Noncollision singularities in a simplified planar four-body problem

Jinxin Xue

Universities of Maryland
joint with D.Dolgopyat

New Perspectives in N-body problem, Banff

Outline

Outline

The model

- Two fixed centers $Q_1 = (-\chi, 0)$, $Q_2 = (0, 0)$. $m_1 = m_2 = 1$.
- Two small particles Q_3 and Q_4, $\quad m_3 = m_4 = \mu \ll 1$
- Q_3 is captured by Q_2 and Q_4 is a messenger traveling between Q_1 and Q_2.

The model

- Two fixed centers $Q_1 = (-\chi, 0)$, $Q_2 = (0, 0)$. $m_1 = m_2 = 1$.
- Two small particles Q_3 and Q_4, $\quad m_3 = m_4 = \mu \ll 1$
- Q_3 is captured by Q_2 and Q_4 is a messenger traveling between Q_1 and Q_2.

The model

- Two fixed centers $Q_1 = (-\chi, 0)$, $Q_2 = (0, 0)$. $m_1 = m_2 = 1$.
- Two small particles Q_3 and Q_4, $\quad m_3 = m_4 = \mu \ll 1$
- Q_3 is captured by Q_2 and Q_4 is a messenger traveling between Q_1 and Q_2.

- The Hamiltonian

$$H = \frac{|P_3|^2}{2\mu} + \frac{|P_4|^2}{2\mu} - \frac{\mu}{|Q_3|} - \frac{\mu}{|Q_3 - (-\chi, 0)|}$$
$$- \frac{\mu}{|Q_4|} - \frac{\mu}{|Q_4 - (-\chi, 0)|} - \frac{\mu^2}{|Q_3 - Q_4|}.$$

-

$$P \to \mu v, \quad H \to H/\mu.$$

- The Hamiltonian

$$H = \frac{|P_3|^2}{2\mu} + \frac{|P_4|^2}{2\mu} - \frac{\mu}{|Q_3|} - \frac{\mu}{|Q_3 - (-\chi, 0)|} - \frac{\mu}{|Q_4|} - \frac{\mu}{|Q_4 - (-\chi, 0)|} - \frac{\mu^2}{|Q_3 - Q_4|}.$$

- $$P \to \mu v, \quad H \to H/\mu.$$

Singular solutions

Let $\omega = \{\omega_j\}_{j=1}^{\infty}$ be a sequence of 3s and 4s.

Definition

We say that $(Q_3(t), Q_4(t))$ is a **singular solution with symbolic sequence** ω if there exists a positive increasing sequence $\{t_j\}_{j=0}^{\infty}$ such that

- $t^* = \lim_{j\to\infty} t_j < \infty$.
- $|\dot{Q}_i(t)| \to \infty$ as $t \to t^*$.
- $|Q_3(t_j) - Q_2| \leq C$, $|Q_4(t_j) - Q_2| \leq C$.
- If $\omega_j = 4$ then for $t \in [t_{j-1}, t_j]$, $|Q_3(t) - Q_2| \leq C$ and $\{Q_4(t)\}_{t\in[t_{j-1},t_j]}$ winds around Q_1 exactly once.
  If $\omega_j = 3$ then for $t \in [t_{j-1}, t_j]$, $|Q_4(t) - Q_2| \leq C$ and $\{Q_3(t)\}_{t\in[t_{j-1},t_j]}$ winds around Q_1 exactly once.

Singular solutions

Let $\omega = \{\omega_j\}_{j=1}^{\infty}$ be a sequence of 3s and 4s.

Definition

We say that $(Q_3(t), Q_4(t))$ is a **singular solution with symbolic sequence** ω if there exists a positive increasing sequence $\{t_j\}_{j=0}^{\infty}$ such that

- $t^* = \lim_{j\to\infty} t_j < \infty$.
- $|\dot{Q}_i(t)| \to \infty$ as $t \to t^*$.
- $|Q_3(t_j) - Q_2| \le C$, $|Q_4(t_j) - Q_2| \le C$.
- If $\omega_j = 4$ then for $t \in [t_{j-1}, t_j]$, $|Q_3(t) - Q_2| \le C$ and $\{Q_4(t)\}_{t\in[t_{j-1},t_j]}$ winds around Q_1 exactly once.
  If $\omega_j = 3$ then for $t \in [t_{j-1}, t_j]$, $|Q_4(t) - Q_2| \le C$ and $\{Q_3(t)\}_{t\in[t_{j-1},t_j]}$ winds around Q_1 exactly once.

