Oscillatory motions for the restricted planar circular three body problem

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The circular restricted planar three body problem (RCP3BP)

- Three bodies of masses $1-\mu$, μ and 0 under the effects of the Newtonian gravitational force.
- The bodies with mass (primaries) are not influenced by the massless one.
- They form a two body problem.
- Assume they move on circles
- Goal: understand the motion of a massless body under the influence of the other two.

The equations of the RCP3BP

- The motion of the massless body q is described by

$$\frac{d^2q}{dt^2} = \frac{(1-\mu)(q_1(t)-q)}{|q_1(t)-q|^3} + \frac{\mu(q_2(t)-q)}{|q_2(t)-q|^3},$$

where $q_1(t) = -\mu q_0(t)$, $q_2(t) = (1-\mu)q_0(t)$ and

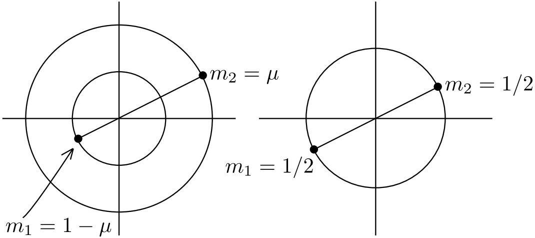
$$q_0(t) = (\cos t, \sin t)$$

correspond to the circular motion of the primaries.

- This is a 2π-periodic in time Hamiltonian system (2 and 1/2 degrees of freedom) with Hamiltonian

$$\mathcal{H}(q,p,t;\mu) = \frac{p^2}{2} - \frac{(1-\mu)}{|q-q_1(t)|} - \frac{\mu}{|q-q_2(t)|}.$$

- The parameter $\mu \in [0, 1/2]$ is not necessarily small.



- Observation When $\mu = 1/2$, the two bodies move in the same circle at diametrally oposed points.
- The Hamiltonian is π-periodic in time.

Types of asymptotic motion in the RCP3BP

- Chazy (1922) gave a classification of all possible states that a three body problem can approach as time tends to infinity.
- For the restricted three body problem the possible final states are reduced to four:
  - $H^{\pm}$ (hyperbolic): $\|q(t)\| \to \infty$ and $\|\dot{q}(t)\| \to c > 0$ as $t \to \pm\infty$.
  - $P^{\pm}$ (parabolic): $\|q(t)\| \to \infty$ and $\|\dot{q}(t)\| \to 0$ as $t \to \pm\infty$.
  - $B^{\pm}$ (bounded): $\limsup_{t\to\pm\infty} \|q\| < +\infty$.
  - $OS^{\pm}$ (oscillatory): $\limsup_{t\to\pm\infty} \|q\| = +\infty$ and $\liminf_{t\to\pm\infty} \|q\| < +\infty$.
- Examples of all types of motion except oscillatory were already known by Chazy.

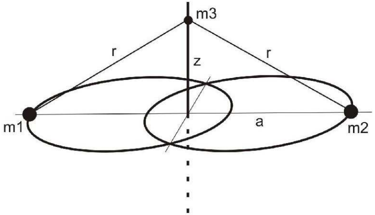
Limiting case $\mu \to 0$

- The massless body is only influenced by one body.
- Its motion is governed by Kepler laws.
- It moves on conic sections.
- Then,
  - $H^{\pm}$ (hyperbolic): motion on hyperbolas.
  - $P^{\pm}$ (parabolic): motion on parabolas.
  - $B^{\pm}$ (bounded): motion on ellipses.
- Oscillatory motions cannot exist.

Existence of oscillatory motions

Oscillatory motions were first proved by

- Sitnikov (1960) considered the restricted spatial elliptic three body problem.
- More concretely,
  - The primaries have mass $\mu = 1/2$ and move on ellipses of small enough eccentricity.
  - The massless moves on the (invariant) vertical axis.
- For these parameters, he proved existence of oscillatory motions.
- Moser (1973) gave a new proof of Sitnikov results.



