Asymptotic formation of self-organized states

Cucker-Smale and Kuramoto models

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Outline

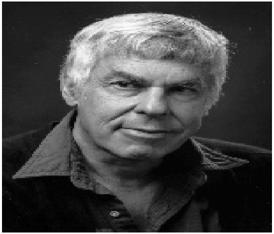
Biological complex systems



"http://cognition.ups-tlse.fr/dynactom/images, blogs.discovermagazine.com"

from "google-image"

The Cucker-Smale model



- The Cucker-Smale model: IEEE Trans. Automat. Control (2007):

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = \frac{K}{N} \underbrace{\sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i)}_{\text{relaxation}}.$$

where ψ is a communication weight (modeling issue):

C-S communication rate : $\psi(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^{\beta}}, \qquad \beta \geq 0.$

$$\Gamma_0 := \frac{1}{2}\sum_{1\le i,j\le N}|x_{i0}-x_{j0}|^2, \quad \Lambda_0 := \frac{1}{2}\sum_{1\le i,j\le N}|v_{i0}-v_{j0}|^2,$$
$$\psi(r) = \frac{1}{(1+r^2)^\beta}, \quad \beta \ge 0.$$

- **Theorem**: (Cucker-Smale '07) Asymptotic flocking occurs if

$$(i)\ \beta < \frac{1}{2}: \text{ No conditions on } \Gamma_0, \Lambda_0;$$
$$(ii)\ \beta \ge \frac{1}{2}: \text{ Conditions on } \Gamma_0, \Lambda_0:$$

Remark. N-dependence and special ψ.

- References: Cucker-Smale, Shen, Ha-Tadmor, Ha-Liu, Ha-Lee-Levy, Carrillo-D'Orsogna-Panferov, Degond-Motsch, Duan-Fornasier-Toscani, Carrillo-Fornasier-Rosado-Toscani, Ahn-Ha, Bolley-Canizo-Carrillo, Motsch-Tadmor, Agueh-Illner-Richardson, Ha-Jung-Slemrod, $\cdots$

Synchronization

"Synchronization (=syn (same, common) + chronous (time))" is an adjustment of rhythms of oscillating objects due to their weak interaction.

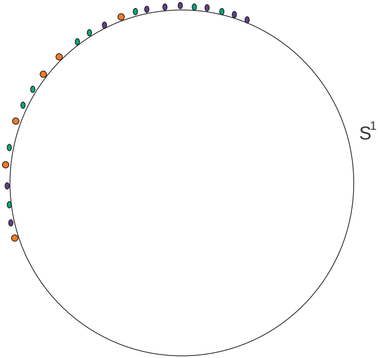
◇ Examples:

- Flashing of fireflies in South-East Asia
- Firing of coupled cardiac pacemaker cells (heart's contraction)
- Synchronous firing of many neurons (Parkinson's disease)
- Hands clapping in a concert

The Kuramoto model

The Kuramoto model

Consider a weakly coupled limit-cycle oscillators $\{x_i = e^{\sqrt{-1}\theta_i}\}$ rotating along S^1 with natural frequency Ω_i which is randomly drawn from some probability distribution with a density $g(\Omega)$.



- Dynamics on S^1: (Phase dynamics)

$$x_k \in S^1 : \text{ position of } k\text{-th oscillator} = e^{i\theta_k}.$$

State of system is determined by the dynamics of θ_k.

When there is no coupling(interaction) $K = 0$,

$$\dot{\theta}_k = \Omega_k (\text{natural frequency}), \quad \text{i.e.} \quad \theta_k(t) = \theta_{k0} + \Omega_k t.$$

- The Kuramoto model ('75).

