Stability of elliptic Lagrangian solutions of the classical three body problem via index theory

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Contents

1. A brief introduction on the Lagrangian solution and studies on its linear stability;
2. New results and main ideas in the proof.
3. Open problems

Based on the recent joint work of:

Xijun Hu, Yiming Long and Shanzhong Sun:

Linear stability of elliptic Lagrangian solutions of the classical planar three-body problem via index theory. arXiv: 1206:6162v1, 2012

We consider the classical planar three-body problem in celestial mechanics. Denote by $q_1, q_2, q_3 \in \mathbf{R}^2$ the position vectors of three particles with masses $m = (m_1, m_2, m_3) \in (\mathbf{R}^+)^3$ respectively. By Newton's second law and the law of universal gravitation, the system of equations for this problem is

$$m_i \ddot{q}_i = \frac{\partial U(q)}{\partial q_i}, \qquad \text{for} \quad i = 1, 2, 3, \tag{1}$$

where

$$U(q) = U(q_1, q_2, q_3) = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|q_i - q_j|}$$

is the potential function by using the standard norm $|\cdot|$ of vector in $\mathbf{R}^2$.

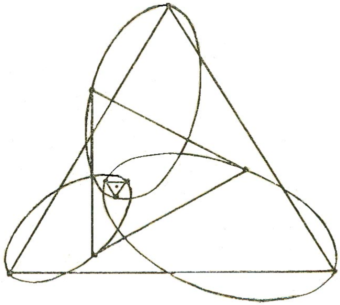
For periodic solutions with period $\tau > 0$, the system is the Euler-Lagrange equation of the action functional

$$\mathcal{A}_\tau(q) = \int_0^\tau \left[\sum_{i=1}^{3} \frac{m_i |\dot{q}_i(t)|^2}{2} + U(q(t)) \right] dt$$

defined on the loop space $W^{1,2}(\mathbf{R}/\tau\mathbf{Z}, X)$, where

$$X \equiv \left\{ q = (q_1, q_2, q_3) \in (\mathbf{R}^2)^3 \;\middle|\; \sum_{i=1}^{3} m_i q_i = 0,\ q_i \neq q_j,\ \forall i \neq j \right\}$$

is the configuration space of the planar three-body problem. Each τ-periodic solution to (1) appears to be a critical point of the action functional $\mathcal{A}_\tau$.



In 1772, J. Lagrange discovered his τ-periodic elliptic solutions of the 3-BP (ELS for short): $q(t) = r(t)R(\theta(t))q(0)$, with $q(0) \in (\mathbf{R}^2)^3$, $r(t) > 0$, and $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $\theta \in \mathbf{R}$.

Here, $q(0)$ and consequently $q(t)$ always form an equilateral triangle (central configuration), and $(r(t)\cos\theta(t), r(t)\sin\theta(t))$ in $\mathbf{R}^2$ describes elliptic curves depending on the period, masses, and eccentricity, which are solutions of the two body Kepler problem, if $q(0)$ is not collinear. Denote these τ-periodic ELS by $q_{m,e}(t)$.

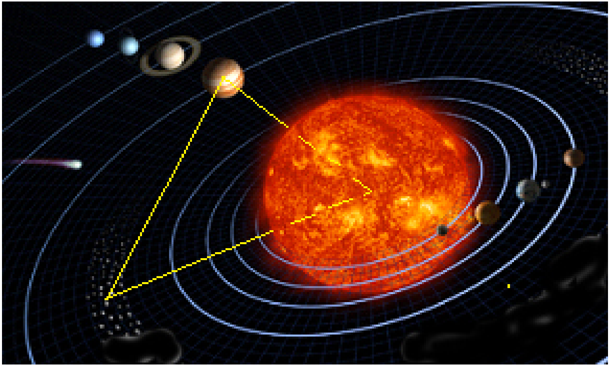


Figure: Sun, Jupiter and Trojan stars

We write the 3-BP system (1) into a Hamiltonian system:

$$\dot{z} = JH'(z), \quad z(\tau) = z(0). \tag{2}$$

with $z = (p, q) = (p_1, p_2, p_3, q_1, q_2, q_3) \in (\mathbf{R}^2)^6$, $p(t) = \bar{M}\dot{q}(t)$, and

$$H(z) = H(p, q) = \sum_{i=1}^{3} \frac{|p_i|^2}{2m_i} - U(q), \quad J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix},$$

with $\bar{M} = \mathrm{diag}(m_1, m_1, m_2, m_2, m_3, m_3)$. The linearized Hamiltonian system at $z_{m,e}(t) = (\bar{M}\dot{q}_{m,e}(t), q_{m,e}(t)) \in (\mathbf{R}^2)^6$ is given by

$$\dot{y}(t) = JH''(z_{m,e}(t))y(t), \quad y(\tau) = y(0), \tag{3}$$

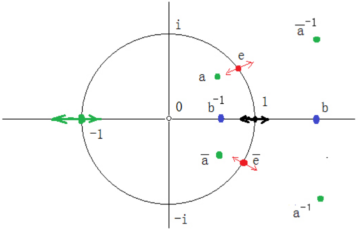
whose fundamental solution $\psi = \psi_{m,e}(t)$ satisfies $\psi(0) = I_{12}$ and $\psi_{m,e}(t) \in \mathrm{Sp}(12) = \{M \in \mathrm{GL}(\mathbf{R}^{12}) \,|\, M^T JM = J\}$ for all $t \in [0, \tau]$.

Our main concern is the linear stability of these ELS, which is determined by $\psi_{m,e}(\tau)$ and its eigenvalues. Let $\mathbf{U} = \{z \in \mathbf{C} \mid |z| = 1\}$.

Definition:

$$\begin{aligned}
&M \in \mathrm{Sp}(2n) \text{ is spectrally stable}, \quad \text{if } \ \sigma(M) \subset \mathbf{U}, \\
&M \in \mathrm{Sp}(2n) \text{ is linearly stable}, \\
&\qquad\qquad \text{if } \ \sigma(M) \subset \mathbf{U} \ \text{ and } M \text{ is semi} - \text{simple}, \\
&\qquad\qquad \text{if and only if } \ \sup_{m \geq 1} \|M^m\| < +\infty. \\
&M \in \mathrm{Sp}(2n) \text{ is strongly linearly stable}, \\
&\qquad\qquad \text{if } \exists \epsilon > 0 \ \text{ such that } N \text{ is linearly stable} \\
&\qquad\qquad\quad \text{whenever } \|M - N\| < \epsilon.
\end{aligned}$$

M is *semi-simple*, if its minimal polynomial is the product of relatively prime irreducible polynomials.