Singular solutions

Let $\omega = \{\omega_j\}_{j=1}^{\infty}$ be a sequence of 3s and 4s.

Definition

We say that $(Q_3(t), Q_4(t))$ is a **singular solution with symbolic sequence** ω if there exists a positive increasing sequence $\{t_j\}_{j=0}^{\infty}$ such that

- $t^* = \lim_{j\to\infty} t_j < \infty$.
- $|\dot{Q}_i(t)| \to \infty$ as $t \to t^*$.
- $|Q_3(t_j) - Q_2| \le C$, $|Q_4(t_j) - Q_2| \le C$.
- If $\omega_j = 4$ then for $t \in [t_{j-1}, t_j]$, $|Q_3(t) - Q_2| \le C$ and $\{Q_4(t)\}_{t\in[t_{j-1},t_j]}$ winds around Q_1 exactly once.
  If $\omega_j = 3$ then for $t \in [t_{j-1}, t_j]$, $|Q_4(t) - Q_2| \le C$ and $\{Q_3(t)\}_{t\in[t_{j-1},t_j]}$ winds around Q_1 exactly once.

Outline

Main theorem

We denote by Σ_ω the set of initial conditions of singular orbits with symbolic sequence ω.

Theorem (Dolgopyat, X.)

There exists $\mu_* \ll 1$ such that for $\mu < \mu_*$ the set $\Sigma_\omega \neq \emptyset$. Moreover there is an open set U in the phase space and a foliation of U by two-dimensional surfaces such that for any leaf S of our foliation $\Sigma_\omega \cap S$ is a Cantor set.

Outline

Conjectures on noncollision singularities

- Conjecture

  The set of non-collision singularities has zero measure for all $N > 3$.

- Conjecture (Painlevé)

  The set of non-collision singularities is non-empty for all $N > 3$.

Conjectures on noncollision singularities

- Conjecture

  The set of non-collision singularities has zero measure for all $N > 3$.

- Conjecture (Painlevé)

  The set of non-collision singularities is non-empty for all $N > 3$.

Previous works

- 1979 Mather, McGehee: collinear 4-body problem.
- Infinite number of binary collisions.

Previous works

- 1979 Mather, McGehee: collinear 4-body problem.
- Infinite number of binary collisions.

- 1994 Xia: the spacial 5-body problem
- 1991 Gerver: planar 3N body problem

OPEN : $N = 4$?

Outline

Poincaré's second species solution

Second species solution:
periodic orbits converging to collision chains as $\mu \to 0$.

- Restricted three body problem:
  Bolotin, MacKay.
  Fonts, Nunes, Simo.
- Full three-body problem:
  Bolotin

Our work:

- Positive masses,
- infinitely long collision chain,
- new mechanism of producing hyperbolicity.

- Restricted three body problem:
  Bolotin, MacKay.
  Fonts, Nunes, Simo.
- Full three-body problem:
  Bolotin

Our work:

- Positive masses,
- infinitely long collision chain,
- new mechanism of producing hyperbolicity.

Outline

Gerver's model: the setting

- $\mu = 0, \chi = \infty$.
- Q_3 ellipse is always vertical.
- Q_4 hyperbola has always horizontal asymptotes.
- Interaction of Q_3 and Q_4 is elastic collision.