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t > 0, \quad i = 1, \cdots, N.$$

If there is no coupling $K = 0$,

$$\theta_i(t) = \theta_i(0) + \Omega_i t, \quad \text{no synchronization.}$$

Similarity between the C-S and Kuramoto models

- The C-S model

$$\dot{x}_i = v_i, \quad \dot{v}_i = \frac{K}{N}\sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i).$$

- The Kuramoto model

$$\dot{\theta}_i = \omega_i, \quad \dot{\omega}_i = \frac{K}{N}\sum_{j=1}^{N} \cos(\theta_j - \theta_i)(\omega_j - \omega_i).$$

- C-S v.s. Kuramoto

$$(x_i, v_i) \quad \Longleftrightarrow \quad (\theta_i, \omega_i).$$

Topics to address today

- From a mechanical model to C-S model
- The complete synchronization problem
- The local flocking problem

From a mechanical model to the C-S model

C-S model is a phenomenological model, but it can be derived from some mechanical model via singular perturbation.

• A mechanical model of particles on a line with a constant friction coefficient.

$$\frac{dx_i}{dt} = v_i, \qquad \varepsilon \frac{dv_i}{dt} = \frac{K}{N} \sum_{j=1}^{N} \varphi'(x_j - x_i) - v_i, \quad \varphi\text{: convex.}$$

Taking the time-derivative of the second equation

$$\varepsilon \frac{d^2 v_i}{dt^2} = \frac{K}{N} \sum_{j=1}^{N} \varphi''(x_j - x_i)(v_j - v_i) - \frac{dv_i}{dt}.$$

In the formal zero mass limit $\varepsilon \to 0$, we recover the C-S model:

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = \frac{K}{N}\sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i),$$
$$\psi(x_j - x_i) := \varphi''(x_j - x_i).$$

cf. Ha-Slemrod: J. Dynam. Differential Equations '10 via the Tikhonov theory

- Extension to the multi-D setting: (Ha-Lattanzio-Rubino-Slemrod '10)

$$\frac{dx_i}{dt} = v_i, \qquad \varepsilon \frac{dv_i}{dt} = \frac{K}{N}\sum_{j=1}^{N} \nabla_x \varphi(x_j - x_i) - v_i, \quad \varphi(x) = \tilde{\varphi}(|x|).$$

The same formal procedure yields

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \\ \frac{dv_i}{dt} &= \frac{K}{N}\sum_{j=1}^{N}\Big[\frac{\tilde{\varphi}'(r_{ji})}{r_{ji}}(v_j - v_i) + \frac{1}{r_{ji}}\Big(\frac{\tilde{\varphi}'(r_{ji})}{r_{ji}}\Big)'((x_j - x_i)\cdot(v_j - v_i))(x_j - x_i)\Big] \\ &= \frac{K}{N}\sum_{j=1}^{N}\Big[\tilde{\psi}(r_{ij})(v_j - v_i) + \frac{\tilde{\psi}'(r_{ij})}{r_{ji}}((x_j - x_i)\cdot(v_j - v_i))(x_j - x_i)\Big]. \end{aligned}$$

For a harmonic potential $\tilde{\varphi}(|x|) = \frac{|x|^2}{2}$, we have

$$\tilde{\psi}(|x|) = 1 : \quad \text{C-S model.}$$

Note that

$$\tilde{\varphi}(r) = \int_0^r s\tilde{\psi}(s)ds.$$

We impose conditions for $\tilde{\psi}$:

$$\tilde{\psi} > 0, \quad \tilde{\psi}' > 0, \quad \lim_{r \to 0+} \tilde{\psi}(r) = R_1 > 0.$$

Examples.

$$\tilde{\psi}(r) = e^{-\frac{1}{1+r}}, \quad \tilde{\varphi}(r) = r^2 + r^4.$$

The complete synchronization problem

Consider the Kuramoto model:

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \cdots, N,$$

subject to initial data

$$\theta_i(0) = \theta_{i0}.$$

where $\Omega_i, K > 0, N$ are given constants satisfying

$$\sum_{i=1}^{N}\Omega_i = 0.$$

- Static questions:

$$\Omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i) = 0, \quad \sum_{i=1}^{N}\Omega_i = 0, \quad i = 1, \cdots, N.$$

1. Are there solutions for the above system ? **(Existence)**
2. If yes, how do they look like ? **(Structure)**

- Dynamic questions:

