Steady states and asymptic limits for a nonlocal aggregation model

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January 24, 2012

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Dynamic evolution: Numerical Methods

$$\rho_t = -\nabla \cdot (\rho v) = \nabla \cdot (\rho \nabla K * \rho).$$

$$K(x) = \frac{1}{n(n-2)\omega_n |x|^{n-2}} + \frac{1}{q}|x|^q, \quad F(r) = r^{q-1} - \frac{1}{n\omega_n r^{n-1}}.$$

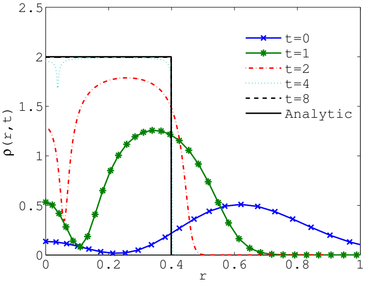
Particle system:

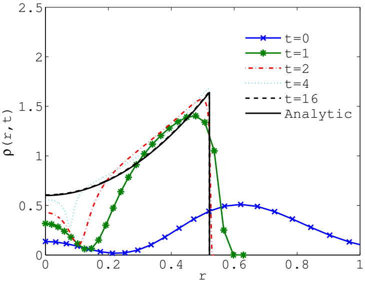
$$\frac{dX_i}{dt} = \frac{1}{N} \sum_{\substack{j=1\dots N \\ j \neq i}} F(|X_i - X_j|) \frac{X_i - X_j}{|X_i - X_j|}, \qquad i = 1 \dots N$$

Continuous density: (in radial coordinate)

$$\frac{dr}{dt} = -\partial_r K * \rho(r), \qquad \frac{d\rho}{dt} = \rho \Delta_r K * \rho.$$

Dynamic evolution: numerical results



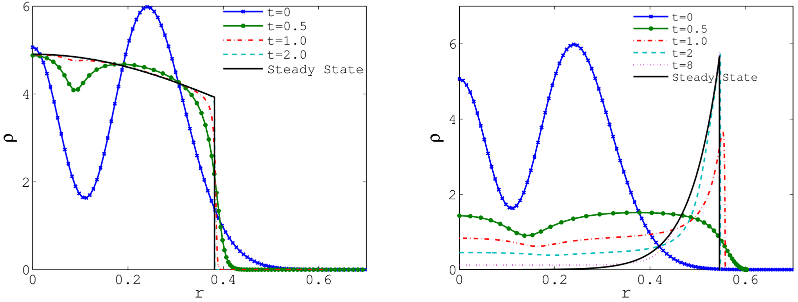


Time evolution of the radially symmetric solution.

Left: The case $q = 2$: the solution approaches asymptotically a **constant, compactly supported** steady state.

Right: The case $q = 4$: the solution approaches asymptotically a **nonconstant, compactly supported** steady state.

Dynamic evolution: numerical results



Time evolution of the radially symmetric solution.

Left: The case $q = 1.5(> 2)$. The steady state is concave on the support.

Right: The case $q = 20(> 2)$. The steady state is convex on the support.

Non-constant steady states

Assume the model admits **a radial steady state supported on a ball** $B(0, R)$.

Equilibria supported on $B(0, R)$, $\nabla \cdot (\rho v) = 0$:

$$v = -\nabla K * \rho = 0,$$

$$\text{hence } \operatorname{div} v = -\Delta K * \rho = \rho - \Delta(\frac{1}{q}|x|^q) * \rho$$

A steady state $\bar{\rho}$ satisfies for $x \in B(0, R)$

$$\boxed{\bar{\rho}(x) - (n + q - 2) \int_{B(0,R)} |x - y|^{q-2} \bar{\rho}(y) dy = 0}$$

Use radial symmetry $\bar{\rho}(x) = \bar{\rho}(r)$.