Let $M \in \mathrm{Sp}(2n)$. Then possible eigenvalue distributions of M are:

$$1 \quad \text{is of even multiplicities;} \quad -1 \quad \text{is of even multiplicities;}$$

$$e, \ \bar{e} \quad \in \quad \mathbf{U} \setminus \mathbf{R}; \qquad b, \ b^{-1} \quad \in \quad \mathbf{R} \setminus \{0, \pm 1\};$$

$$a, \ a^{-1}, \ \bar{a}, \ \bar{a}^{-1} \quad \in \quad \mathbf{C} \setminus (\mathbf{U} \cup \mathbf{R}).$$

Thus there are 3 possible ways for eigenvalues to escape from $\mathbf{U}$ as shown in the Figure.

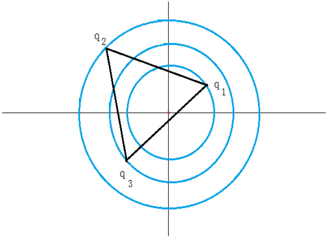


Figure: Circular solution of the 3-body problem with $e = 0$

Earlier studies on the linear stability:
M.Gascheau (1843) and E.Routh (1875) for circular orbits, i.e., $e = 0$.
J.Danby (1964), G.Roberts (2003): for $e \geq 0$ sufficiently small, by perturbation method.

Consider the linearized Hamiltonian system at $z_{m,e}(t)$:

$$\dot{y}(t) = JH''(z_{m,e}(t))y(t), \quad y(\tau) = y(0),$$

with fundamental solution $\psi_{m,e}(t) \in \mathrm{Sp}(12)$ and $\psi_{m,e}(0) = I_{12}$.

First integrals of (1):

(i) (Integral of the center of masses)
$\sum_{i=1}^{n} m_i q_i(t) = V_1 t + V_2 = 0$. (2-dim.)

(ii) (Integral of the linear momentum) $\sum_{i=1}^{3} m_i \dot{q}_i(t) = V_1 = 0$. (2-dim.)

(iii) (Integral of the energy) $\frac{1}{2}\sum_{i=1}^{n} m_i |\dot{q}_i(t)|^2 - U(q(t)) = h$.
(periodic solution) $\ddot{z}(t) = JH''(z(t))\dot{z}(t)$. (in total 2-dim.)

(iv) (Integral of the angular momentum) $\sum_{i=1}^{n} m_i q_i \times \dot{q}_i(t) = 0$. (2-dim.)

Thus $1 \in \sigma(\psi_{m,e}(\tau))$ has always algebraic multiplicity at least 8 in total.

K.Meyer and D.Schmidt (2005): Using the central configuration coordinates, they decomposed the linearized Hamiltonian system at ELS into two parts symplectically:

$$\psi_{m,e}(\tau) = P^{-1}\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \diamond \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \diamond \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \diamond I_2 \diamond M\right] P.$$

(i) the 8 eigenvalue 1 stays always for all $(m, e) \in (\mathbf{R}^+)^3 \times [0, 1)$;
(ii) the other part corresponding to M is the 4-dim. essential part for the linear stability, which can be transformed to a linear system with coefficient matrix:

$$\bar{B}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & \frac{2e\cos\theta - 1 - \sqrt{9-\beta}}{2(1+e\cos\theta)} & 0 \\ 1 & 0 & 0 & \frac{2e\cos\theta - 1 + \sqrt{9-\beta}}{2(1+e\cos\theta)} \end{pmatrix},$$

where $t \in [0, \tau]$ is transformed to the true anomaly $\theta \in [0, 2\pi]$. They studied also the linear stability for $e \geq 0$ small enough.

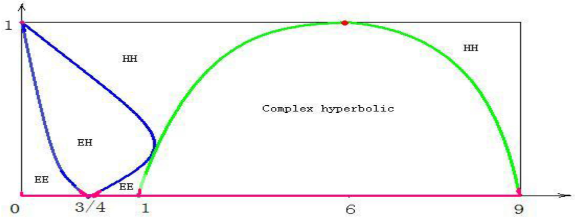
Rewrite Meyer and Schmidt's essential part (4-dim. linearized Hamiltonian system) for ELS as (use t to replace θ):

$$\dot{y}(t) = JB_{\beta,e}(t)y(t), \quad y(2\pi) = y(0), \tag{4}$$
$$B_{\beta,e}(t) = \begin{pmatrix} I_2 & -J \\ J & I_2 - K_{\beta,e}(t) \end{pmatrix},$$

with $K_{\beta,e}(t) = \begin{pmatrix} \frac{3-\sqrt{9-\beta}}{2(1+e\cos t)} & 0 \\ 0 & \frac{3+\sqrt{9-\beta}}{2(1+e\cos t)} \end{pmatrix}$, and the mass parameter β and the eccentricity e satisfy

$$\beta = \frac{27(m_1 m_2 + m_1 m_3 + m_2 m_3)}{(m_1 + m_2 + m_3)^2} \in [0, 9], \quad e \in [0, 1).$$

Denote the fundamental solution of this system by $\gamma_{\beta,e}(t) \in \mathrm{Sp}(4)$, which satisfies $\gamma_{\beta,e}(0) = I_4$. The linear stability of $z_{\beta,e} \equiv z_{m,e}(t)$ is determined by $\gamma_{\beta,e}(2\pi) \in \mathrm{Sp}(4)$.



R.Martínez, A.Samà and C.Simó (2004-2006) Perturbation method for $e \geq 0$ small enough + numerical method:

EE: $\sigma(\gamma_{\beta,e}(2\pi)) = \{\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2\}$ with $\omega_i \in \mathbf{U} \setminus \mathbf{R}$ for $i = 1, 2$;

EH: $\sigma(\gamma_{\beta,e}(2\pi)) = \{\lambda, \lambda^{-1}, \omega, \overline{\omega}\}$ for some $-1 \neq \lambda < 0$ and $\omega \in \mathbf{U} \setminus \mathbf{R}$;

HH: $\sigma(\gamma_{\beta,e}(2\pi)) = \{\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}\}$ for some $\lambda_i \in \mathbf{R} \setminus \{0, \pm 1\}$ with $i = 1, 2$;

Complex hyperbolic: $\sigma(\gamma_{\beta,e}(2\pi)) \subset \mathbf{C} \setminus (\mathbf{U} \cup \mathbf{R})$.

We are not aware of any rigorous mathematical method on this linear stability problem which works for the full range of parameters $(\beta, e) \in [0, 9] \times [0, 1)$ before 2010 !

Difficulty: due to the substantial dependence of the coefficients on t when $0 < e < 1$:

$$
\begin{aligned}
\dot{y}(t) &= J \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & \frac{2e\cos(t)-1-\sqrt{9-\beta}}{2(1+e\cos(t))} & 0 \\ 1 & 0 & 0 & \frac{2e\cos(t)-1+\sqrt{9-\beta}}{2(1+e\cos(t))} \end{pmatrix} y(t), \\
y(2\pi) &= y(0).
\end{aligned}
$$

Preparations for further results:

W.Gordon (1977) *The Kepler elliptic orbit $q = q(t)$ is the solution of the equation*

$$\ddot{q}(t) = -\frac{q(t)}{|q(t)|^3}.$$

The functional

$$f(q) = \int_0^\tau \left(\frac{1}{2}|\dot{q}(t)|^2 + \frac{1}{|q(t)|}\right) dt$$

attains its minimum in $W^{1,2}(\mathbf{R}/(\tau\mathbf{Z}), \mathbf{R}^2 \setminus \{0\})$ on Kepler elliptic orbits.