Gerver's model: the first collision

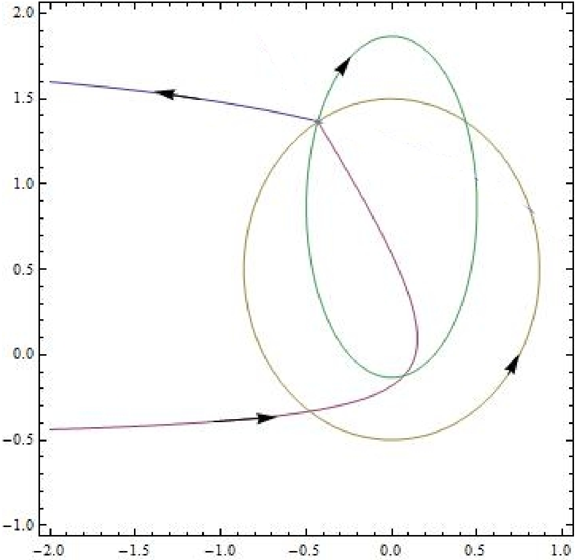

Figure: Angular momentum transfer collision

Gerver's model: the second collision

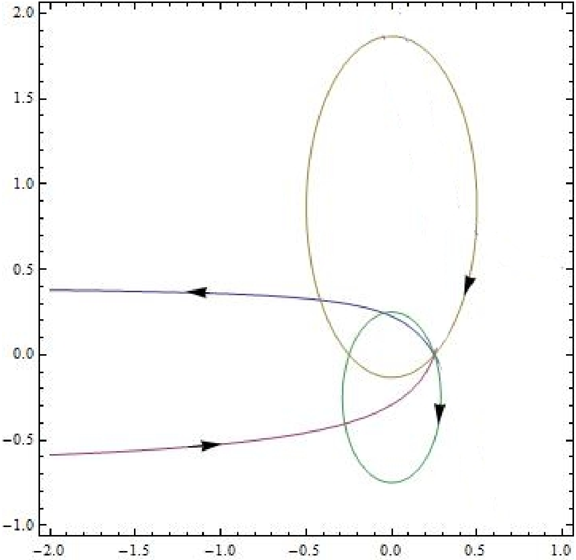

Figure: Energy transfer collision

Gerver's model: Main conclusion

- After two steps of collisions,

the same eccentricity

smaller semimajor

- For elliptic motion, $E = -\frac{1}{2a}$.
-

$$E_3 \sim -\lambda^n, \quad E_4 \sim \lambda^n, \quad \lambda > 1.$$

$$v_4 \sim \lambda^{n/2}, \quad \Delta t \sim \lambda^{-n/2}.$$

Gerver's model: Main conclusion

- After two steps of collisions,

  the same eccentricity

  smaller semimajor

- For elliptic motion, $E = -\dfrac{1}{2a}$.
-

$$E_3 \sim -\lambda^n, \quad E_4 \sim \lambda^n, \quad \lambda > 1.$$

$$v_4 \sim \lambda^{n/2}, \quad \Delta t \sim \lambda^{-n/2}.$$

Gerver's model: Main conclusion

- After two steps of collisions,

  the same eccentricity

  smaller semimajor

- For elliptic motion, $E = -\frac{1}{2a}$.
-
$$E_3 \sim -\lambda^n, \quad E_4 \sim \lambda^n, \quad \lambda > 1.$$
$$v_4 \sim \lambda^{n/2}, \quad \Delta t \sim \lambda^{-n/2}.$$

Outline

Poincaré Sections

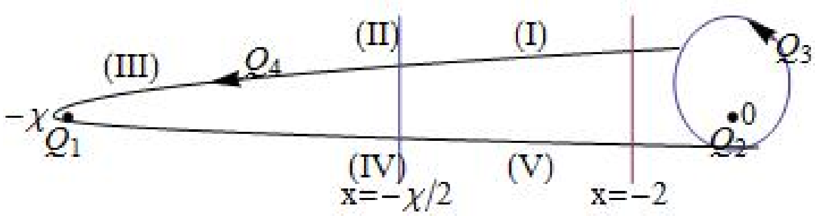

Figure: Poincaré sections

Local map

Lemma

If the y coordinates of the incoming and outgoing orbits of Q_4 are bounded, then there exist a linear functional $\hat{\mathbf{l}}_i$ and a vector $\hat{u}_i$ such that

$$d\mathbb{L}(\boldsymbol{x}) = \frac{1}{\mu} u(\boldsymbol{x}) \otimes \mathbf{l}(\boldsymbol{x}) + B(\boldsymbol{x}) + o(1). \quad \mu \to 0, \chi \to \infty.$$