Oscillatory motions for the RPC3BP

- First results by Llibre and Simó, 1980.
- They follow Moser's approach.

Theorem (Llibre-Simó)

Fix μ small enough. Then, there exists an orbit $(q(t), p(t))$ of RCP3BP which is oscillatory. Namely, it satisfies

$$\limsup_{t\to\pm\infty} \|q\| = +\infty \quad \textit{and} \quad \liminf_{t\to\pm\infty} \|q\| < +\infty.$$

Llibre-Simó results

- To state more precisely their results, we need to introduce the Jacobi constant
- The RPC3BP has a first integral called Jacobi constant

$$\mathcal{J}(q,p,t;\mu) = \mathcal{H}(q,p,t;\mu) - (q_1 p_2 - q_2 p_1).$$

- They obtain oscillatory motions in each $\mathcal{J}(q,p,t;\mu) = J_0$ for large enough J_0 and μ exponentially small with respect to J_0:

$$\mu \ll e^{-\frac{J_0^3}{3}}$$

Other results in oscillatory motions

1. Xia (1992), following Llibre-Simó shows that for RPC3BP there are oscillatory motions for every $\mu \in (0, 1/2]$ except a finite number of values.
2. J. Galante and V. Kaloshin (2011) use Aubry-Mather theory to prove the existence of orbits which initially are in the range of our Solar System and become oscillatory as time tends to infinity for the RPC3BP with $\mu = 10^{-3}$ (realistic for the Jupiter-Sun).
3. Other results proving existence of oscillatory motions by Alexeev and Llibre-Simó.

Abundance of the different types of motions

- All possible combinations $X^- \cap Y^+$ for $X, Y = H, P, B, OS$ exist.
- How abundant is each type of motion in the measure sense?
- It is known for each of them whether they have positive or zero measure except for $OS^- \cap OS^+$.
- Conjecture (Kolmogorov, Alexeev): Lebesgue measure of $OS^- \cap OS^+$ is zero.
- Kaloshin and Gorodetski (2011): study the Hausdorff dimension of oscillatory motions for both the Sitnikov problem and the RPC3BP.
- For the RPC3BP:
  - Fix J_0 large enough. For a a Baire generic set in an open set of mass ratio μ, oscillatory motions have maximal Hausdorff dimension in $\mathcal{J}(q, p, t; \mu) = J_0$.
  - Fix $\mu \in (0, 1/2]$. For a a Baire generic set in an open set of Jacobi constants J_0, the oscillatory motions have maximal Hausdorff dimension in $\mathcal{J}(q, p, t; \mu) = J_0$.

Oscillatory motions in the RCP3BP

- Our goal: generalize Llibre-Simó and Xia results to any value $\mu \in (0, 1/2]$.
- Recall that for $\mu = 0$ they cannot exist.

Theorem

Fix any $\mu \in (0, 1/2]$. Then, there exists an orbit $(q(t), p(t))$ of RCP3BP which is oscillatory. Namely, it satisfies

$$\limsup_{t\to\pm\infty} \|q\| = +\infty \quad \textit{and} \quad \liminf_{t\to\pm\infty} \|q\| < +\infty.$$

More precisely,

Theorem

Fix any $\mu \in (0, 1/2]$. *Then, there exists* $J_0 > 0$ *big enough, such that for any* $J > J_0$ *there exists an orbit* $(q_J(t), p_J(t))$ *of RCP3BP in the hypersurface* $\mathcal{J}(q, p, t; \mu) = J$ *which is oscillatory. Namely, it satisfies*

$$\limsup_{t\to\pm\infty} \|q_J\| = +\infty \quad \textit{and} \quad \liminf_{t\to\pm\infty} \|q_J\| < +\infty.$$

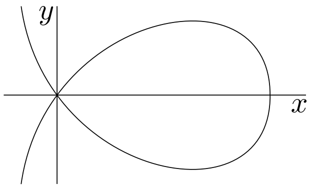
- These orbits satisfy $\liminf_{t\to\pm\infty} \|q_J\| \sim J^2$.
- They are far from the primaries (far from collision).