A.Venturelli (2001), S.Zhang-Q.Zhou (2001) *ELS is a global minimizer of the action $\mathcal{A}(q)$ on the loops in the non-trivial homology class of $W^{1,2}(\mathbf{R}/\tau\mathbf{Z}, X)$. Specially its Morse index satisfies*

$$i_1(\mathrm{ELS}) = 0.$$

For any $M, N \in \mathrm{Sp}(2n)$, we write $M \approx N$ if $\exists P \in \mathrm{Sp}(2n)$ s.t. $M = P^{-1}NP$ holds.

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

$$N_1(\lambda, a) = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \quad N_2(e^{\sqrt{-1}\theta}, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix},$$

$$N_2(-1, c) = \begin{pmatrix} -1 & 1 & c_1 & 0 \\ 0 & -1 & c_2 & c_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix},$$

where $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, and $\lambda, a, \theta, b_i, c_i \in \mathbf{R}$.

Theorem. (X.Hu and S.Sun, 2010, Advances in Math.)

(I) $2 \leq i_1(z^2_{\beta,e}) \leq 4$ *holds always;*

Suppose $\gamma_{\beta,e}(2\pi)^2$ *is non-degenerate, i.e.,* $1 \notin \sigma(\gamma_{\beta,e}(2\pi)^2)$. *Then*

(II-1) If $i_1(z^2_{\beta,e}) = 4$, *then* $\gamma_{\beta,e}(2\pi) \approx R(\theta_1)\diamond R(\theta_2)$ *holds for some* θ_1 *and* $\theta_2 \in (\pi, 2\pi)$, *and ELS is linearly stable;*

(II-2) If $i_1(z^2_{\beta,e}) = 3$, *then* $\gamma_{\beta,e}(2\pi) \approx D(\lambda)\diamond R(\theta)$ *for some* $-1 \neq \lambda < 0$ *and* $\theta \in (\pi, 2\pi)$, *and ELS is linearly unstable;*

(II-3) If $i_1(z^2_{\beta,e}) = 2$ *and* $\exists k \geq 3$ *such that* $i_1(z^k_{\beta,e}) > 2(k-1)$, *then* $\gamma_{\beta,e}(2\pi) \approx R(2\pi - \theta_1)\diamond R(\theta_2)$ *holds with* $0 < \theta_1 < \theta_2 < \pi$, *and ELS is linearly stable;*

(II-4) If $i_1(z^k_{\beta,e}) = 2(k-1)$ *for all* $k \in \mathbf{N}$, *then* $\gamma_{\beta,e}(2\pi)$ *and ELS are hyperbolic or spectrally stable and linearly unstable.*

As usual, $z^k_{\beta,e}(t) = z_{\beta,e}(kt)$ is used for all $k \in \mathbf{N}$.

Advantage of Hu-Sun's Theorem:

$\langle 1 \rangle$ The first method which works for the full range of parameters $(\beta, e) \in [0, 9] \times [0, 1)$;
$\langle 2 \rangle$ Based on different iterated Morse indices of regions, some regions of linear stability are given (not all).

Further understanding needed after Hu-Sun's Theorem:

$\langle 1 \rangle$ The non-degeneracy assumption (on $\gamma_{\beta,e}(2\pi)^2$) needs to be understood.
$\langle 2 \rangle$ The classification is based on the values of iterated Morse indices, but is not directly related to the two parameters;
$\langle 3 \rangle$ No information is given on properties of the shape of the curves which separate the linear stability regions and their behaviors in the rectangle $(\beta, e) \in [0, 9] \times [0, 1)$.
$\langle 4 \rangle$ The (II-4) case is not clear.

A brief introduction on ω-index theory of symplectic matrix paths starting from the identity matrix I

Let $M \in \mathrm{Sp}(2)$, Then we have:

$$M = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \leftrightarrow (r, \theta, z) \in \mathbf{R}^3 \backslash \{z\text{−axis}\}.$$

$$1 \in \sigma(M) \Leftrightarrow \det(M - I) = 0 \Leftrightarrow (r^2 + z^2 + 1)\cos\theta = 2r.$$

$$\begin{aligned} \mathrm{Sp}(2)_1^0 &= \{M \in \mathrm{Sp}(2) \mid 1 \in \sigma(M)\} \\ &= \{(r, \theta, z) \in \mathbf{R}^3 \setminus \{z - \text{axis}\} \mid (r^2 + z^2 + 1)\cos\theta = 2r\}. \end{aligned}$$

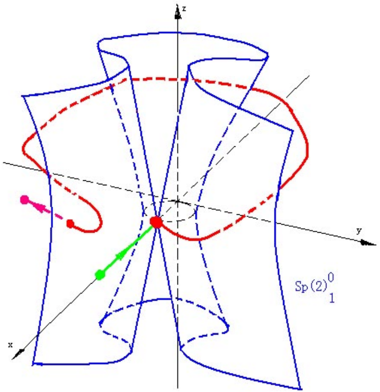


Figure: Graph of $\mathrm{Sp}(2)_1^0$

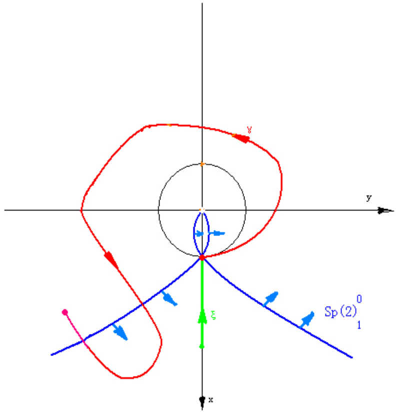


Figure: Graph of $\mathrm{Sp}(2)^0_1$ when $z = 0$

For $\gamma \in C([0,\tau], \mathrm{Sp}(2n))$ with $\gamma(0) = I$, we define

$$\begin{aligned}
\nu_1(\gamma) &= \dim \ker(\gamma(\tau) - I), \\
i_1(\gamma) &= [\gamma * \xi : \mathrm{Sp}(2n)_1^0], \qquad \text{if } \ \nu_1(\gamma) = 0, \\
i_1(\gamma) &= \min\{i_1(\phi) \mid \nu_1(\phi) = 0 \text{ and } \phi \text{ is suff. close to } \gamma\}, \\
&\qquad\quad \text{if } \nu_1(\gamma) > 0.
\end{aligned}$$