Global map

Lemma

Let $\boldsymbol{x}$ and $\boldsymbol{y} = \mathbb{G}(\boldsymbol{x})$ be such that $|y(\boldsymbol{x})| \leq C$, $|y((\boldsymbol{y}))| \leq C$ and Q_4 passes within distance $\tilde{C}/\chi$ from Q_1. Then there exist linear functionals $\bar{\mathbf{l}}(\boldsymbol{x})$ and $\bar{\bar{\mathbf{l}}}(\boldsymbol{x})$ and vectorfields $\bar{u}(\boldsymbol{y})$ and $\bar{\bar{u}}(\boldsymbol{y})$ such that

$$d\mathbb{G}(\boldsymbol{x}) = \chi^2 \bar{u}(\boldsymbol{y}) \otimes \bar{\mathbf{l}}(\boldsymbol{x}) + \chi \bar{\bar{u}}(\boldsymbol{y}) \otimes \bar{\bar{\mathbf{l}}}(\boldsymbol{x}) + O(\mu^2 \chi).$$

Nondegeneracy

Lemma

The following non degeneracy conditions are satisfied.

- $$span(u, BY) \pitchfork (Ker(\bar{\mathbf{l}}) \cap Ker(\bar{\bar{\mathbf{l}}}))$$
  where $Y = (\mathbf{l}\bar{u})\bar{\bar{u}} - (\mathbf{l}\bar{\bar{u}})\bar{u} \in span(\bar{u}, \bar{\bar{u}}) \cap Ker\mathbf{l}$.
- $$\det \begin{pmatrix} \bar{\mathbf{l}}(u) & \bar{\mathbf{l}}BY \\ \bar{\bar{\mathbf{l}}}(u) & \bar{\bar{\mathbf{l}}}BY \end{pmatrix} \neq 0.$$

Nondegeneracy

Lemma

The following non degeneracy conditions are satisfied.

- $$span(u, BY) \pitchfork (Ker(\bar{\mathbf{l}}) \cap Ker(\bar{\bar{\mathbf{l}}}))$$

  where $Y = (\mathbf{l}\bar{u})\bar{\bar{u}} - (\mathbf{l}\bar{\bar{u}})\bar{u} \in span(\bar{u}, \bar{\bar{u}}) \cap Ker\mathbf{l}$.

- $$\det \begin{pmatrix} \bar{\mathbf{l}}(u) & \bar{\mathbf{l}}BY \\ \bar{\bar{\mathbf{l}}}(u) & \bar{\bar{\mathbf{l}}}BY \end{pmatrix} \neq 0.$$

Cone family, Hyperbolicity

Definition

$U(\delta)$: a δ neighbhourhood of Gerver's collision point in the phase space.

Lemma

There are cone families $\mathcal{K}_1$ *on* $T_x(T^*\mathbb{T}^3),\ x \in U_1(\delta)$ *and* $\mathcal{K}_2$ *on* $T_x(T^*\mathbb{T}^3),\ x \in U_2(\delta)$*, each of which contains a two dimensional plane, such that*

- *Invariance:* $d\mathcal{P}(\mathcal{K}_1) \subset \mathcal{K}_2$, $d(\mathcal{R} \circ \mathcal{P})(\mathcal{K}_2) \subset \mathcal{K}_1$.
- *Expansion: If* $v \in \mathcal{K}_1$*, then* $||d\mathcal{P}(v)|| \geq c\chi||v||$.
  If $v \in \mathcal{K}_2$*, then* $||d(\mathcal{R} \circ \mathcal{P})(v)|| \geq c\chi||v||$.

Lemma

There are cone families $\mathcal{K}_1$ on $T_x(T^*\mathbb{T}^3),\ x \in U_1(\delta)$ and $\mathcal{K}_2$ on $T_x(T^*\mathbb{T}^3),\ x \in U_2(\delta)$, each of which contains a two dimensional plane, such that

- *Invariance:* $d\mathcal{P}(\mathcal{K}_1) \subset \mathcal{K}_2$, $d(\mathcal{R} \circ \mathcal{P})(\mathcal{K}_2) \subset \mathcal{K}_1$.
- *Expansion: If* $v \in \mathcal{K}_1$, *then* $||d\mathcal{P}(v)|| \geq c\chi||v||$.
  If $v \in \mathcal{K}_2$, *then* $||d(\mathcal{R} \circ \mathcal{P})(v)|| \geq c\chi||v||$.