1. Are phase-locked states stable ? **(Stability)**
2. If phase-locked states can emerge from initial configurations, how does the relaxation process look like ? **(Relaxation)**

cf. Mirollo-Strogatz('05, '07), Aeyels-Rogge '04, De Smet-Aeyels '07, $\cdots$

- **Simple observation**:

1. If $|\Omega_i| > K$, then there is no solution at all.
2. One solution generates a one-parameter family of solutions:

$$\theta = (\theta_1, \cdots, \theta_N) : \text{ solution } \quad \Longrightarrow \quad \theta + c\mathbf{1} = (\theta_1 + c, \cdots, \theta_N + c) \ : \ \text{solution.}$$

Definition: Let $\mathcal{P} = \{(\theta_i, \omega_i := \dot{\theta}_i)\}_{i=1}^N$ be the system of Kuramoto oscillators.

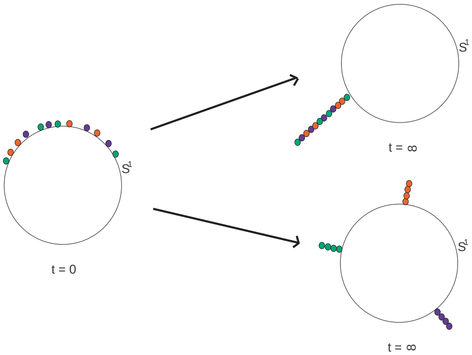
1. The system $\mathcal{P}$ has asymptotic complete phase synchronization if and only if the following condition holds.

$$\lim_{t \to \infty} |\theta_i(t) - \theta_j(t)| = 0, \qquad \forall \; i \neq j.$$

2. The augmented Kuramoto system $\mathcal{P}$ has asymptotic complete frequency synchronization if and only if the following condition holds.

$$\lim_{t \to \infty} |\omega_i(t) - \omega_j(t)| = 0, \qquad \forall \; i \neq j.$$

- Formation of asymptotic synchronization



Problem and Strategy

- **Problem**:

Find conditions for initial configurations and parameters leading to complete synchronization.

- **Strategy**:

1. Consider the Lyapunov functionals: diameters of the phase and frequency configurations:

$$\begin{aligned} D(\theta(t)) &:= \max_{1\le i,j\le N} |\theta_i(t)-\theta_j(t)|, \quad \text{if } D(\theta(t)) \le \pi, \\ D(\omega(t)) &:= \max_{1\le i,j\le N} |\omega_i(t)-\omega_j(t)|. \end{aligned}$$

2. Derive Gronwall's inequalities for $D(\theta)$ and $D(\omega)$.

3. Finally we show

$$\lim_{t\to\infty} D(\theta(t)) = 0, \qquad \lim_{t\to\infty} D(\omega(t)) = 0.$$

A framework for exponential complete synchronization

- (H1) The oscillators are non-identical and the coupling strength is sufficiently large such that

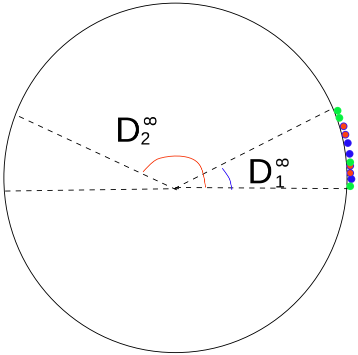
$$D(\Omega) > 0, \qquad K > D(\Omega).$$

- (H2) Initial phase fluctuation is sufficiently small so that

$$D_0 < D_1^\infty,$$

where $D_i^\infty \in (0, \frac{\pi}{2}), \;\; i = 1, 2$ are the roots of the following trigonometric equation:

$$\frac{D(\Omega)}{K} = \sin x, \qquad x \in \Big(0, \pi\Big).$$


D_2^∞
D_1^∞

- **Theorem**: H-Ha-Kim '10

Suppose the main assumptions (H1) -(H2) hold, and let (θ_i) be the smooth solutions to the Kuramoto system with initial phase configurations (θ_{i0}). Then we have a complete frequency synchronization:

$$D(\omega(t)) \leq D(\omega(0)) \exp\Big(-K(\cos D_1^\infty)t \Big), \quad t \geq 0.$$