Oscillatory motions in the RCP3BP: Moser and Llibre-Simó approach

- Moser approach:
  - McGehee coordinates send infinity to zero. Then, infinity becomes a parabolic critical point.
  - Consider the invariant manifolds of infinity
  - Prove that they intersect transversally.
  - Establish symbolic dynamics close to these invariant manifolds.
  - It leads to the existence of oscillatory motions.
- Main difficulty in applying the approach to RPC3BP: prove the transversality of the invariant manifold of infinity.

Transversality of the invariant manifolds of infinity in Llibre-Simó

- For $\mu = 0$ the invariant manifolds coincide (parabolic orbits).
- In McGehee coordinates we have a homoclinic to a critical point.
- For $0 < \mu \ll 1$, expand in μ and compute the first order of the difference between the manifolds (Melnikov Theory).
- We know only how to compute it provided $J_0 \gg 1$.



- Llibre-Simó are only able to prove the transversality provided $J_0 \gg 1$ and $\mu \leq e^{-J_0^3/3}$.
- We prove the transversality for any $\mu \in (0, 1/2]$ and $J_0 \gg 1$.

RPC3BP in rotating polar coordinates

- Fix the primaries at the x axis.
- Polar coordinates for the third body: $q = (r\cos\phi, r\sin\phi)$.
  - y simplectic conjugate to r (radial velocity).
  - G symplectic conjugate to ϕ (angular momentum).
- Hamiltonian:

$$H(r,\phi,y,G;\mu) = \frac{y^2}{2} + \frac{G^2}{2r^2} - U(r,\phi;\mu),$$

- $U(r,\phi;\mu)$ is the Newtonian potential, which satisfies $U(r,\phi;0) = \frac{1}{r}$.
- The system has two degrees of freedom.
- Conservation of energy corresponds to conservation of the Jacobi constant.

Infinity

- Equations

$$\begin{aligned} \dot{r} &= y \\ \dot{y} &= \frac{G^2}{r^3} + \partial_r U(r,\phi;\mu) \\ \dot{\phi} &= -1 + \frac{G}{r^2} \\ \dot{G} &= \partial_\phi U(r,\phi;\mu) \end{aligned}$$

- Recall $U(r,\phi;0) \sim \frac{1}{r}$
- For any value of G_0, the "infinity":

$$(r, y, \phi, G) = (\infty, 0, \phi_0 - t, G_0), t \in \mathbb{T}$$

  is a periodic solution.
- At infinity, energy coincides with angular momentum: $H = -G_0$.

The two body problem: $\mu \to 0$

- When $\mu = 0$, the massless body is only influenced by one primary, located at the origin
- Hamiltonian (in rotating coordinates)

$$H(r, \phi, y, G, s; 0) = \frac{y^2}{2} + \frac{G^2}{2r^2} - G - \frac{1}{r},$$

- H and G are first integrals.
- Fixing $G = G_0$, the variables (r, y) form a Hamiltonian system of one degree of freedom

$$H_0(r, y; G_0) = \frac{y^2}{2} + \frac{G_0^2}{2r^2} - \frac{1}{r}.$$

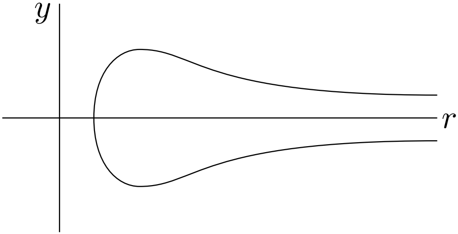
- The invariant manifolds of infinity coincide forming a separatrix.
- We show that the separatrix splits when we add the perturbation.