Cones

Definition

We now take $\mathcal{K}$ to be the set of vectors which make an angle less than a small constant η with span$(\bar{u}, \bar{\bar{u}})$.

Admissible surface and Cantor set construction

- Definition
  We call a C^1 surface $S_1 \subset U_1(\delta)$ (respectively $S_2 \subset U_2(\delta)$) **admissible** if $TS_1 \subset \mathcal{K}_1$ (respectively $TS_2 \subset \mathcal{K}_2$).
  - The Cantor set:
  $$\lim_j (\mathcal{R}\mathcal{P}^2)^{-j} S_{2j}.$$

Admissible surface and Cantor set construction

- Definition
  We call a C^1 surface $S_1 \subset U_1(\delta)$ (respectively $S_2 \subset U_2(\delta)$) **admissible** if $TS_1 \subset \mathcal{K}_1$ (respectively $TS_2 \subset \mathcal{K}_2$).
  - The Cantor set:

$$\lim_j (\mathcal{R}\mathcal{P}^2)^{-j} S_{2j}.$$

Local map: Poincaré section

The Poincaré section

$$|Q_3 - Q_4| = \mu^\kappa, \quad 1/3 < \kappa < 1/2.$$

Local map, the C^0 estimate

Close to elastic collision.

- $$\begin{cases} v_3^+ = \frac{1}{2}R(\alpha)(v_3^- - v_4^-) + \frac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3}), \\ v_4^+ = -\frac{1}{2}R(\alpha)(v_3^- - v_4^-) + \frac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3}), \\ Q_3^+ = Q_3^- + O(\mu^\kappa), \\ Q_4^+ = Q_4^- + O(\mu^\kappa), \end{cases}$$

  where $R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$,

- $$\alpha = \arctan \frac{d|v_3^- - v_4^-|^2}{\mu}.$$

- $$d = (Q_3^- - Q_4^-) \times \frac{v_3^- - v_4^-}{|v_3^- - v_4^-|} : \quad \text{impact parameter.}$$

Local map, the C^0 estimate

Close to elastic collision.

- $$\begin{cases} v_3^+ = \frac{1}{2}R(\alpha)(v_3^- - v_4^-) + \frac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3}), \\ v_4^+ = -\frac{1}{2}R(\alpha)(v_3^- - v_4^-) + \frac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3}), \\ Q_3^+ = Q_3^- + O(\mu^\kappa), \\ Q_4^+ = Q_4^- + O(\mu^\kappa), \end{cases}$$

  where $R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$,

- $$\alpha = \arctan\frac{d|v_3^- - v_4^-|^2}{\mu}.$$

- $$d = (Q_3^- - Q_4^-) \times \frac{v_3^- - v_4^-}{|v_3^- - v_4^-|} : \quad \text{impact parameter.}$$

Local map, the C^0 estimate

Close to elastic collision.

- $$\begin{cases} v_3^+ = \frac{1}{2}R(\alpha)(v_3^- - v_4^-) + \frac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3}), \\ v_4^+ = -\frac{1}{2}R(\alpha)(v_3^- - v_4^-) + \frac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3}), \\ Q_3^+ = Q_3^- + O(\mu^\kappa), \\ Q_4^+ = Q_4^- + O(\mu^\kappa), \end{cases}$$

  where $R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$,

- $$\alpha = \arctan \frac{d|v_3^- - v_4^-|^2}{\mu}.$$

- $$d = (Q_3^- - Q_4^-) \times \frac{v_3^- - v_4^-}{|v_3^- - v_4^-|} : \quad \text{impact parameter.}$$

- Energy conservation & momentum conservation.
  - $d = O(\mu)$ if α is bounded away from 0 and π.
  - $\frac{\partial \alpha}{\partial d} = O(1/\mu)$.
- Lemma

  The C^1 calculation is the same as taking derivatives of the C^0 expression directly.
  - $\frac{\partial +}{\partial -} = \frac{c}{\mu}\frac{\partial +}{\partial \alpha} \otimes \frac{\partial d}{\partial -}$+(derivative involving no d)+ o(1).

- Energy conservation & momentum conservation.
- $d = O(\mu)$ if α is bounded away from 0 and π.
- $\dfrac{\partial\alpha}{\partial d} = O(1/\mu)$.