Radial steady states

$$\boxed{\bar{\rho}(x) - (n+q-2)\int_{B(0,R)} |x-y|^{q-2}\bar{\rho}(y)dy = 0}$$

The density $\bar{\rho}$ satisfies the Fredholm integral equation

$$\boxed{\bar{\rho}(r) = c(q,n)\int_0^R (r')^{n-1}\bar{\rho}(r')I(r,r')dr' \equiv T_R\bar{\rho}(r), \qquad 0 \le r < R,}$$

$$I(r,r') = \int_0^\pi (r^2 + (r')^2 - 2rr'\cos\theta)^{q/2-1}\sin^{n-2}\theta d\theta.$$

The eigenvalue problem: find $\bar{\rho}$ **and** the radius R of the support for $\bar{\rho} = T_R\bar{\rho}$.

Exact steady states: q even

Kernel $|x-y|^{q-2}$ is separable when q is even.
Define the i-th order moments of the density ($m_0 = M$):

$$m_i = n\omega_n \int_0^R r^{n+i-1}\bar{\rho}(r)dr. \tag{1}$$

Example: $q = 4$

$$I(r, r') = (r^2 + (r')^2)\int_0^\pi \sin^{n-2}\theta d\theta$$

and

$$\begin{aligned}\bar{\rho}(r) &= n(n+2)\omega_n \int_0^R (r')^{n-1}(r^2 + (r')^2)\bar{\rho}(r')dr' \\ &= (n+2)m_0 r^2 + (n+2)m_2 \end{aligned} \tag{2}$$

Plug (2) into (1): linear system to find R and m_2

$$\begin{pmatrix} m_0 \\ m_2 \end{pmatrix} = \begin{pmatrix} n\omega_n R^{n+2} & (n+2)\omega_n R^n \\ \frac{n(n+2)}{n+4}\omega_n R^{n+4} & n\omega_n R^{n+2} \end{pmatrix} \begin{pmatrix} m_0 \\ m_2 \end{pmatrix}$$

General q even: $\bar{\rho}(r)$ is a polynomial of even powers of r, of degree $q-2$.

Exact steady states: $q = 4 - n (n > 4)$

$$\bar{\rho}(x) = 2 \int_{B(0,R)} |x-y|^{2-n} \bar{\rho}(y) dy \tag{3}$$

Taking the Laplacian in r,

$$\bar{\rho}'' + \frac{n-1}{r} \bar{\rho}' + \kappa^2 \bar{\rho} = 0, \qquad \kappa = \sqrt{2n(n-2)\omega_n}.$$

The solution can be written in terms of Bessel function,

$$\bar{\rho}(r) = c r^{1-n/2} J_{n/2-1}(\kappa r).$$

Evaluating (3) at $x = 0$,

$$J_{n/2-2}(\kappa R) = 0,$$

the radius of support R is chosen such that κR is the first nonzero zero of $J_{n/2-2}$.

Some properties

$$\bar{\rho}(r) = T_R\bar{\rho}(r) \equiv c(q,n)\int_0^R (r')^{n-1} I(r,r')\bar{\rho}(r')dr',$$

The kernel $c(q,n)(r')^{n-1}I(r,r')$ is nonnegative.
T_R is a linear, strongly positive, compact operator that maps the space of continuous functions $C([0,1],\mathbb{R})$ into itself.

Krein-Rutman theorem for existence and uniqueness: For fixed R, there exists a *positive* eigenfunction $\bar{\rho}$ such that

$$T_R\bar{\rho} = \lambda\bar{\rho} \tag{4}$$

$\lambda(q,n,R)$ is the spectral radius of T_R; it is a simple eigenvalue and there is no other eigenvalue with a positive eigenvector (**Perron–Frobenius theorem** for nonnegative square matrices).