Similarly, for every $\omega \in \mathbf{U}$ we define

$$\begin{aligned}
\nu_\omega(\gamma) &= \dim_{\mathbf{C}} \ker_{\mathbf{C}}(\gamma(\tau) - \omega I), \\
i_\omega(\gamma) &= [\gamma * \xi : \mathrm{Sp}(2n)_\omega^0], \qquad \text{if } \ \nu_\omega(\gamma) = 0, \\
i_\omega(\gamma) &= \min\{i_1(\phi) \mid \nu_\omega(\phi) = 0 \text{ and } \phi \text{ is suff. close to } \gamma\}, \\
&\qquad\quad \text{if } \nu_\omega(\gamma) > 0.
\end{aligned}$$

$$(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \ldots, 2n\}, \qquad \forall\, \omega \in \mathbf{U}.$$

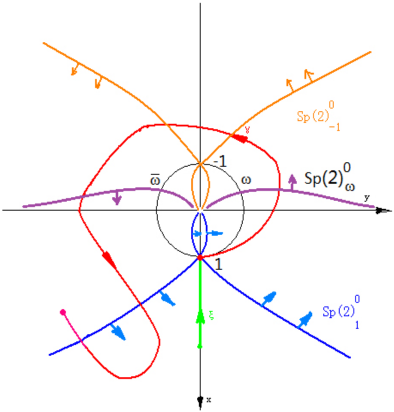


Figure: Graph of $\mathrm{Sp}(2)^0_\omega$ when $z = 0$

For second order Hamiltonian system, the following theorem on the relation of the Morse index $i_\omega(q_{m,e}, \mathcal{A}_\tau)$ and nullity $\nu_\omega(q_{m,e}, \mathcal{A}_\tau)$ of $\mathcal{A}_\tau$ at $q_{m,e}$ and the ω-index $i_\omega(\psi_{m,e})$ and ω-nullity $\nu_\omega(\psi_{m,e})$ of $\psi_{m,e}$) hold:

Theorem. ([Viterbo,1990], [An-Long, 1998], [Long-An,1998]) *For every $\omega \in \mathbf{U}$, there hold*

$$i_\omega(q_{m,e}, \mathcal{A}_\tau) = i_\omega(\psi_{m,e}), \quad \nu_\omega(q_{m,e}, \mathcal{A}_\tau) = \nu_\omega(\psi_{m,e}).$$

Lemma. ([Hu-Sun,2010]) *For every $\omega \in \mathbf{U}$, there hold*

$$\begin{aligned} i_\omega(\gamma_{\beta,e}) &= i_\omega(\psi_{m,e}) = i_\omega(q_{m,e}, \mathcal{A}_\tau), \\ \nu_\omega(\gamma_{\beta,e}) &= \nu_\omega(\psi_{m,e}) = \nu_\omega(q_{m,e}, \mathcal{A}_\tau). \end{aligned}$$

Specially

$$i_1(\gamma_{\beta,e}) = i_1(\psi_{m,e}) = i_1(q_{m,e}, \mathcal{A}_\tau) = 0, \quad \forall\, (\beta, e) \in [0, 9] \times [0, 1).$$

Such index theories were defined by

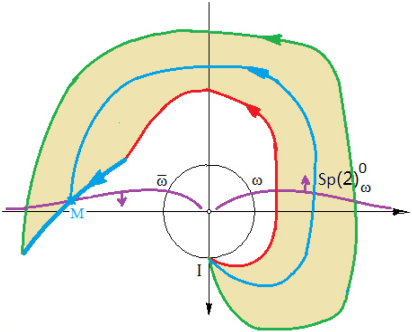
1984, C. Conley-E. Zehnder: for any path γ in $\mathrm{Sp}(2n)$ with $n \geq 2$ and γ being 1-non-degenerate, i.e., $(i_1(\gamma), \nu_1(\gamma))$ with $\nu_1(\gamma) = 0$;

1990, Y. Long-E. Zehnder: for any path γ in $\mathrm{Sp}(2)$ and γ being 1-non-degenerate, i.e., $(i_1(\gamma), \nu_1(\gamma))$ with $\nu_1(\gamma) = 0$;

1990, Y. Long, C. Viterbo (independently): for any path γ in $\mathrm{Sp}(2n)$ and γ may be 1-degenerate, i.e., $(i_1(\gamma), \nu_1(\gamma))$ with $\nu_1(\gamma) \geq 0$;

1999, Y. Long: for any path γ in $\mathrm{Sp}(2n)$ with respect to every $\omega \in \mathbf{U}$, i.e., $(i_\omega(\gamma), \nu_\omega(\gamma))$ with $\nu_\omega(\gamma) \geq 0$.

Important observation:



ω-index change implies the existence of some eigenvalue ω

$$i_\omega(\xi) - i_\omega(\gamma) \neq 0 \Rightarrow \omega \in \sigma(\gamma_{\beta,e}(2\pi))$$

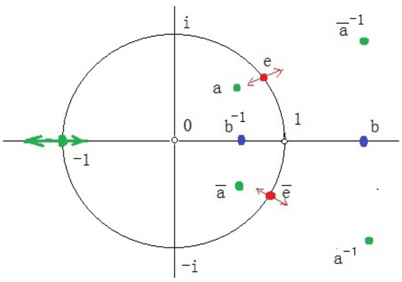
for some point (β, e), where $M = \gamma_{\beta,e}(2\pi)$.

Main results of Hu-Long-Sun, 2012:

Main Theorem 1. (X. Hu-Y. Long-S. Sun) *The ELS is 1-nondegenerate when* $(\beta, e) \in (0,9] \times [0,1)$. *Specially we have*

$$i_1(\gamma_{\beta,e}) = 0 \quad \textit{and} \quad \nu_1(\gamma_{\beta,e}) = \begin{cases} 3, & \text{if } \beta = 0, \\ 0, & \text{if } \beta \in (0,9], \end{cases} \qquad e \in [0,1).$$

Thus no eigenvalues of $\gamma_{\beta,e}(2\pi)$ can escape from **U** at 1 as $\beta > 0$!

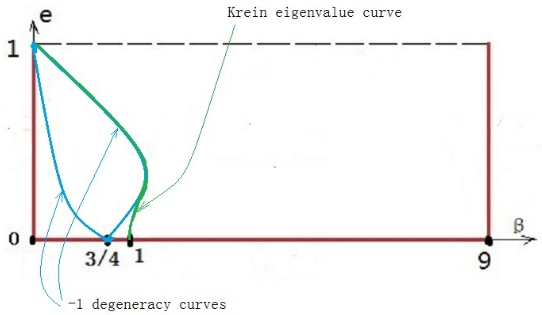


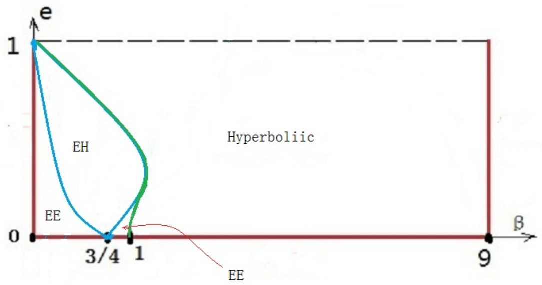
Main results of Hu-Long-Sun, 2012:

Main Theorem 2. (X. Hu-Y. Long-S. Sun) *In the* (β, e) *rectangle* $(0, 9] \times [0, 1)$ *there exist three distinct continuous curves from left to right: two* -1*-degeneracy curves* Γ_s *and* Γ_m *going up from* $(3/4, 0)$ *with tangents* $-\sqrt{33}/4$ *and* $\sqrt{33}/4$ *respectively and converges to* $(0, 1)$*, and the Krein collision eigenvalue curve* Γ_k *going up from* $(1, 0)$ *and converges to* $(0, 1)$ *as* e *increases from* 0 *to* 1*; each of them intersects every horizontal segment* $e = \mathrm{constant} \in [0, 1)$ *only once.*

Moreover the linear stability pattern of $\gamma_{\beta,e}(2\pi)$ *as well as that of the ELS* $z_{\beta,e}$ *changes if and only if* (β, e) *passes through one of these three curves* Γ_s*,* Γ_m *and* Γ_k*.*

Three separating curves and linear stability subregions





New observations and ideas (I) Reduction to a 2nd order OD operator.

