to the Resonance (1,1) for Reversed 3BP

In the reduced coordinates coordinates H_{3BP} takes the form

$$\begin{aligned} H_{RED} &= h(\hat{L};G) + \bar{f}_{RED}(\hat{L},\hat{u},\hat{u}^\star;G) + \mu \hat{f}_{RED}(\hat{L},\hat{l},\hat{u},\hat{u}^\star;G) \\ &= h(\hat{L};G) + \underbrace{(w^{(1)} - w^{(2)}) \sum_{j=1}^{3} i\hat{u}_j\hat{u}_j^\star}_{\text{elliptic term}} + \\ &+ \underbrace{\mu(\hat{Q}_{0h}(\hat{L};G)\cdot(\hat{v},\hat{v}^\star)^2 + \hat{Q}_{0v}(\hat{L};G)\cdot(\hat{w}_1,\hat{w}_1^\star)^2)}_{\text{partially hyperbolic term}} + O_4(\hat{u},\hat{u}^\star;\hat{L},G) \\ &+ \underbrace{\mu \hat{f}_{RED}(\hat{L},\hat{l},\hat{u},\hat{u}^\star;G)}_{\int_T \hat{f}_{RED} d\hat{l} = 0} \end{aligned}$$

where $\begin{cases} h(\hat{L};G) := (h_{Kep} + \mu\bar{f}_0)(\Lambda) \\ w^{(i)} := \partial_{\Lambda_i}(h_{Kep} + \mu\bar{f}_0)(\Lambda) \\ \hat{Q}_{0h}(\hat{L};G) = \hat{Q}_h(\Lambda) \quad \hat{Q}_{0v}(\hat{L};G) = \hat{Q}_v(\Lambda) \end{cases} \Bigg|_{\Lambda_1 = \frac{1}{2}(\hat{L}+G),\ \Lambda_2 = \frac{1}{2}(\hat{L}-G)}$

Two Natural Questions

We would like to know if

- Equation $w^{(1)} - w^{(2)} = 0$ has a solution $\hat{L} = R(G)$ for any fixed G. If so, on a suitable small neighborhood of R the quadratic part would be partially hyperbolic (with 2 unstable directions)

- a domain D for the variables $(\hat{L}, \hat{l}, \hat{u}, \hat{u}^{\star})$ exists, with $\Pi_{\hat{L}} D$ including R such, that, on D, H_{RED} can be conjugated to a new Hamiltonian (normal form) having the aspect as H_{RED}, but with a much smaller remainder.

The Resonant Set

Lemma

Let

$$\mathtt{w} = (\mathtt{w}^{(1)}, \mathtt{w}^{(2)}) := \left(\frac{\mathrm{d}}{\mathrm{d}\Lambda_1}, \frac{\mathrm{d}}{\mathrm{d}\Lambda_2}\right)(h_{kep} + \mu \bar{f}_0)(\Lambda) \ .$$

There exists a one--dimensional set R of semi--major axes such that

$$\mathtt{w}^{(1)} - \mathtt{w}^{(2)}\Big|_R = 0 \ .$$

This set may be described by

$$R = \left\{ a_1 > a_2 : \ \frac{a_2}{a_1} = 1 - \mathrm{const}\sqrt{\mu}(1 + o(1)) \right\} \ \mathrm{const} := \sqrt{\frac{4}{3\pi} \frac{m_1 + m_2}{m_0}}$$

Note:

There is a displacement of R with respect to the value we would have ($= \mathrm{const}_1\, \mu$) without including the perturbation term $\mu \bar{f}_0$ into w. This is due to the singularity of $\mu \bar{f}_0$ for $a_1 = a_2$.

Remark:

The existence of normal form is not obvious. In Poschel's normal form theorem (optimal) the smallness condition which is required to achieve it is

$$\mathrm{const}_{\,2}\frac{|\mathrm{f}_{\mathrm{NBP}}|}{\rho\sigma} < 1 \qquad\qquad (**) \ .$$

In our case, if

$$\mathrm{r} := \inf(1 - \frac{\mathrm{a}_2}{\mathrm{a}_1})$$

we have

$$\mathrm{f}_{\mathrm{NBP}} \sim \frac{\mu}{\mathrm{r}} \ . \quad \rho \sim \mathrm{r} \ , \quad \sigma \sim \mathrm{r} \ .$$

So, condition (**) would be satisfied for

$$\mathrm{const}_{\,2}\frac{\mathrm{P}}{\rho\sigma} = \mathrm{const}_{\,3}\frac{\mu}{\mathrm{r}^3} < 1$$

which means $\mathrm{r} \geq \mathrm{const}_4\mu^{1/3}$, while we need $\mathrm{r} = \mathrm{const}\mu^{1/2}$ to be included.