Moving-plane method for convex/concave solution $\bar{\rho}$

Eigenvalue problem $\bar\rho = T_R\bar\rho$: Reduction to $R = 1$

Define (λ, ρ_1) to be the solution to (for $r \in [0,1]$)

$$T_1\bar\rho_1 = \lambda\bar\rho_1$$

Introduce $\bar\rho(r) = \bar\rho_1(r/R)$:

$$T_R\bar\rho(r) = R^{n+q-2}\lambda\,\bar\rho(r)$$

Ask that $\bar\rho$ is an eigenfunction of T_R corresponding to e-value 1:

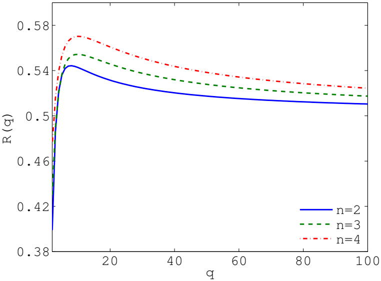
$$R = \lambda^{-\frac{1}{n+q-2}}$$

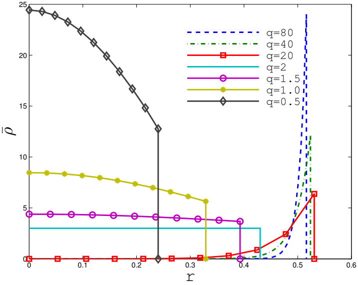
Power method for

$$\lambda_1\bar\rho_1(x) = T_1\bar\rho_1(x) = (n+q-2)\int_{B(0,1)}|x-y|^{q-2}\bar\rho_1(y)dy$$

$$\bar\rho^{(m+1)} = \frac{T_1\bar\rho^{(m)}}{\|T_1\bar\rho^{(m)}\|}, \qquad \lambda^{(m)} = \|T_1\bar\rho^{(m)}\|/\|\bar\rho^{(m)}\|.$$

Steady states: numerical results





Left: Plot of the radius of the support R of the steady states as a function of the exponent q, for various space dimensions n.

Right: Normalized radially symmetric steady states $\bar{\rho}(r)$ various values of the exponent q in 3D.

Limiting behavior for $q \to \infty$ and $q \to 2 - n$?

Asymptotic Steady States: $q \to \infty$

In one dimension:

$$\lambda \bar{\rho}_1(x) = (q-1) \int_{-1}^{1} |x-y|^{q-2} \bar{\rho}_1(y) dy$$

For $x < 0$, the dominant contribution of the integral comes from $y \approx 1$,

$$\begin{aligned} (q-1) \int_{-1}^{1} |x-y|^{q-2} \bar{\rho}_1(y) dy &\approx (q-1) \bar{\rho}_1(1) \int_{-1}^{1} |x-y|^{q-2} dy \\ &= \bar{\rho}_1(1)((1-x)^{q-1} + (1+x)^{q-1}) \end{aligned}$$

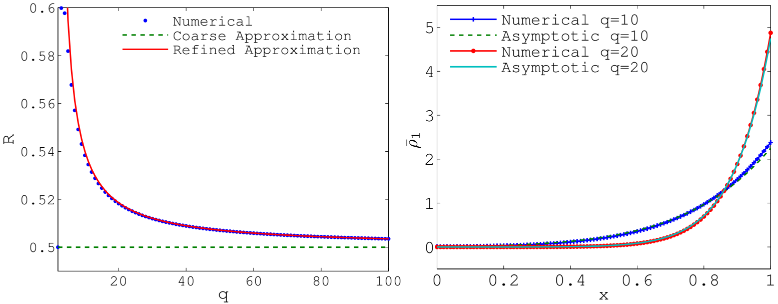
Evaluating at $x = -1$:

$$\lambda \bar{\rho}_1(-1) \approx \bar{\rho}_1(1) 2^{q-1}.$$

Therefore, the asymptotic behavior the eigen-pair is

$$\boxed{\lambda = 2^{q-1}, \quad R = 1/2, \quad \bar{\rho}_1(x) = \tfrac{q}{2^{q+1}}((1+x)^{q-1} + (1-x)^{q-1}).}$$

Asymptotic Steady States: $q \to \infty$



Refined approximation of the eigenvalue in one dimension:

$$(q-1)\int_{-1}^{1}|x-y|^{q-2}\bar{\rho}_1(y)dy \approx \frac{q(q-1)}{2^{q+1}}\int_{-1}^{1}(1+y)^{2q-3}dy = q2^q$$

This gives $\lambda = 2^{q-2}$ or $R = 2^{-(q-2)/(q-1)}$.

