Twisted Higher Rank Graph C*-algebras

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Introduction

We define the C*-algebra $C^*_\varphi(\Lambda)$ of a higher rank graph Λ twisted by a 2-cocycle φ which takes values in $\mathbb{T}$ and derive some basic properties.

Examples of this construction include all noncommutative tori, crossed products of Cuntz algebras by quasifree automorphisms and Heegaard quantum 3-spheres (see [BHMS]).

We also discuss the cohomology theory, where the twisting cocycle φ resides, and the homology theory on which it is based.

Our definition of the homology of a k-graph Λ is modeled on the cubical singular homology of a topological space (see [Mas91, §VII.2]).

It agrees with the homology of the associated cubical set (see [Gr05]).

This talk is based on joint work with David Pask and Aidan Sims of the University of Wollongong. Many of the the results discussed here were obtained while I was also employed there.

k-graphs

Definition (see [KP00])

Let Λ be a countable small category and let $d : \Lambda \to \mathbb{N}^k$ be a functor. Then (Λ, d) is a *k-graph* if it satisfies the factorization property:
For every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that

$$d(\lambda) = m + n$$

there exist unique $\mu, \nu \in \Lambda$ satisfying:

- $d(\mu) = m$ and $d(\nu) = n$,
- $\lambda = \mu\nu$.

Set $\Lambda^n := d^{-1}(n)$ and identify $\Lambda^0 = \mathrm{Obj}\,(\Lambda)$, the set of *vertices*.

An element $\lambda \in \Lambda^{e_i}$ is called an *edge*.

Remarks and Examples

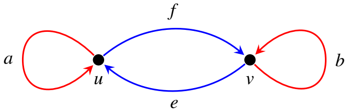
Let Λ be a k-graph.

- If $k = 0$, then d is trivial and Λ is just a set.
- If $k = 1$, then Λ is the path category of a directed graph.
- If $k \geq 2$, think of Λ as generated by k graphs of different colors that share the same set of vertices Λ^0.

Commuting squares form an essential piece of structure for $k \geq 2$.

Let C_m denote the directed cycle with m vertices viewed as a 1-graph.

Example of a 2-graph Λ: Only the edges, Λ^{e_1} and Λ^{e_2}, are shown.



Note that $\Lambda \cong C_2 \times C_1$.

More examples

The k-graph $T_k := \mathbb{N}^k$ is regarded as the k-graph analog of a torus.

Here is a simple k-graph with an infinite number of vertices:

$$\Delta_k := \{(m,n) \in \mathbb{Z}^k \times \mathbb{Z}^k \mid m \leq n\}$$

with structure maps

$$\begin{aligned} s(m,n) &= n \\ r(m,n) &= m \\ d(m,n) &= n - m \\ (\ell,n) &= (\ell,m)(m,n). \end{aligned}$$

This may be regarded as the k-graph analog of Euclidean space.

Cubes and Faces

Let Λ be a k-graph. For $0 \le n \le k$ an element $\lambda \in \Lambda$ with

$$d(\lambda) = e_{i_1} + \cdots + e_{i_n} \quad \text{where} \quad i_1 < \cdots < i_n$$

is called an *n-cube*. Let $Q_n(\Lambda)$ denote the set of n-cubes.

Note that 0-cubes are vertices and 1-cubes are edges.

For $n < 0$ or $n > k$, we have $Q_n(\Lambda) = \emptyset$.

Let $\lambda \in Q_n(\Lambda)$. We define the *faces* $F_j^0(\lambda), F_j^1(\lambda) \in Q_{n-1}(\Lambda)$, where $1 \le j \le n$, to be the unique elements such that

$$\lambda = F_j^0(\lambda)\lambda_0 = \lambda_1 F_j^1(\lambda)$$

where $d(\lambda_\ell) = e_{i_j}$ for $\ell = 0, 1$.

Fact: If $i < j$, then $F_i^\ell \circ F_j^m = F_{j-1}^m \circ F_i^\ell$.

Homology complex

For $1 \leq n \leq k$ define $\partial_n : \mathbb{Z}Q_n(\Lambda) \to \mathbb{Z}Q_{n-1}(\Lambda)$ such that for $\lambda \in Q_n(\Lambda)$

$$\partial_n(\lambda) = \sum_{j=1}^{n} \sum_{\ell=0}^{1} (-1)^{j+\ell} F_j^{\ell}(\lambda).$$

It is straightforward to show that $\partial_{n-1} \circ \partial_n = 0$.

Hence, $(\mathbb{Z}Q_*(\Lambda), \partial_*)$ is a complex and we define the homology of Λ by

$$H_n(\Lambda) = \ker \partial_n / \operatorname{Im} \partial_{n+1}.$$

The assignment $\Lambda \mapsto H_*(\Lambda)$ is a covariant functor.