A Normal Form Theorem

Theorem

There exist a number b and a domain $D = A_1 \times T \times B^6$ verifying

(i) $R \in A_1$

(ii) H_{RED} is real--analytic on D

and a real--analytic transformation having D as image--set such that, on D, such that H_{RED} is analytically and symplectically conjugated to

$$H_{NF} = h(\hat{L}; G) + \bar{f}_{NF}(\hat{L}, \hat{v}, \hat{v}^{\star}; G) + \mu f_{\star}(\hat{L}, \hat{l}, \hat{v}, \hat{v}^{\star}; G)$$

where $f_{\star}$ is exponentially small

$$|f_{\star}| \leq \text{const}\, e^{-1/\delta^b}$$

and $\bar{f}_{NF}$ is close to $\bar{f}_{RED}$. Here, δ denotes

$$\delta := |\frac{m_1}{m_2} - 1| \ .$$

Idea of proof.

1^{st} step. The one--dimensional set of (Λ_1, Λ_2) solving

$$\begin{cases} \dfrac{a_2}{a_1} = \dfrac{M_1\bar{m}_1^2}{M_2\bar{m}_2^2}\dfrac{\Lambda_2^2}{\Lambda_1^2} \leq 1 - r \\ \Lambda_1 - \Lambda_2 = \text{const} \end{cases}$$

becomes unbounded ad $\hat{d} := \frac{M_1\bar{m}_1^2}{M_2\bar{m}_2^2} - 1 \to 0$. This set defines an interval for the variable $\hat{L}$ such that, if $\hat{L} \in A_1$, the coefficients of the Taylor expansion of f_{RED} around $(\hat{u}, \hat{u}^\star) = 0$ are real--analytic. It can be proved that A_1 is an interval whose extremes are of order δ^{-1} and such that, for $(\hat{L}, \hat{l})$ in the *complex domain* $(A_1)_\rho \times T_r$, where $\rho = \dfrac{r}{\delta^2}$, the coefficients remain analytic. The whole perturbation in analytic on the complex domain $(A_1)_\rho \times T_r \times B_\varepsilon^6$, with $\varepsilon = \frac{r}{\sqrt{\delta}}$, whose real part will be identified with D.

$2^{\rm nd}$ step. Consider the Hamiltonian

$$\mathtt{h}_{\rm Kep}(\hat{\mathtt{L}};\mathtt{G})+\mu\mathtt{f}_0(\hat{\mathtt{L}},\hat{\mathtt{l}};\mathtt{G})\quad\text{where}\quad \mathtt{f}_0(\hat{\mathtt{L}},\hat{\mathtt{l}};\mathtt{G}):=\mathtt{f}_{\rm RED}\Big|_{(\hat{\mathtt{u}},\hat{\mathtt{u}}^\star)=0}$$

and aim to integrate it (it has one degree of freedom). This is done via a quantitative version of Arnold--Liouville Theorem whose smallness condition goes as

$$\mathtt{const}\,\frac{|\mathtt{f}_0|}{\rho}\le 1\quad \mathtt{const}\,\frac{|\bar{\mathtt{f}}_0|}{\rho\sigma}\le 1\cdots$$

where $\bar{\mathtt{f}}_0:=\int_{\mathtt{T}}\mathtt{f}_0$. Due to the rescaling rules of the analyticity radii ρ, σ and to the smallness condition which allow to ‘‘gain one power in r’’. That is, now we can average over the set with $\mathtt{r}\ge\mathtt{const}_5\mathtt{r}^{1/2}$, but this could be even not enough to include $\mathtt{R}=\mathtt{const}\,\mathtt{r}^{1/2}$. Requiring almost equal masses finally gives the result, because, by the previous discussion, $\mathtt{const}_5$ goes to 0 with δ.

3^{rd} step. The rest of the perturbation

$$f_1 = f_{\mathrm{RED}} - f_0$$

(starting with linear terms in $(\hat{u}, \hat{u}^\star)$) is finally averaged out. This is done via many steps (as many as some power of $\frac{r}{\delta}$) of Averaging Theory [Arnold]. A suitable statement of this theory is built up, in order to fit with the parameters involved. In particular, it is essential that averaging involves one angle only, so as to obtain a suitable smallness condition allowing to apply. q.e.d.

Challenging Goals

- Does this normal form allow to prove existence of quasi--periodic motions evolving on hyperbolic tori of co--dimension 1 for the planar problem, 2 for the spatial one?

- If so, does this setting allow to prove existence of Arnold instability for semi--axes? (Numerically, it seems to be detected: [Quillen, preprint 2011] and references therein)

(Thanks!)

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