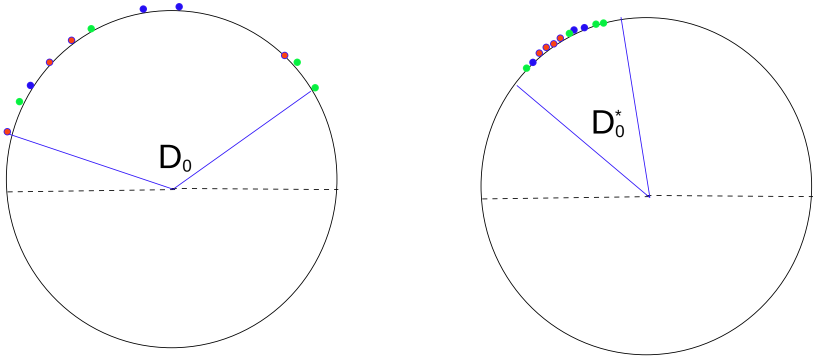
Proof. Derive the Gronwall's inequality

$$\frac{d}{dt} D(\omega) \leq -K(\cos D_1^\infty) D(\omega).$$

cf. 1. Preliminary result: H-Lattanzio-Rubino-Slemrod '10: N-dependence

2. Chopra-Spong '09, Choi-Ha-Yun '10, Dorfler-Bullo '11

Numerical simulation shows that for sufficiently large K,



Derivation of the Adler inequality

• **Lemma**. Suppose initial data, natural frequencies and coupling strength satisfy

$$0 < D_0 < \pi, \quad D(\Omega) > 0, \quad K > K_e := \frac{D(\Omega)}{\sin D_0}.$$

Then $D(\theta)$ satisfies

$$(i)\ \frac{d}{dt}D(\theta) \leq D(\Omega) - K \sin D(\theta).$$
$$(ii)\ D(\theta(t)) \leq D_0^*, \quad t \geq t_0.$$

- **Theorem** (Formation of phase-locked states): Choi-H-Jung-Kim '12

Suppose initial data, natural frequencies and coupling strength satisfy

$$0 < D_0 < \pi, \quad D(\Omega) > 0, \quad K > K_e := \frac{D(\Omega)}{\sin D_0},$$

and let θ be the solution to the Kuramoto model with initial data θ_0. Then there exists t_0 such that

$$D(\omega(t_0))e^{-K(t-t_0)} \leq D(\omega(t)) \leq D(\omega(t_0))e^{-K(\cos D_0^*)(t-t_0)}, \quad t \geq t_0.$$

The flocking problem

"Finding sufficient conditions on parameters and initial data" leading to flocking behaviors)".

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = \frac{K}{N}\sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i).$$

where

$$\psi(\cdot) \geq 0 : \text{ Lipschitz continuous, non-increasing},$$
$$\sum_{i=1}^{N} x_i = 0, \quad \sum_{i=1}^{N} v_i = 0.$$

• **Definition**

An interacting N-particle system $\{(x_i(t), v_i(t))\}_{i=1}^N$ exhibits (velocity) asymptotic flocking $\Longleftrightarrow$

1. Formation of a group.

$$\sup_{0\le t<\infty} |x_i(t) - x_j(t)| < \infty, \quad \forall\, i \neq j.$$

2. Velocity alignment.

$$\lim_{t\to\infty} |v_i(t) - v_j(t)| = 0, \quad \forall\, i \neq j.$$

Lyanpunov functional approach

We set

$$||x|| := \Big(\sum_{i=1}^{N} ||x_i||^2\Big)^{\frac{1}{2}}, \quad ||v|| := \Big(\sum_{i=1}^{N} ||v_i||^2\Big)^{\frac{1}{2}}.$$

Then $||x||$ and $||v||$ satisfy

$$\Big|\frac{d||x||}{dt}\Big| \le ||v||, \quad \frac{d||v||}{dt} \le -K\psi(2||x||)||v||.$$