- Lemma

  The C^1 calculation is the same as taking derivatives of the C^0 expression directly.

  - $\dfrac{\partial +}{\partial -} = \dfrac{c}{\mu}\dfrac{\partial +}{\partial\alpha} \otimes \dfrac{\partial d}{\partial -}$ +(derivative involving no d)+ o(1).

- Energy conservation & momentum conservation.
- $d = O(\mu)$ if α is bounded away from 0 and π.
- $\frac{\partial \alpha}{\partial d} = O(1/\mu)$.

- Lemma

  The C^1 calculation is the same as taking derivatives of the C^0 expression directly.

  - $\frac{\partial +}{\partial -} = \frac{c}{\mu}\frac{\partial +}{\partial \alpha} \otimes \frac{\partial d}{\partial -} +$(derivative involving no d)+ o(1).

- Energy conservation & momentum conservation.
- $d = O(\mu)$ if α is bounded away from 0 and π.
- $\dfrac{\partial \alpha}{\partial d} = O(1/\mu)$.

- Lemma

  The C^1 calculation is the same as taking derivatives of the C^0 expression directly.

  - $\dfrac{\partial +}{\partial -} = \dfrac{c}{\mu} \dfrac{\partial +}{\partial \alpha} \otimes \dfrac{\partial d}{\partial -}$+(derivative involving no d)+ o(1).

- Energy conservation & momentum conservation.
- $d = O(\mu)$ if α is bounded away from 0 and π.
- $\frac{\partial \alpha}{\partial d} = O(1/\mu)$.

- Lemma

  The C^1 calculation is the same as taking derivatives of the C^0 expression directly.

  - $\frac{\partial +}{\partial -} = \frac{c}{\mu}\frac{\partial +}{\partial \alpha} \otimes \frac{\partial d}{\partial -}$+(derivative involving no d)+ o(1).

Coordinates: Delaunay coordinates

$$dP \wedge dQ = dL \wedge d\ell + dG \wedge dg.$$

The Hamiltonian

$$H = \frac{|P|^2}{2} - \frac{1}{|Q|},$$

$$\to H = -\frac{1}{2L^2}, \qquad \text{elliptic motion},$$

$$\to H = \frac{1}{2L^2}, \qquad \text{hyperbolic motion}.$$

The Hamiltonian in Delaunay coordinates

- The LEFT

$$H_L = -\frac{1}{2L_3^2} + \frac{1}{2L_4^2} - \frac{1}{|Q_4|} - \frac{1}{|Q_3 - (-\chi, 0)|} - \frac{\mu}{|Q_3 - Q_4|}.$$

- The RIGHT

$$H_R = -\frac{1}{2L_3^2} + \frac{(1+\mu)^2}{2L_4^2} - \frac{1}{|Q_3 + (\chi, 0)|} - \frac{1}{|Q_4 + (\chi, 0)|}$$

$$-\frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O\left(\frac{\mu}{|Q_4|^3}\right).$$

Coordinates for the Poincaré map

- Eliminate L_4 by fixing an energy level.
- Treat ℓ_4 as the new time.
- Coordinates for the Poincaré map:

$$(L_3, \ell_3, G_3, g_3, G_4, g_4).$$

Coordinates for the Poincaré map

- Eliminate L_4 by fixing an energy level.
- Treat ℓ_4 as the new time.
- Coordinates for the Poincaré map:

$$(L_3, \ell_3, G_3, g_3, G_4, g_4).$$

The first expanding direction of the Global map: hyperbolicity from parabolicity

The Hamiltonian for elliptic motion

$$H_3 = -\frac{1}{2L_3^2}.$$

The Hamiltonian equations

$$\begin{cases} \dot{L}_3 = 0, \\ \dot{\ell}_3 = \dfrac{1}{L_3^3}, \\ \dot{G}_3 = 0, \\ \dot{g}_3 = 0. \end{cases} \Longrightarrow \begin{cases} L_3(T) = L_3(0), \\ \ell_3(T) = \ell_3(0) + \dfrac{T}{L_3^3(0)}, \\ G_3(T) = G_3(0), \\ g_3(T) = g_3(0). \end{cases}$$

The derivative matrix

$$\frac{\partial(L,\ell,G,g)_3(T)}{\partial(L,\ell,G,g)_3(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\dfrac{3T}{L_3^4(0)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= -\frac{3T}{L_3^4(0)} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \otimes [1,0,0,0] + O(1).$$

We have estimate

$$-\frac{3T}{L_3^4(0)} = O(\chi).$$

Hyperbolicity created from parabolicity

- Parabolic matrix

$$\begin{bmatrix} 1 & 0 \\ \chi & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \chi & \chi + 1 \end{bmatrix}$$

- Two eigenvalues, $O(\chi)$ and $O(1/\chi)$.