The whole processes can be interpreted as the *power iteration* starting from a constant density.

Asymptotic Steady States: $q \to \infty$

In higher dimensions, similarly $(r = |x|)$

$$\lambda \bar{\rho}_1(x) \approx (n+q-2)\bar{\rho}_1(1) \int_{B(0,1)} |x-y|^{q-2} dy$$
$$= \frac{c(q,n)\bar{\rho}_1(1)}{\int_0^\pi \sin^{n-2}\theta d\theta} \int_0^\pi \sin^{n-2}\theta (\sqrt{1-r^2\sin^2\theta} - r\cos\theta)^{n+q-2} d\theta$$

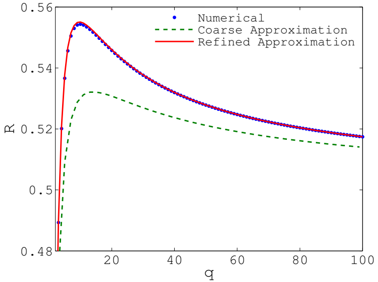
Evaluating at $|x| = r = 1$,

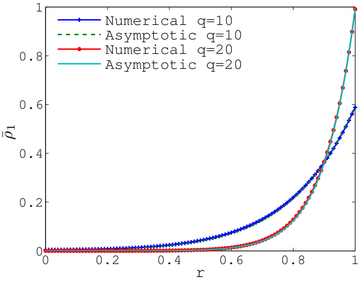
$$\lambda \approx \frac{n\omega_n}{\int_0^\pi \sin^{n-2}\theta d\theta} \int_0^\pi \sin^{n-2}\theta (|\cos\theta| - \cos\theta)^{n+q-2} d\theta$$

$$\bar{\rho}_1(r) \approx C \int_0^\pi \sin^{n-2}\theta (\sqrt{1-r^2\sin^2\theta} - r\cos\theta)^{n+q-2} d\theta$$

Asymptotic Steady States: $q \to \infty$ $(n = 3)$

$$\lambda_1 \bar{\rho}_1 = (n + q - 2) \int_{B(0,1)} |x - y|^{n+q-2} \bar{\rho}_1(y) dy$$





Left: The comparison of the radius of support: numerical versus asymptotic

Right: The normalized steady states: numerical versus asymptotic

In all cases, the radius of support $R \to 1/2$ and the steady state concentrate on the edge of the support (but a very slow rate).

Asymptotic Steady States: $q \to 2 - n$

In one dimension, let $q = 1 + \epsilon$,

$$\lambda_\epsilon \bar{\rho}^\epsilon(x) = \epsilon \int_{-1}^{1} |x - y|^{\epsilon - 1} \bar{\rho}^\epsilon(y) dy$$

Asymptotic Steady States: $q \to 2 - n$

In one dimension, let $q = 1 + \epsilon$,

$$\lambda_\epsilon \bar{\rho}^\epsilon(x) = \epsilon \int_{-1}^{1} |x - y|^{\epsilon - 1} \bar{\rho}^\epsilon(y) dy$$

Using the identity

$$(1 - x)^\epsilon + (1 + x)^\epsilon = \epsilon \int_{-1}^{1} |x - y|^{\epsilon - 1} dy,$$

the above equation can be written as

$$\lambda_\epsilon \bar{\rho}(x) - [(1 + x)^\epsilon + (1 - x)^\epsilon] \bar{\rho}^\epsilon(x) = \epsilon \int_{-1}^{1} |x - y|^{\epsilon - 1} (\bar{\rho}^\epsilon(y) - \bar{\rho}^\epsilon(x)) dy$$