We now introduce Lyapunov type functionals:

$$\mathcal{E}(t) := ||v(t)||_\infty \pm K \int_0^{||x(t)||} \psi(2s)ds.$$

- **Theorem**: Ha-Liu '10: If (x_0, v_0) satisfies

$$||v_0|| < K \int_{||x_0||_\infty}^{\infty} \psi(2s)ds,$$

then for any global smooth solutions (x, v), there exists r_M such that

$$\sup_{t\geq 0} ||x(t)|| \leq r_M, \quad ||v(t)|| \leq ||v_0|| e^{-K\psi(2r_M)t}.$$

- Question: If initial configuration satisfies

$$|v_0|| \geq K \int_{||x_0||_\infty}^{\infty} \psi(2s)ds.$$

what will happen ? Numerical simulations suggest the formation of local flockings, but how to prove it rigorously ?

Existence of local flocking

Consider the short range communication weight

$$\int_0^\infty \psi(s)ds < \infty, \quad \text{e.g.} \quad \psi(s) = \frac{1}{(1+s^2)^{\frac{\beta}{2}}}, \ \ \beta > 1, \ \ e^{-s^2}.$$

• Simple observation: Bi-flocking configurations for C-S model can not achieved in finite time.

$$0 = \dot{v}_i = \frac{K}{N}\sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i) \neq 0.$$

- A two-particle system on a line:

$$\dot{x}_1 = v_1, \quad \dot{x}_2 = v_2, \quad t > 0,$$
$$\dot{v}_1 = \frac{K}{2}(|x_2 - x_1|)(v_2 - v_1), \quad \dot{v}_1 = \frac{K}{2}(|x_2 - x_1|)(v_2 - v_1).$$

Then the differences $x := x_1 - x_2, \;\; v := v_1 - v_2$ satisfy

$$\dot{x} = v, \quad \dot{v} = -K\psi(|x|)v,$$

or equivalently

$$v(t) = v_0 - K\int_{x_0}^{x(t)} \psi(|y|)dy.$$

- **Easy theorem**: If (x_0, v_0) satisfies

$$v_0 \geq K \int_{x_0}^{\infty} \psi(|s|)ds,$$

then there is colorredno global flocking (formation of bi-flocking configuration).

Proof. Suppose that

$$v_0 \geq K \int_{x_0}^{\infty} \psi(|s|)ds.$$

Then we have

$$v(t) = v_0 - K \int_{x_0}^{x(t)} \psi(|y|)dy \geq K \int_{x(t)}^{\infty} \psi(|y|)dy.$$

This implies the incompatibility of

$$\sup_{t \geq 0} x(t) < \infty, \quad \lim_{t \to \infty} v(t) = 0.$$

A formulation for bi-flocking groups

$$\mathcal{G}_1 = \{1, \cdots, N_1\}, \quad \mathcal{G}_2 = \{1, \cdots, N_2\}, \quad N_1 + N_2 = N,$$
$(x_{\alpha i}, v_{\alpha i})$: the state of i-th member in $\mathcal{G}_\alpha$.

- The C-S model:

$$\begin{aligned}
\dot{x}_{1i} &= v_{1i}, \quad \dot{x}_{2j} = v_{2j}, \quad i = 1, \cdots, N_1, \quad j = 1, \cdots, N_2, \\
\dot{v}_{1i} &= \frac{K}{N} \sum_{k \in \mathcal{G}_1} \psi(||x_{1k} - x_{1i}||)(v_{1k} - v_{1i}) + \frac{K}{N} \sum_{k \in \mathcal{G}_2} \psi(||x_{2k} - x_{1i}||)(v_{2k} - v_{1i}), \\
\dot{v}_{2j} &= \frac{K}{N} \sum_{k \in \mathcal{G}_2} \psi(||x_{2k} - x_{2j}||)(v_{2k} - v_{2j}) + \frac{K}{N} \sum_{k \in \mathcal{G}_1} \psi(||x_{1k} - x_{2j}||)(v_{1k} - v_{2j}).
\end{aligned}$$