The second expanding direction of the Global map: hyperbolicity near collision

- Define the angle of asymptotes

$$f = g \pm \arctan \frac{G}{L} = O(1/\chi), \quad v_4 \simeq (1, f).$$

- Coordinates changes from the Right to the Left

$$(G, g)_R \xrightarrow{(i)} (G, f)_R \xrightarrow{(ii)} (G, f)_L \xrightarrow{(iii)} (G, g)_L.$$

- The maps $(i), (ii), (iii)$:

$$(i) : G_R = G_R, \quad f_R = g_R - \arctan \frac{G_R}{L_R}.$$

$$(ii) : G_L = G_R + \chi f_R, \quad f_L = f_R.$$

$$(iii) : G_L = G_L, \quad g_L = f_L - \arctan \frac{G_L}{L_L}.$$

The second expanding direction of the Global map: hyperbolicity near collision

- Define the angle of asymptotes

$$f = g \pm \arctan \frac{G}{L} = O(1/\chi), \quad v_4 \simeq (1, f).$$

- Coordinates changes from the Right to the Left

$$(G, g)_R \xrightarrow{(i)} (G, f)_R \xrightarrow{(ii)} (G, f)_L \xrightarrow{(iii)} (G, g)_L.$$

- The maps $(i), (ii), (iii)$:

$$(i) : G_R = G_R, \quad f_R = g_R - \arctan \frac{G_R}{L_R}.$$

$$(ii) : G_L = G_R + \chi f_R, \quad f_L = f_R.$$

$$(iii) : G_L = G_L, \quad g_L = f_L - \arctan \frac{G_L}{L_L}.$$

The second expanding direction of the Global map: hyperbolicity near collision

- Define the angle of asymptotes

$$f = g \pm \arctan \frac{G}{L} = O(1/\chi), \quad v_4 \simeq (1, f).$$

- Coordinates changes from the Right to the Left

$$(G, g)_R \xrightarrow{(i)} (G, f)_R \xrightarrow{(ii)} (G, f)_L \xrightarrow{(iii)} (G, g)_L.$$

- The maps $(i), (ii), (iii)$:

$$(i) : G_R = G_R, \quad f_R = g_R - \arctan \frac{G_R}{L_R}.$$

$$(ii) : G_L = G_R + \chi f_R, \quad f_L = f_R.$$

$$(iii) : G_L = G_L, \quad g_L = f_L - \arctan \frac{G_L}{L_L}.$$

- The derivatives for $(II) = (iii)(ii)(i)$

$$D[(iii)(ii)(i)] = \begin{bmatrix} 1 & 0 \\ \sharp & 1 \end{bmatrix} \begin{bmatrix} 1 & \chi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sharp & 1 \end{bmatrix}.$$

- For matrix (IV) going from the Left to the Right, we get

$$D[(iii')(ii')(i')] = \begin{bmatrix} 1 & 0 \\ \sharp & 1 \end{bmatrix} \begin{bmatrix} 1 & -\chi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sharp & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ \sharp & 1 \end{bmatrix} \begin{bmatrix} 1 & -\chi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sharp & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \sharp & 1 \end{bmatrix} \begin{bmatrix} 1 & \chi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sharp & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \sharp & 1 \end{bmatrix} \begin{bmatrix} 1 + \sharp\chi & -\sharp\chi^2 \\ \sharp & -\sharp\chi + 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sharp & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 + \sharp\chi & -\sharp\chi^2 \\ \sharp & -\sharp\chi + 1 \end{bmatrix} = \sharp\chi^2 \begin{bmatrix} 1 \\ 1/\chi \end{bmatrix} \otimes [1/\chi, 1] + O(1).$$