Example: Recall that C_m is a cycle with m vertices. One may check that

$$H_n(C_m) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

The Künneth Theorem

Using basic homological algebra one may prove:

Theorem (Künneth Formula)

Let Λ_i be a k_i-graph for $i = 1, 2$. For $n \geq 0$ there is an exact sequence:

$$0 \to \sum_{m_1+m_2=n} H_{m_1}(\Lambda_1) \otimes H_{m_2}(\Lambda_2) \xrightarrow{\alpha} H_n(\Lambda_1 \times \Lambda_2) \xrightarrow{\beta}$$

$$\sum_{m_1+m_2=n-1} \mathrm{Tor}(H_{m_1}(\Lambda_1), H_{m_2}(\Lambda_2)) \to 0.$$

Let Λ be the 2-graph example above and recall that $\Lambda \cong C_2 \times C_1$.

By the Künneth Theorem we have

$$H_0(\Lambda) \cong \mathbb{Z}, \qquad H_1(\Lambda) \cong \mathbb{Z}^2, \qquad H_2(\Lambda) \cong \mathbb{Z}.$$

Acyclic k-graphs and free actions

A k-graph Λ is said to be *acyclic* if $H^0(\Lambda) \cong \mathbb{Z}$ and $H^n(\Lambda) = 0$ for $n > 0$.

Theorem

Let Λ be an acyclic k-graph and suppose that there is a free action of the group G on Λ. Then for each $n \geq 0$ there is an isomorphism:

$$H_n(\Lambda/G) \cong H_n(G).$$

Example. Take $\Lambda = \Delta_k$ and let $G = \mathbb{Z}^k$ act on Δ_k by translation.
It is easy to show that Δ_k is acyclic. We have $\Delta_k/\mathbb{Z}^k \cong T_k$ and so

$$H_n(T_k) \cong H_n(\mathbb{Z}^k) \cong \mathbb{Z}^{\binom{k}{n}}.$$

If E is a connected 1-graph with finitely many vertices and edges, then $H_1(E) \cong \mathbb{Z}^b$ where $b = |E^1| - |E^0| + 1$ (i.e. the first Betti number of E).

Cohomology

Let Λ be a k-graph and let A be an abelian group. For $n \in \mathbb{N}$ set

$$C^n(\Lambda, A) = \operatorname{Hom}(\mathbb{Z}Q_n(\Lambda), A)$$

and define

$$\delta^n : C^n(\Lambda, A) \to C^{n+1}(\Lambda, A) \quad \text{by} \quad \delta^n(\varphi) = \varphi \circ \partial_{n+1}.$$

It is straightforward to show that $(C^*(\Lambda, A), \delta^*)$ is a complex.

We define the cohomology of Λ by

$$H^n(\Lambda, A) := Z^n(\Lambda, A)/B^n(\Lambda, A),$$

where $Z^n(\Lambda, A) := \ker \delta^n$ and $B^n(\Lambda, A) := \operatorname{Im} \delta^{n-1}$.

Note $\Lambda \mapsto H^*(\Lambda, A)$ is a contravariant functor (it is covariant in A).

The UCT and a long exact sequence.

Theorem (Universal Coefficient Theorem)

Let Λ be a k-graph and let A be an abelian group. Then for $n \geq 0$, there is a short exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(\Lambda), A) \to H^n(\Lambda, A) \to \mathrm{Hom}(H_n(\Lambda), A) \to 0.$$

By a standard argument, a short exact sequence of coefficient groups

$$0 \to A \to B \to C \to 0$$

gives rise to a long exact sequence

$$0 \to H^0(\Lambda, A) \to H^0(\Lambda, B) \to H^0(\Lambda, C) \to H^1(\Lambda, A) \to \cdots$$
$$\cdots \to H^{n-1}(\Lambda, C) \to H^n(\Lambda, A) \to H^n(\Lambda, B) \to H^n(\Lambda, C) \to \cdots$$

The C^*-algebra $C^*_\varphi(\Lambda)$

Suppose that Λ satisfies $(*)$: For all $v \in \Lambda^0, n \in \mathbb{N}^k$, $v\Lambda^n$ is finite and nonempty where $v\Lambda^n := r^{-1}(v) \cap \Lambda^n$.