Asymptotic expansion:

$$\lambda_\epsilon = \lambda_0 + \lambda_1 \epsilon + \lambda_2 \epsilon^2 + \cdots$$

$$\bar{\rho}_1(x) = \bar{\rho}^{(0)}(x) + \epsilon \bar{\rho}^{(1)}(x) + \epsilon^2 \bar{\rho}^{(2)}(x) + \cdots$$

Asymptotic Steady States: $q \to 2 - n$

$$\lambda_\epsilon \bar{\rho}^\epsilon(x) = \epsilon \int_{-1}^{1} |x-y|^{\epsilon-1} \bar{\rho}^\epsilon(y) dy$$

The expansions:

$$O(1): \qquad \lambda_0 = 2$$

$$O(\epsilon): \qquad \int_{-1}^{1} |y-x|^{-1}(\bar{\rho}^{(0)}(y) - \bar{\rho}^{(0)}(x))dy = (\lambda_1 - \ln(1-x^2))\bar{\rho}^{(0)}(x))$$

$$O(\epsilon^2): \qquad \int_{-1}^{1} |y-x|^{-1}(\bar{\rho}^{(1)}(y)) - \bar{\rho}^{(1)}(x)))dy - (\lambda_1 - \ln(1-x^2))\bar{\rho}^{(1)}(x))$$

$$= \left(\lambda_2 - \frac{\ln^2(1-x) + \ln^2(1+x)}{2}\right) \bar{\rho}^{(0)}(x))$$

$$- \int_{-1}^{1} \frac{\ln|x-y|}{|y-x|} (\bar{\rho}^{(0)}(y)) - \bar{\rho}^{(0)}(x)))dy$$

Asymptotic Steady States: $q \to 2 - n$

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$$= \left(\lambda_2 - \frac{\ln^2(1-x) + \ln^2(1+x)}{2} \right) \bar{\rho}^{(0)}(x))$$

$$- \int_{-1}^{1} \frac{\ln|x-y|}{|y-x|} (\bar{\rho}^{(0)}(y)) - \bar{\rho}^{(0)}(x))) dy$$

The eigenvalue problem for $(\lambda_1, \bar{\rho}^{(0)})$ is solved by **inverse iteration**, with the eigevalue estimated by

$$\lambda_1 \int_{-1}^{1} [\bar{\rho}^{(0)}(x)]^2 dx = \int_{-1}^{1} \ln(1-x^2) [\bar{\rho}^{(0)}(x)]^2 dx$$

Asymptotic Steady States: $q \to 2 - n$

$$\lambda_\epsilon \bar{\rho}^\epsilon(x) = \epsilon \int_{-1}^{1} |x-y|^{\epsilon-1} \bar{\rho}^\epsilon(y)dy$$

The expansions:

$$O(1): \quad \lambda_0 = 2$$

$$O(\epsilon): \quad \int_{-1}^{1} |y-x|^{-1}(\bar{\rho}^{(0)}(y) - \bar{\rho}^{(0)}(x))dy = (\lambda_1 - \ln(1-x^2))\bar{\rho}^{(0)}(x))$$

$$O(\epsilon^2): \quad \int_{-1}^{1} |y-x|^{-1}(\bar{\rho}^{(1)}(y)) - \bar{\rho}^{(1)}(x)))dy - (\lambda_1 - \ln(1-x^2))\bar{\rho}^{(1)}(x))$$

$$= \left(\lambda_2 - \frac{\ln^2(1-x) + \ln^2(1+x)}{2}\right) \bar{\rho}^{(0)}(x))$$