Let

$$\xi_{\beta,e}(t) = \begin{pmatrix} R(t) & 0 \\ 0 & R(t) \end{pmatrix} \gamma_{\beta,e}(t), \; R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

for all $t \in [0, 2\pi]$. Then $\xi_{\beta,e}(2\pi) = \gamma_{\beta,e}(2\pi)$, $\xi_{\beta,e} \sim \gamma_{\beta,e}$, and it is the fundamental solution of:

$$\dot{y}(t) = J\overline{B}_{\beta,e}(t)y(t), \quad y(2\pi) = y(0),$$

$$\text{with} \qquad \overline{B}_{\beta,e}(t) = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 - R(t)K_{\beta,e}(t)R(t)^T \end{pmatrix},$$

$$\text{Recall}: \qquad B_{\beta,e}(t) = \begin{pmatrix} I_2 & -J \\ J & I_2 - K_{\beta,e}(t) \end{pmatrix}.$$

For $\omega \in \mathbf{U}$, $\overline{B}_{\beta,e}$ corresponds to a self-adjoint linear operator:

$$A(\beta, e) = -\frac{d^2}{dt^2} I_2 - I_2 + R(t)K_{\beta,e}(t)R(t)^T, \qquad \text{defined on}$$

$$\overline{D}(\omega) = \{y \in W^{2,2}([0, 2\pi], \mathbf{C}^2) \mid y(2\pi) = \omega y(0),\ \dot{y}(2\pi) = \omega \dot{y}(0)\}.$$

New observations and ideas (II) Index monotonicity.

Fix $e \in [0,1)$ and $\omega \in \mathbf{U}$. On $\overline{D}(\omega)$ we have:

$$\begin{aligned} A(\beta,e) &= -\frac{d^2}{dt^2}I_2 - I_2 + R(t)K_{\beta,e}(t)R(t)^T \\ &= -\frac{d^2}{dt^2}I_2 - I_2 + \frac{1}{2(1+e\cos t)}(3I_2 + \sqrt{9-\beta}S(t)) \\ &\equiv \sqrt{9-\beta}\,\hat{A}(\beta,e), \end{aligned}$$

where for $\beta \in [0,9)$,

$$\hat{A}(\beta,e) = \frac{A(9,e)}{\sqrt{9-\beta}} + \frac{S(t)}{2(1+e\cos t)}, \quad S(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{pmatrix}.$$

New observations and ideas (II) Index monotonicity.

Main Lemma 1. For β near β_0, the eigenvalues $\lambda(\beta)$ near $\lambda(\beta_0) = 0$ of $\hat{A}(\beta, e)$ satisfies

$$\frac{d}{d\beta}\lambda(\beta)|_{\beta=\beta_0} > 0.$$

In fact, we have

$$\lambda(\beta) = \lambda(\beta)\xi(\beta) \cdot \xi(\beta) = \hat{A}(\beta, e)\xi(\beta) \cdot \xi(\beta).$$

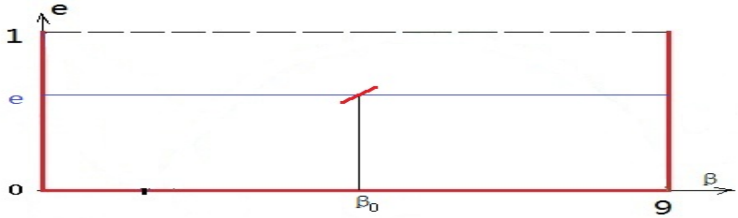
Differentiating both sides yields

$$\begin{aligned} \frac{d}{d\beta}\lambda(\beta)|_{\beta=\beta_0} &= (\frac{\partial}{\partial\beta}\hat{A}(\beta, e))\xi(\beta) \cdot \xi(\beta)|_{\beta=\beta_0} \\ &\quad + 2\hat{A}(\beta, e)\xi(\beta) \cdot (\frac{d}{d\beta}\xi(\beta))|_{\beta=\beta_0} \\ &= \frac{A(9, e)\xi(\beta) \cdot \xi(\beta)}{2(9-\beta)^{3/2}}|_{\beta=\beta_0} > 0. \end{aligned}$$

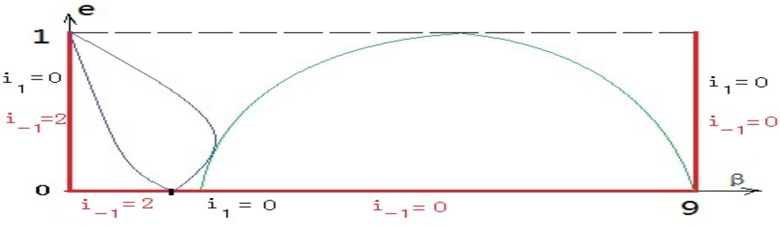
Main Lemma 2. Fix $e \in [0,1)$. For any $\omega \in \mathbf{U}$, when β increases in $(0,9]$, the index $i_\omega(\gamma_{\beta,e})$ is non-increasing, i.e.,

$$^{\#}\{\text{negative eigenvalues of } A(\beta,e)\} \text{ is non} - \text{increasing.}$$

Here $i_\omega(\gamma_{\beta,e}) = i_\omega(A(\beta,e)) = i_\omega(\hat{A}(\beta,e))$
$$= \; ^{\#}\{\text{negative eigenvalues of } \hat{A}(\beta,e)|_{\overline{D}(\omega)}\}.$$



New observations and ideas (III) Studies on the three boundary segments of $[0,9]\times[0,1)$



On the boundary segment $\{0\}\times[0,1)$

For every $e\in[0,1)$, we have

$$\gamma_{0,e}(2\pi)\approx I_2\diamond\begin{pmatrix}1&1\\0&1\end{pmatrix},$$

$$i_\omega(\gamma_{0,e})=\begin{cases}0,\\2,\end{cases}\quad \nu_\omega(\gamma_{0,e})=\begin{cases}3, & \text{when } \omega=1,\\0, & \text{when } \omega\in\mathbf{U}\setminus\{1\}.\end{cases}$$