Remaining issues

- Exclude collisions.
- Control the shape: Two phases ψ_1, ψ_2. We need

$$\det\left(\frac{\partial(g_3, e_3)}{\partial(\psi_1, \psi_2)}\right) \neq 0.$$

- Check nondegeneracy: Essentially

$$\det\begin{pmatrix} \bar{\mathbf{I}}(u) & \bar{\mathbf{I}}BY \\ \bar{\bar{\mathbf{I}}}(u) & \bar{\bar{\mathbf{I}}}BY \end{pmatrix} \neq 0 \Leftrightarrow \frac{\partial L_3^+}{\partial \psi} \neq 0.$$

Remaining issues

- Exclude collisions.
- Control the shape: Two phases ψ_1, ψ_2. We need

$$\det\left(\frac{\partial(g_3, e_3)}{\partial(\psi_1, \psi_2)}\right) \neq 0.$$

- Check nondegeneracy: Essentially

$$\det\begin{pmatrix} \bar{\mathbf{I}}(u) & \bar{\mathbf{I}}BY \\ \bar{\bar{\mathbf{I}}}(u) & \bar{\bar{\mathbf{I}}}BY \end{pmatrix} \neq 0 \Leftrightarrow \frac{\partial L_3^+}{\partial \psi} \neq 0.$$

Remaining issues

- Exclude collisions.
- Control the shape: Two phases ψ_1, ψ_2. We need

$$\det\left(\frac{\partial(g_3, e_3)}{\partial(\psi_1, \psi_2)}\right) \neq 0.$$

- Check nondegeneracy: Essentially

$$\det\begin{pmatrix} \bar{\mathbf{I}}(u) & \bar{\mathbf{I}}BY \\ \bar{\bar{\mathbf{I}}}(u) & \bar{\bar{\mathbf{I}}}BY \end{pmatrix} \neq 0 \Leftrightarrow \frac{\partial L_3^+}{\partial \psi} \neq 0.$$

Planar four-body problem, in progress.

Planar 4 body problem,

- 8 degrees of freedom = 16 dimensional phase space.
- Remove the translation invariance $16 - 4 = 12$.
- Remove the rotation invariance $12 - 2 = 10$.
- Pick an energy level and take a Poincaré section, $10 - 2 = 8$ dimensional Poincaré map.
- We expect that similarly to the problem at hand the Poincaré map have only two strongly expanding directions dominating all other directions.

Planar four-body problem, in progress.

Planar 4 body problem,

- 8 degrees of freedom $=$ 16 dimensional phase space.
- Remove the translation invariance $16 - 4 = 12$.
- Remove the rotation invariance $12 - 2 = 10$.
- Pick an energy level and take a Poincaré section, $10 - 2 = 8$ dimensional Poincaré map.
- We expect that similarly to the problem at hand the Poincaré map have only two strongly expanding directions dominating all other directions.

Planar four-body problem, in progress.

Planar 4 body problem,

- 8 degrees of freedom $=$ 16 dimensional phase space.
- Remove the translation invariance $16 - 4 = 12$.
- Remove the rotation invariance $12 - 2 = 10$.
- Pick an energy level and take a Poincaré section, $10 - 2 = 8$ dimensional Poincaré map.
- We expect that similarly to the problem at hand the Poincaré map have only two strongly expanding directions dominating all other directions.

Planar four-body problem, in progress.

Planar 4 body problem,

- 8 degrees of freedom $=$ 16 dimensional phase space.
- Remove the translation invariance $16 - 4 = 12$.
- Remove the rotation invariance $12 - 2 = 10$.
- Pick an energy level and take a Poincaré section, $10 - 2 = 8$ dimensional Poincaré map.
- We expect that similarly to the problem at hand the Poincaré map have only two strongly expanding directions dominating all other directions.

Planar four-body problem, in progress.

Planar 4 body problem,

- 8 degrees of freedom $=$ 16 dimensional phase space.
- Remove the translation invariance $16 - 4 = 12$.
- Remove the rotation invariance $12 - 2 = 10$.
- Pick an energy level and take a Poincaré section, $10 - 2 = 8$ dimensional Poincaré map.
- We expect that similarly to the problem at hand the Poincaré map have only two strongly expanding directions dominating all other directions.

THANK YOU!