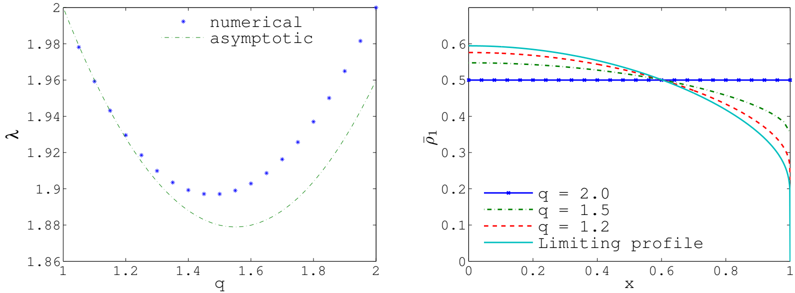
$$- \int_{-1}^{1} \frac{\ln|x-y|}{|y-x|}(\bar{\rho}^{(0)}(y)) - \bar{\rho}^{(0)}(x)))dy$$

The 2nd order correction λ_2 can be obatined:

$$\lambda_2 \int_{-1}^{1} \left[\bar{\rho}^{(0)}(x)\right]^2 dx = \int_{-1}^{1} \frac{\ln|y-x|}{|y-x|}(\bar{\rho}^{(0)}(y) - \bar{\rho}^{(0)}(x))\bar{\rho}^{(0)}(x)dydx$$

$$+ \int_{-1}^{1} \frac{\ln^2(1-x) + \ln^2(1+x)}{2} \left[\bar{\rho}^{(0)}(x)\right]^2 dx.$$

Asymptotic Steady States: $q \to 2 - n$



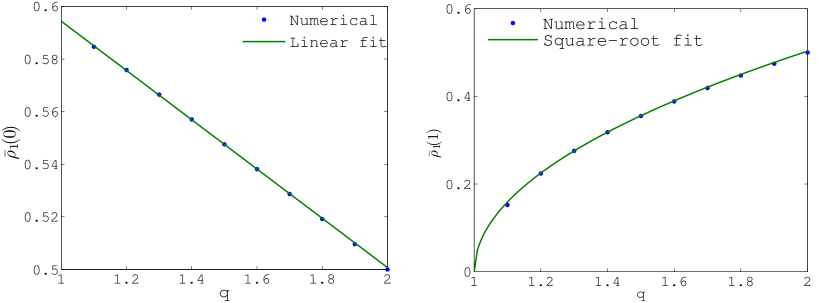
Left: The eigenvalue from the numerics (star) and the asymptotic expansion up to second order (dash).

Right: The steady states (normalized on $[-1, 1]$) when q is close to 1.

The radius of support $R_\epsilon = \lambda_\epsilon^{-1/\epsilon}$ shrinks exponentially fast to zero.

Asymptotic Steady States: $q \to 2 - n$

Nonuniformity of the expansion near the boundary:



Left: The asymptotic expansion $\bar{\rho}$ near the origin is uniform, $\bar{\rho}_1^{\epsilon}(0) \approx 0.5944 - 0.0936\epsilon$.

Right: The asymptotic expansion $\bar{\rho}$ near the boundary $|x| = 1$ is non-uniform, $\bar{\rho}_1^{\epsilon}(1) \approx \sqrt{0.2530\epsilon}$.

Asymptotic Steady States: $q \to 2 - n$

In higher dimensions, the eigenvalue problem $(q = 2 - n + \epsilon)$ is expanded in the form

$$(\lambda_\epsilon - k_\epsilon(|x|))\bar{\rho}_1^\epsilon(x) = \epsilon \int_{B(0,1)} (\bar{\rho}_1^\epsilon(y) - \bar{\rho}_1^\epsilon(x))|x - y|^{\epsilon - n} dy,$$

where

$$k_\epsilon(|x|) = \epsilon \int_{B(0,1)} |x - y|^{\epsilon - n} dy.$$

$$\begin{aligned} k_\epsilon(r) &= \frac{n\omega_n}{\int_0^\pi \sin^{n-2}\theta d\theta} \int_0^\pi \sin^{n-2}\theta (\sqrt{1 - r^2 \sin^2\theta} - r\cos\theta)^\epsilon d\theta \\ &= k^{(0)}(r) + \epsilon k^{(1)}(r) + \epsilon^2 k^{(2)}(r) + \cdots \end{aligned}$$