Reduction to a Poincaré map

- Restrict to $H = -G_0$ with $G_0 \gg 1$ and consider a section $\phi = \phi_0$.
- Area preserving Poincaré map

$$\begin{array}{rccc} \mathcal{P}_{\phi_0} & : & \{\phi = \phi_0\} & \longrightarrow \{\phi = \phi_0\} \\ & & (r, y) & \mapsto \ \mathcal{P}_{\phi_0}(r, y) \end{array}$$

- $(r, y) = (\infty, 0)$ is a fixed point with 1 dim. invariant manifolds.
- $\mu = 0$: they coincide forming a separatrix.



- $\mu > 0$: we measure their distance in a section transversal to the unperturbed separatrix.

The difference between the manifolds

Theorem

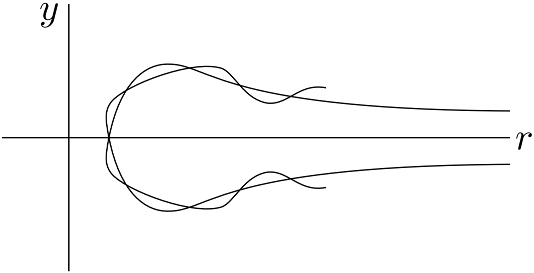
Consider the invariant manifolds of infinity of the Poincaré map $\mathcal{P}_{\phi_0}$. *Then, there exists* $G_0^* > 0$ *such that for any* $G_0 > G_0^*$ *and* $\mu \in (0, 1/2]$, *in a suitable section the distance* d *between these curves along this section is given by*

$$d = C\mu(1-\mu)\sqrt{\pi}\left[\frac{1-2\mu}{2\sqrt{2}} G_0^{3/2} e^{-\frac{G_0^3}{3}} \sin(f(\phi_0)) + 8 G_0^{7/2} e^{-\frac{2G_0^3}{3}} \sin(2f(\phi_0)) + \mathcal{O}\left((1-2\mu) G_0 e^{-\frac{G_0^3}{3}} + G_0^3 e^{-\frac{2G_0^3}{3}}\right)\right],$$

where $C > 0$ *and* $f(\phi)$ *are an explicit constant and an explicit function.*

Theorem

Fix $\mu \in (0, 1/2]$. Then, there exists $G^* > 0$ such that for any $G_0 > G^*$, the invariant manifolds of infinity of $\mathcal{P}_{\phi_0}$ intersect transversally.



This result allow us to proof the existence of oscillatory motions.

Theorem

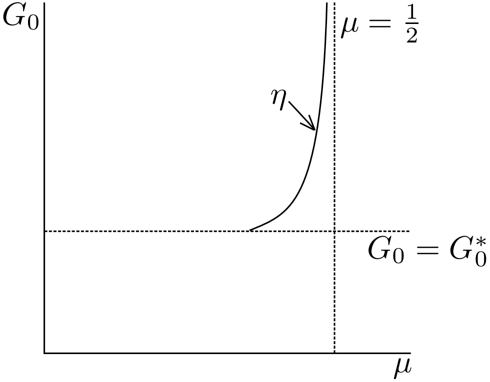
There exist G_0^* and a curve η in the parameter region

$$(\mu, G_0) \in \left(0, \frac{1}{2}\right] \times (G_0^*, +\infty),$$

of the form

$$\mu = \mu^*(G_0) = \frac{1}{2} - 16\sqrt{2}G_0^2 e^{-\frac{G_0^3}{3}} \left(1 + O\left(G_0^{-1/2}\right)\right),$$

such that, for $(\mu, G_0) \in \eta$, the invariant manifolds of infinity of $\mathcal{P}_{\phi_0}$ have a cubic homoclinic tangency and a transversal homoclinic point.



Bifurcation curve η in the parameter space where the homoclinic tangency is undergone.