On the boundary segment $\{9\} \times [0,1)$

For every $e \in [0,1)$, we have

$$\gamma_{9,e}(2\pi) \approx D(\lambda) \diamond D(\lambda) \quad \text{with some } 0 < \lambda \neq 1,$$
$$i_\omega(\gamma_{9,e}) = 0, \quad \nu_\omega(\gamma_{9,e}) = 0, \qquad \forall\, \omega \in \mathbf{U}.$$

On the boundary segment $(0,9] \times \{0\}$

We have

$$\text{For } 0 < \beta < 3/4: \ \gamma_{\beta,0}(2\pi) \approx R(\theta_1) \diamond R(\theta_2) \text{ with } \theta_1, \theta_2 \in (\pi, 2\pi),$$
$$i_1(\gamma_{\beta,0}) = 0, \ i_{-1}(\gamma_{\beta,0}) = 2, \ \nu_{\pm 1}(\gamma_{\beta,0}) = 0,$$
$$\text{For } \beta = 3/4: \ \gamma_{3/4,0}(2\pi) \approx -I_2 \diamond R(\theta_2) \text{ with } \theta_2 \in (\pi, 2\pi),$$
$$i_{\pm 1}(\gamma_{3/4,0}) = 0, \ \nu_1(\gamma_{3/4,0}) = 0, \ \nu_{-1}(\gamma_{3/4,0}) = 3,$$
$$\text{For } 3/4 < \beta \leq 1: \ \sigma(\gamma_{\beta,0}(2\pi)) \subset \mathbf{U} \setminus \{\pm 1\},$$
$$i_{\pm 1}(\gamma_{\beta,0}) = 0, \ \nu_{\pm 1}(\gamma_{\beta,0}) = 0;$$
$$\text{For } 1 < \beta \leq 9: \quad \sigma(\gamma_{\beta,0}(2\pi)) \cap \mathbf{U} = \emptyset,$$
$$i_{\pm 1}(\gamma_{\beta,0}) = 0, \ \nu_{\pm 1}(\gamma_{\beta,0}) = 0.$$

Main new results

Main Theorem 1 (Hu-Long-Sun, 2012).

$$i_1(\gamma_{\beta,e}) = 0, \qquad \forall\ (\beta, e) \in [0,9] \times [0,1),$$

$$\nu_1(\gamma_{\beta,e}) = \begin{cases} 3, & \text{if } \beta = 0, \\ 0, & \text{if } \beta \in (0,9], \end{cases} \qquad e \in [0,1).$$

That is, the ELS is non-degenerate when $\beta > 0$.

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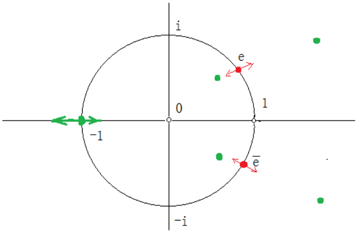
That is, the ELS is non-degenerate when $\beta > 0$.

Idea of the proof.

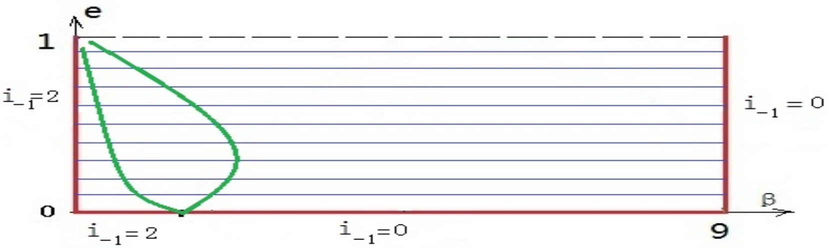
(1) Fix $e \in [0,1)$. By The Main Lemma 2 and our computations of $i_1(\gamma_{\beta,e})$ on the two boundaries $\{\beta = 0\}$ and $\{\beta = 9\}$, we obtain

$$\begin{aligned} & 0 = i_1(\gamma_{0,e}) \geq i_1(\gamma_{\beta,e}) \geq i_1(\gamma_{9,e}) = 0, \\ \text{then} \quad & 0 = i_1(\gamma_{\beta,e}) = i_1(A(\beta,e)) = i_1(\hat{A}(\beta,e)) \quad \forall\, \beta \in [0,9]. \end{aligned}$$

(2) If $\hat{A}(\beta_0, e)$ has an eigenvalue $\lambda(\beta_0) = 0$ for some $\beta_0 \in (0,9)$, then Main Lemma 1 implies $\frac{d}{d\beta}\lambda(\beta_0) > 0$, and thus $i_1(\hat{A}(\beta,e)) > 0$ for some $\beta < \beta_0$ close to β_0. Contradiction !



Because $1 \notin \sigma(\gamma_{\beta,e}(2\pi))$ for $\beta > 0$, there are only 2 possible ways for eigenvalues to escape from $\mathbf{U}$ as shown in the Figure, i.e., from -1 or from Krein collision eigenvalues.



Theorem 3 (Hu-Long-Sun, 2012). *For every $e \in [0,1)$, the -1 index $i_{-1}(\gamma_{\beta,e})$ is non-increasing, and strictly decreasing precisely on two values of $\beta = \beta_1(e)$ and $\beta = \beta_2(e) \in (0,9)$, at which $-1 \in \sigma(\gamma_{\beta,e}(2\pi))$ holds. For $e \in [0,1)$, define*

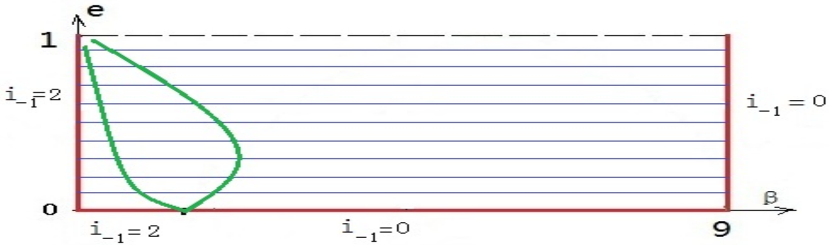
$$\beta_s(e) = \min\{\beta_1(e), \beta_2(e)\} \text{ and } \beta_m(e) = \max\{\beta_1(e), \beta_2(e)\},$$
$$\Gamma_s = \{(\beta_s(e), e) \mid e \in [0,1)\} \text{ and } \Gamma_m = \{(\beta_m(e), e) \mid e \in [0,1)\}.$$

They form the two -1-degeneracy curves in $[0,9] \times [0,1)$.

Idea of the proof.

Because $i_{-1}(\gamma_{0,e}) = 2$ and $i_{-1}(\gamma_{9,e}) = 0$, there exist two $\beta_1(e)$ and $\beta_2(e)$ such that $i_{-1}(\gamma_{\beta,e})$ strictly decreases by 1 when β passes $\beta_i(e)$. Here it is possible that $\beta_1(e) = \beta_2(e)$ and $i_{-1}(\gamma_{\beta,e})$ strictly decreases by 2 when β passes $\beta_1(e)$.