Definition

Let $\varphi \in Z^2(\Lambda, \mathbb{T})$. Define $C^*_\varphi(\Lambda)$ to be the universal C^*-algebra generated by a family of operators $\{t_\lambda : \lambda \in \Lambda^{e_i}, 1 \leq i \leq k\}$ and a family of orthogonal projections $\{p_v : v \in \Lambda^0\}$ satisfying:

1. For $\lambda \in \Lambda^{e_i}$, $t_\lambda{}^* t_\lambda = p_{s(\lambda)}$.
2. Suppose $\mu\nu = \nu'\mu'$ where $d(\mu) = d(\mu') = e_i$, $d(\nu) = d(\nu') = e_j$ and $i < j$. Then
$$t_{\nu'} t_{\mu'} = \varphi(\mu\nu) t_\mu t_\nu.$$
3. For $v \in \Lambda^0$ and $i = 1, \ldots, k$,
$$p_v = \sum_{\lambda \in v\Lambda^{e_i}} t_\lambda t_\lambda{}^*.$$

Main Results

Fact: The isomorphism class of $C^*_\varphi(\Lambda)$ only depends on $[\varphi] \in H^2(\Lambda, \mathbb{T})$.

There is a gauge action γ of $\mathbb{T}^k$ on $C^*_\varphi(\Lambda)$: For all $z \in \mathbb{T}^k$

$$\gamma_z(p_v) = p_v \qquad \text{for all } v \in \Lambda^0,$$
$$\gamma_z(t_\lambda) = z_i t_\lambda \qquad \text{for all } \lambda \in \Lambda^{e_i}, i = 1, \dots, k.$$

Moreover, the fixed point algebra $C^*_\varphi(\Lambda)^\gamma$ is AF (cf. [KP00]).

Theorem (Gauge Invariant Uniqueness Theorem)

Let $\pi : C^*_\varphi(\Lambda) \to B$ be an equivariant $*$-homomorphism. Then π is injective iff $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$.

Theorem

There is a $\mathbb{T}$-valued groupoid 2-cocycle σ_φ on $\mathcal{G}_\Lambda$ such that

$$C^*_\varphi(\Lambda) \cong C^*(\mathcal{G}_\Lambda, \sigma_\varphi).$$

Rotation algebras

Recall that $T_k = \mathbb{N}^k$.

There is precisely one 2-cube in T_2, namely $(1,1)$.

Fix $\theta \in [0,1)$. Let $\varphi \in Z^2(T_2, \mathbb{T})$ be given by $\varphi(1,1) = e^{2\pi i\theta}$.

Then $C^*_\varphi(T_2)$ is the universal C^*-algebra generated by unitaries t_{e_1} and t_{e_2} satisfying

$$t_{e_2} t_{e_1} = e^{2\pi i\theta} t_{e_1} t_{e_2}.$$

That is, $C^*_\varphi(T_2)$ is the rotation algebra A_θ.

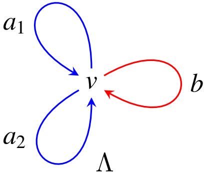
When $\theta = 0$, $C^*_\varphi(T_2) \cong C(\mathbb{T}^2)$.

When θ is irrational, $C^*_\varphi(T_2)$ is the well-known irrational rotation algebra.

More generally, every noncommutative torus arises as a twisted k-graph C^*-algebra $C^*_\varphi(T_k)$.

Crossed products of Cuntz algebras

Let $\Lambda = B_2 \times C_1$ where B_2 is the 1-graph with one vertex and two edges. Note that $C^*(B_2) \cong \mathcal{O}_2$ and so $C^*(\Lambda) \cong \mathcal{O}_2 \otimes C(\mathbb{T})$.



There are two 2-cubes in Λ, $a_j b$ for $j = 1, 2$.

The boundary maps are trivial; so we have $Z^2(\Lambda, \mathbb{T}) = H^2(\Lambda, \mathbb{T}) \cong \mathbb{T}^2$ where

$$Z^2(\Lambda, \mathbb{T}) \ni \varphi \mapsto (\varphi(a_1 b), \varphi(a_2 b))$$

Fix $\varphi \in Z^2(\Lambda, \mathbb{T})$, say $\varphi(a_j b) = z_j$. $C^*_\varphi(\Lambda)$ is isomorphic to the universal C^*-algebra generated by two isometries, s_1, s_2, and a unitary u such that

$$s_1 {s_1}^* + s_2 {s_2}^* = 1 \quad \text{and} \quad u s_j = z_j s_j u.$$

So $C^*_\varphi(\Lambda) \cong \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}$ where $\alpha(S_j) = z_j S_j$. Hence, every crossed product of $\mathcal{O}_2$ by a quasifree automorphism is isomorphic to one of the form $C^*_\varphi(\Lambda)$.

Heegaard quantum 3-spheres

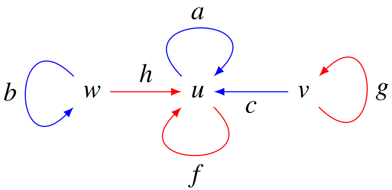
The quantum 3-sphere $S^3_{pq\theta}$ where $p, q, \theta \in [0, 1)$ is defined in [BHMS].

The authors prove that $S^3_{pq\theta} \cong S^3_{00\theta}$.