We introduce local averages and local fluctuations

$$\begin{aligned}
x_{1c} &:= \frac{1}{|\mathcal{G}_1|} \sum_{i \in \mathcal{G}_1} x_{1i}, \quad x_{2c} = \frac{1}{|\mathcal{G}_2|} \sum_{j \in \mathcal{G}_2} x_{2j}, \\
v_{1c} &:= \frac{1}{|\mathcal{G}_1|} \sum_{i \in \mathcal{G}_1} v_{1i}, \quad v_{2c} = \frac{1}{|\mathcal{G}_2|} \sum_{j \in \mathcal{G}_2} v_{2j} \\
\hat{x}_{\alpha i} &:= x_{\alpha i} - x_{\alpha c}, \qquad \hat{v}_{\alpha i} := v_{\alpha i} - x_{\alpha c}, \quad \alpha = 1, 2.
\end{aligned}$$

- **Macro-micro decomposition**:

$$
\begin{aligned}
&\dot{x}_{1c} = v_{1c}, \quad \dot{x}_{2c} = v_{2c},\\
&|\mathcal{G}_1|\dot{v}_{1c} = -\frac{K}{N}\Psi(t)(v_{1c} - v_{2c}) - \frac{K}{N}\sum_{i\in\mathcal{G}_1}\sum_{j\in\mathcal{G}_2}\psi(||x_{2j} - x_{1i}||)(\hat{v}_{1i} - \hat{v}_{2j}),\\
&|\mathcal{G}_2|\dot{v}_{2c} = -\frac{K}{N}\Psi(t)(v_{2c} - v_{1c}) - \frac{K}{N}\sum_{i\in\mathcal{G}_1}\sum_{j\in\mathcal{G}_2}\psi(||x_{2j} - x_{1i}||)(\hat{v}_{2j} - \hat{v}_{1i}),
\end{aligned}
$$

where $\Psi(t) := \sum_{i\in\mathcal{G}_1}\sum_{j\in\mathcal{G}_2}\psi(||x_{2j} - x_{1i}||)$ and

$$
\begin{aligned}
&\dot{\hat{x}}_{1i} = \hat{v}_{1i}, \quad \dot{\hat{x}}_{2j} = \hat{v}_{2j}, \quad i = 1, \cdots, N_1, \; j = 1, \cdots, N_2,\\
&\dot{\hat{v}}_{1i} = -\hat{v}_{1c} + \frac{K}{N}\sum_{k\in\mathcal{G}_1}\psi(||x_{1k} - x_{1i}||)(\hat{v}_{1k} - \hat{v}_{1i})\\
&\qquad + \frac{K}{N}\sum_{k\in\mathcal{G}_2}\psi(||x_{2k} - x_{1i}||)(\hat{v}_{2k} - \hat{v}_{1i}) + \frac{K}{N}\sum_{k\in\mathcal{G}_2}\psi(||x_{2k} - x_{1i})(v_{2c} - v_{1c}),\\
&\dot{\hat{v}}_{2j} = -\hat{v}_{2c} + \frac{K}{N}\sum_{k\in\mathcal{G}_2}\psi(||x_{2k} - x_{2j}||)(\hat{v}_{2k} - \hat{v}_{2j})\\
&\qquad + \frac{K}{N}\sum_{k\in\mathcal{G}_1}\psi(||x_{1k} - x_{2j}||)(\hat{v}_{1k} - \hat{v}_{2j}) + \frac{K}{N}\sum_{k\in\mathcal{G}_2}\psi(||x_{1k} - x_{2j})(v_{1c} - v_{2c}).
\end{aligned}
$$