Specially $-1 \in \sigma(\gamma_{\beta_i(e),e}(2\pi))$ holds for $i = 1$ and 2.



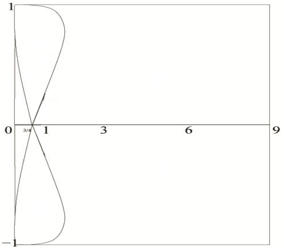
Idea of the proof (continued): Let

$$B(e,\omega) = A(9,e)^{-\frac{1}{2}} \frac{1}{2(1+e\cos(t))} S(t) A(9,e)^{-\frac{1}{2}}.$$

Here $B(e,\omega)$ depends on ω, because $A(9,e)$ is defined on $\overline{D}(\omega, 2\pi)$.

Lemma. *For any ω boundary condition and $e \in (0,9)$, $A(\beta,e)$ is ω degenerate if and only if $\lambda(e,\beta,\omega) = \frac{-1}{\sqrt{9-\beta}} \in \sigma_p(B(e,\omega))$.*

Here $B(\beta,e)$ depends on e analytically. Thus $\lambda(e,\beta,\omega)$ depends on e analytically by Operator Theory ([Kato]). Thus the above Lemma yields the analyticity of $\beta_i(e)$ in $e \in (-1,1)$, and then Γ_s and Γ_m are well defined and have at most isolated intersection points. ∎

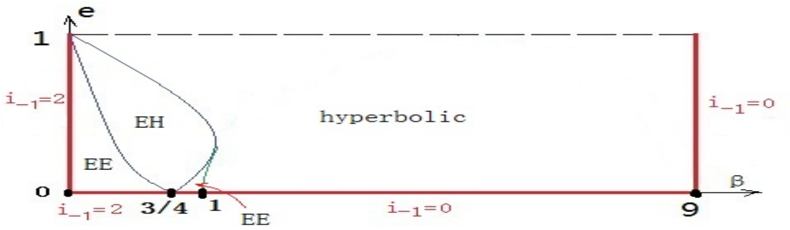


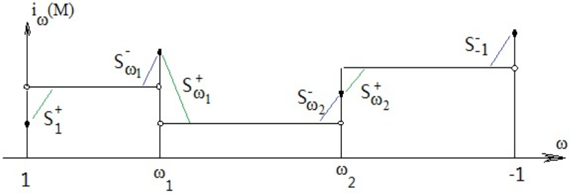
Theorem 4-(I) (Hu-Long-Sun, 2012). *Let $e \in [0,1)$. We have*

$$\text{(i)} \qquad i_{-1}(\gamma_{\beta,e}) = \begin{cases} 2, & \text{if } 0 \le \beta < \beta_s(e), \\ 1, & \text{if } \beta_s(e) \le \beta < \beta_m(e), \\ 0, & \text{if } \beta_m(e) \le \beta \le 9, \end{cases}$$

(ii) $\gamma_{\beta,e}(2\pi) \approx R(\theta_1)\diamond R(\theta_2)$ for some θ_1 and $\theta_2 \in (\pi, 2\pi)$, and thus is strongly linearly stable, when $0 < \beta < \beta_s(e)$;

(iii) $\gamma_{\beta,e}(2\pi) \approx D(\lambda)\diamond R(\theta))$ for some $0 > \lambda \neq -1$ and $\theta \in (\pi, 2\pi)$, and it is hyperbolic-elliptic and thus linearly unstable, when $\beta_s(e) < \beta < \beta_m(e)$.





Idea of the proof Theorem 4-(I)-(ii). When $0 < \beta < \beta_s(e)$, let $M = \gamma_{\beta,e}(2\pi)$. Because $\sigma(M) \subset \mathbf{U} \setminus \mathbf{R}$ when $0 < \beta < \beta_s(e)$ (no eigenvalues ± 1 and hyperbolic ones), we obtain

$$\begin{aligned} 2 &= i_{-1}(\gamma_{\beta,e}) \\ &= i_1(\gamma_{\beta,e}) + S_M^+(1) + \sum_{i=1}^{2}(-S_M^-(\omega_i) + S_M^+(\omega_i)) - S_M^-(-1) \\ &= \sum_{i=1}^{2}(-S_M^-(\omega_i) + S_M^+(\omega_i)) \leq \sum_{i=1}^{2} S_M^+(\omega_i) \leq 2. \end{aligned}$$

Then we get $2 = S_M^+(\omega_1) + S_M^+(\omega_2)$. It implies

$$\gamma_{\beta,e}(2\pi) \approx R(\theta_1) \diamond R(\theta_2) \text{ for some } \theta_1 \text{ and } \theta_2 \in (\pi, 2\pi),$$

and thus is strongly linearly stable.

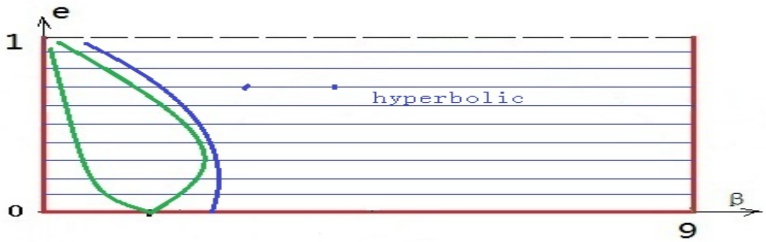
Theorem 5 (Hu-Long-Sun, 2012). *For every $e \in [0,1)$ we define*

$$\beta_k(e) = \inf\{\beta \in [0,9] \mid \sigma(\gamma_{\beta,e}(2\pi)) \cap \mathbf{U} = \emptyset\},$$
$$\Gamma_k = \{(\beta_k(e), e) \in [0,9] \times [0,1) \mid e \in [0,1)\}.$$

Then (i) $\beta_s(e) \le \beta_m(e) \le \beta_k(e) < 9$ *holds for all* $e \in [0,1)$*;*

(ii) Γ_k *is the boundary curve of the hyperbolic region of* $\gamma_{\beta,e}(2\pi)$ *in the* (β, e) *rectangle* $[0,9] \times [0,1)$*;*

(iii) Γ_k *is continuous in* $e \in [0,1)$, *starts from* $(1,0)$ *and goes up,* $\lim_{e\to 1} \beta_k(e) = 0$, *and* Γ_k *is distinct from* Γ_m.



Idea of the proof. (A)

$\gamma_{\beta_1,e}(2\pi)$ is hyperbolic $\Rightarrow$ $i_{-1}(\gamma_{\beta_1,e}) = 0$ by Theorem 4-(I).