Note $S^3_{00\theta}$ is the universal C^*-algebra generated by S and T satisfying

$$(1 - SS^*)(1 - TT^*) = 0, \qquad ST = e^{2\pi i\theta} TS,$$
$$S^*S = T^*T = 1, \qquad ST^* = e^{-2\pi i\theta} T^*S.$$

It was known that S^3_{000} is isomorphic to $C^*(\Lambda)$ where Λ is the 2-graph



But what about $S^3_{00\theta}$?

Quantum spheres are twisted 2-graph C^*-algebras

The degree map gives a homomorphism $f : \Lambda \to T_2$ and the induced map

$$f^* : H^2(T_2, \mathbb{T}) \to H^2(\Lambda, \mathbb{T}).$$

is an isomorphism.

There are three 2-cubes $\alpha = ah = hb$, $\beta = cg = fc$ and $\tau = af = fa$.

Fix $\theta \in [0, 1)$. The 2-cocycle on T_2 determined by $(1, 1) \mapsto e^{-2\pi i\theta}$ pulls back to a 2-cocycle φ on Λ satisfying

$$\varphi(\alpha) = \varphi(\beta) = \varphi(\tau) = e^{-2\pi i\theta}.$$

Let $\{t_\lambda : \lambda \in \Lambda^{e_i}, 1 \leq i \leq k\}$ and $\{p_v : v \in \Lambda^0\}$ be the generators of $C^*_\varphi(\Lambda)$.

By the universal property there is a unique map $\Psi : S^3_{00\theta} \to C^*_\varphi(\Lambda)$ such that $\Psi(S) = t_a + t_b + t_c$ and $\Psi(T) = t_f + t_g + t_h$.

Moreover, Ψ is an isomorphism.

Categorical cocycle cohomology

The categorical cocycle cohomology, $H^*_{\mathrm{cc}}(\Lambda, A)$, is just the usual cocycle cohomology for groupoids (see [Ren80]) extended to small categories.

We have proven that for $n = 0, 1, 2$

$$H^n(\Lambda, A) \cong H^n_{\mathrm{cc}}(\Lambda, A).$$

A map $c : \Lambda * \Lambda \to A$ is a categorical 2-cocycle if for any composable triple $(\lambda_1, \lambda_2, \lambda_3)$ we have

$$c(\lambda_1, \lambda_2) + c(\lambda_1\lambda_2, \lambda_3) = c(\lambda_1, \lambda_2\lambda_3) + c(\lambda_2, \lambda_3)$$

and c is a categorical 2-coboundary if there is $b : \Lambda \to A$ such that

$$c(\lambda_1, \lambda_2) = b(\lambda_1) - b(\lambda_1\lambda_2) + b(\lambda_2).$$

$H^2_{\mathrm{cc}}(\Lambda, A)$ is the quotient group (2-cocycles modulo 2-coboundaries).

The C*-algebra $C^*(\Lambda, c)$

Suppose Λ satisfies $(*)$ and let c be a $\mathbb{T}$-valued categorical 2-cocycle.

Definition (see [KPS])

Let $C^*(\Lambda, c)$ be the universal C*-algebra generated by the set $\{t_\lambda : \lambda \in \Lambda\}$ satisfying:

1. $\{t_v : v \in \Lambda^0\}$ is a family of orthogonal projections.
2. For $\lambda \in \Lambda$, $t_{s(\lambda)} = {t_\lambda}^* t_\lambda$.
3. If $s(\lambda) = r(\mu)$, then $t_\lambda t_\mu = c(\lambda, \mu) t_{\lambda\mu}$.
4. For $v \in \Lambda^0, n \in \mathbb{N}^k$
$$t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda {t_\lambda}^*.$$

If $[\varphi]$ is mapped to $[c]$ in the identification $H^2(\Lambda, \mathbb{T}) \cong H^2_{\mathrm{cc}}(\Lambda, \mathbb{T})$, then

$$C^*_\varphi(\Lambda) \cong C^*(\Lambda, c).$$

Topological realizations

One may construct the topological realization X_Λ of a k-graph Λ (see [KKQS]) by analogy with the geometric realization of a simplicial set.

Let $I = [0, 1]$. For $i = 1, \ldots, n$ and $\ell = 0, 1$ define $\varepsilon_i^\ell : I^{n-1} \to I^n$ by

$$\varepsilon_i^\ell(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, \ell, x_i, \ldots, x_{n-1}).$$

Then the topological realization is the quotient of

$$\bigsqcup_{n=0}^{k} Q_n(\Lambda) \times I^n$$

by the equivalence relation generated by $(\lambda, \varepsilon_i^\ell(x)) \sim (F_i^\ell(\lambda), x)$ where $\lambda \in Q_n(\Lambda)$ and $x \in I^{n-1}$.

We prove that there is a natural isomorphism $H_n(\Lambda) \cong H_n(X_\Lambda)$.

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Thanks!