- Derivation of SDDI:

$$
\begin{aligned}
\mathcal{X} &:= ||\hat{x}_1|| + ||\hat{x}_2||, \quad \mathcal{V} := ||\hat{v}_1|| + ||\hat{v}_2||, \\
\Delta_x &:= ||x_{1c} - x_{2c}||, \quad \Delta_v := ||v_{1c} - v_{2c}||,
\end{aligned}
$$

where

$$
||\hat{x}_\alpha|| := \Big(\sum_{i \in \mathcal{G}_\alpha} ||\hat{x}_{\alpha i}||^2 \Big)^{\frac{1}{2}}, \qquad ||\hat{v}_\alpha|| := \Big(\sum_{i \in \mathcal{G}_\alpha} ||\hat{v}_{\alpha i}||^2 \Big)^{\frac{1}{2}}.
$$

$$
\begin{aligned}
& |\dot{\mathcal{X}}| \le \mathcal{V}, \quad |\dot{\Delta}_x| \le \Delta_v, \\
& \dot{\mathcal{V}} \le -\frac{2KC_*(N)}{N} \psi(\sqrt{2}\mathcal{X})\mathcal{V} + \frac{2K\Psi}{N}(\mathcal{V} + \Delta_v), \\
& \dot{\Delta}_v \le -\frac{K}{|\mathcal{G}_1||\mathcal{G}_2|}\Psi\Delta_v + \frac{K\psi_M}{\sqrt{C_*(N)}}\mathcal{V}, \\
& \dot{\Delta}_v \ge -\frac{K}{|\mathcal{G}_1||\mathcal{G}_2|}\Psi\Delta_v - \frac{K\psi_M}{\sqrt{C_*(N)}}\mathcal{V},
\end{aligned}
$$

•: Admissible class of initial configuration

$$\mathcal{S} := \{(x_0, v_0) \in R^{2dN} \ : \ (\mathcal{H}1), (\mathcal{H}2) \text{ and } (\mathcal{H}3) \text{ hold}\}.$$

- $(\mathcal{H}1)$: Initially two approximate flocking groups are well-separated to satisfy

$$||v_0|| \geq K \int_{||x_0||}^{\infty} \psi(\sqrt{2}r)dr.$$

- $(\mathcal{H}2)$: Initial fluctuation of phase is small to satisfy

$$\left(\frac{2K|\mathcal{G}_1||\mathcal{G}_2|}{N}\right)\left(2\sqrt{M_2(0)} + \Delta_v(0) + \frac{2K\sqrt{M_2(0)}}{\sqrt{C_*(N)}}||\psi||_{L^1}\right) + \mathcal{V}(0) \leq \frac{2KC_*(N)}{N}\int_{\mathcal{X}_0}^{\infty} \psi(\sqrt{2}s)ds.$$

- $(\mathcal{H}3)$: There exists an index $k \in \{1, \cdots, d\}$ such that

$$(i)\ v_{1c}^k - v_{2c}^k \geq \left(1 + \frac{K\sqrt{M_2(0)}}{\sqrt{C_*(N)}}||\psi||_{L^1}\right)e^{K||\psi||_{L^1}},$$
$$(ii)\ x_{1c}^k - x_{2c}^k > \tilde{\mathcal{X}}_M,$$

- **Theorem**: Ha-Jung '12:

Let $(x_{\alpha i}, v_{\alpha i})$ be the solution to the macro-micro system with initial data in $\mathcal{S}$. Then we have

$$\sup_{t\geq 0} \mathcal{X}(t) \leq \mathcal{X}_M, \quad \mathcal{V}(t) \leq \mathcal{O}(1) \max\left\{ e^{-\frac{\psi(\mathcal{X}_M)t}{2}}, \max_{[t/2,t]} \Psi(s) \right\},$$

$$\Delta_v(t) \geq \Delta_v(0) \exp\Big(-K||\psi||_{L^1}\Big) - \frac{2K\sqrt{M_2(0)}}{\sqrt{C_*(N)}}||\psi||_{L^1},$$

$$\Delta_v(t) \leq \Delta_v(0) + \frac{2K\sqrt{M_2(0)}}{\sqrt{C_*(N)}}||\psi||_{L^1},$$

$$\Delta_x(t) \geq \mathcal{O}(1)(1+t).$$

cf. For C-S communication weight, $\mathcal{V}(t) \leq C(1+t)^{-\beta}$ (slow relaxation)

Conclusion

The Cucker-Smale flocking model and The Kuramoto model have lots of common structures, and we provided a one unified Lyapunov functional approach for flocking and synchronization problems.

Thank you for your attention