Similarly $i_\omega(\gamma_{\beta_1,e}) = 0 \; \forall \, \omega \in \mathbf{U}$

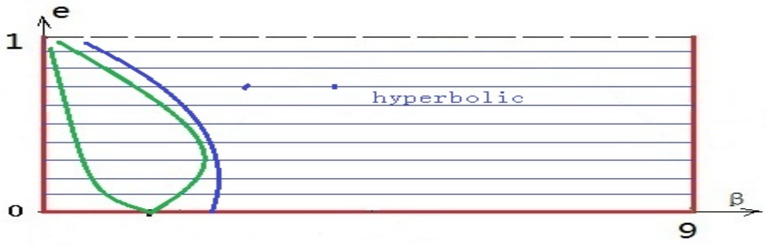
Main Lemma 2 $\Rightarrow$ $i_\omega(\gamma_{\beta,e}) = 0 \; \forall \, \omega \in \mathbf{U}$ and $\beta \in (\beta_1, 9]$

Main Lemma 1 $\Rightarrow$ $\nu_\omega(\gamma_{\beta,e}(2\pi)) = 0 \; \forall \, \omega \in \mathbf{U}$ and $\beta \in (\beta_1, 9]$,

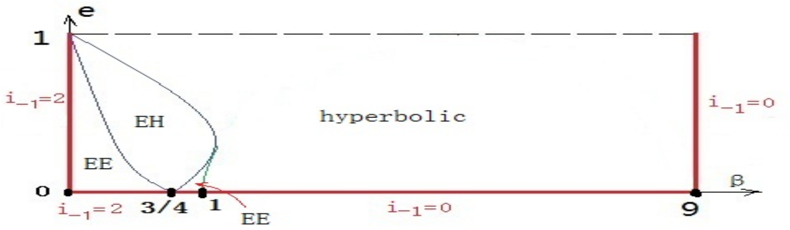
i.e., $\gamma_{\beta,e}(2\pi)$ is hyperbolic,

i.e., the hyperbolic subregion of $\gamma_{\beta,e}(2\pi)$ is connected. Then Γ_k is well-defined and contains one point on each $\{e = \mathrm{const.}\}$.

(B) Other hard parts: to prove the continuity of Γ_k, and $\beta_k(e) \to 0$ as $e \to 1$. ∎



Theorem 4-(II) (Hu-Long-Sun, 2012). *Let $e \in [0,1)$. We have (iv) $\gamma_{\beta,e}(2\pi) \approx R(\theta_1) \diamond R(\theta_2)$ for some $\theta_1 \in (0,\pi)$ and $\theta_2 \in (\pi, 2\pi)$ with $2\pi - \theta_2 < \theta_1$, and thus is strongly linearly stable, when $\beta_m(e) < \beta < \beta_k(e)$.*

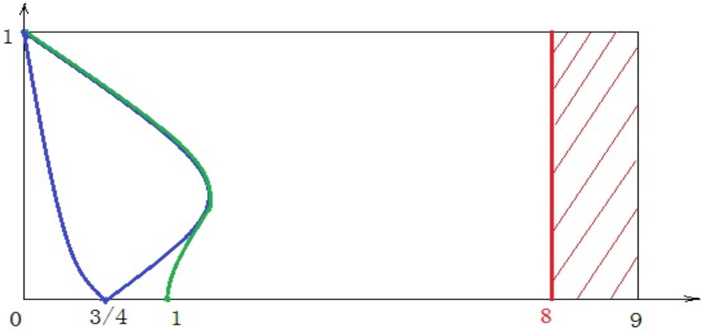


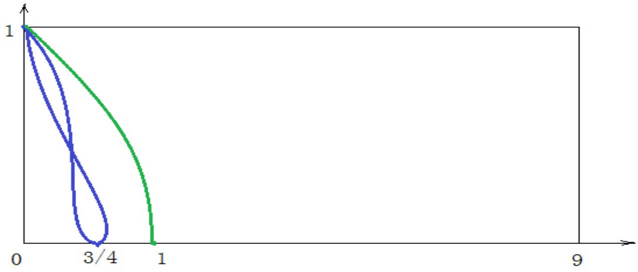
Theorem 6 (Hu-Long-Sun, 2012). *Let $e \in [0,1)$.*
(i) If $\beta_s(e) < \beta_m(e)$, $\gamma_{\beta_s(e),e}(2\pi) \approx N_1(-1,1)\diamond R(\theta)$ for some $\theta \in (\pi, 2\pi)$, and is spectrally stable and linearly unstable;
(ii) If $\beta_s(e) = \beta_m(e) < \beta_k(e)$, $\gamma_{\beta_s(e),e}(2\pi) \approx -I_2\diamond R(\theta)$ for some $\theta \in (\pi, 2\pi)$, and is inearly stable, but not strongly linearly stable;
(iii) If $\beta_s(e) < \beta_m(e) < \beta_k(e)$, $\gamma_{\beta_m(e),e}(2\pi) \approx N_1(-1,-1)\diamond R(\theta)$ for some $\theta \in (\pi, 2\pi)$, and is spectrally stable and linearly unstable;
(iv) If $\beta_s(e) \leq \beta_m(e) < \beta_k(e)$, $\gamma_{\beta_k(e),e}(2\pi) \approx N_2(e^{\sqrt{-1}\theta}, b)$ for some $\theta \in (0, \pi)$ and $(b_2 - b_3)\sin\theta > 0$, and is spectrally stable and linearly unstable;
(v) If $\beta_s(e) < \beta_m(e) = \beta_k(e)$, either $\gamma_{\beta_k(e),e}(2\pi) \approx N_1(-1,1)\diamond D(\lambda)$ for some $-1 \neq \lambda < 0$ and is linearly unstable; or $\gamma_{\beta_k(e),e}(2\pi) \approx N_2(-1,c)$ with $c_1, c_2 \in \mathbf{R}$ and $c_2 \neq 0$, and is spectrally stable and linearly unstable;
(vi) If $\beta_s(e) = \beta_m(e) = \beta_k(e)$, either $\gamma_{\beta_k(e),e}(2\pi) \approx M_2(-1,c)$ with $c_1 \in \mathbf{R}$ and $c_2 = 0$ which possesses basic normal form $N_1(-1,1)\diamond N_1(-1,1)$, or $\gamma_{\beta_k(e),e}(2\pi) \approx N_1(-1,1)\diamond N_1(-1,1)$; and thus is spectrally stable and linearly unstable.

New estimate of Yuwei Ou, 2012:

Theorem. (Y. Ou, 2012) $\gamma_{\beta,e}(2\pi)$ *is hyperbolic for all* (β, e) *in rectangle* $(8, 9] \times [0, 1)$, *i.e.,*

$$\sigma(\gamma_{\beta,e}(2\pi)) \subset \mathbf{C} \setminus \mathbf{U}, \qquad \forall\, (\beta, e) \in (8, 9] \times [0, 1).$$





Further open problems

(i) Precise locations of the three curves Γ_s, Γ_m and Γ_k;

(ii) No intersection of Γ_s and Γ_m;

(iii) The coincidence part of Γ_m and Γ_k;

(iv) Classification of real and complex hyperbolic cases;

(v) Applications to other problems.

Thank you !