Asymptotic Steady States: $q \to 2 - n$

$$k_\epsilon(r) = \frac{n\omega_n}{\int_0^\pi \sin^{n-2}\theta d\theta} \int_0^\pi \sin^{n-2}\theta (\sqrt{1 - r^2 \sin^2\theta} - r\cos\theta)^\epsilon d\theta$$
$$= k^{(0)}(r) + \epsilon k^{(1)}(r) + \epsilon^2 k^{(2)}(r) + \cdots$$

Expansion of K_ϵ:

$$k^{(0)}(r) = n\omega_n,$$

$$k^{(1)}(r) = \int_0^\pi \sin^{n-2}\theta \ln(\sqrt{1 - r^2\sin^2\theta} - r\cos\theta) d\theta$$
$$= \ln(1 - r^2) \int_0^{\pi/2} \sin^{n-2}\theta d\theta,$$

$$k^{(2)}(r) = \frac{n\omega_n}{2\int_0^\pi \sin^{n-2}\theta d\theta} \int_0^\pi \sin^{n-2}\theta \ln^2(\sqrt{1 - r^2\sin^2\theta} - r\cos\theta) d\theta.$$

Asymptotic Steady States: $q \to 2 - n$

Equation at different orders:

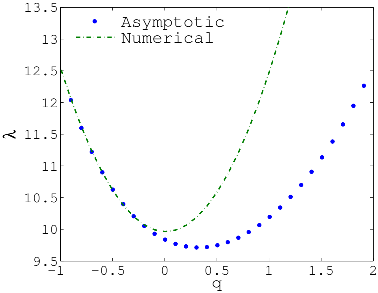
$$O(1): \quad \lambda_0 = n\omega_n$$

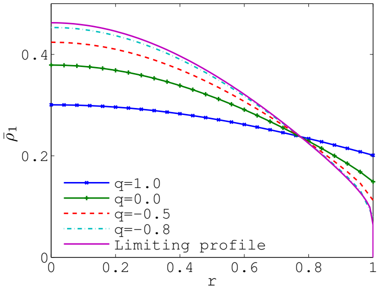
$$O(\epsilon): \quad \left(\lambda_1 - \frac{1}{2} n\omega_n \ln(1 - |x|^2)\right)\bar{\rho}^{(0)}(x) = \int_{B(0,1)} |y - x|^{-n}(\bar{\rho}^{(0)}(x) - \bar{\rho}^{(0)}(y))dy$$

$$O(\epsilon^2): \quad \int_{B(0,1)} |y - x|^{-n}(\bar{\rho}^{(1)}(y) - \bar{\rho}^{(1)}(x))dy - \left(\lambda_1 - \frac{1}{2} n\omega_n \ln(1 - |x|^2)\right)\bar{\rho}^{(1)}(x) =$$

$$\left(\lambda_2 - k^{(2)}(|x|)\right)\bar{\rho}^{(0)}(x) - \int_{B(0,1)} |y - x|^{-n} \ln|y - x|(\bar{\rho}^{(0)}(y) - \bar{\rho}^{(0)}(x))dy.$$

Asymptotic Steady States: $q \to -1$ (in 3D)





Left: The eigenvalue from the numerics (dash) and the asymptotic expansion up to second order (star).

Right: The steady states (normalized on $[-1, 1]$) when q is close to 1.

The radius of support $R_\epsilon = \lambda_\epsilon^{-1/\epsilon}$ shrinks exponentially fast to zero.

Summary

- This nonlocal model does lead to biologically relevant steady equilibria: finite densities, sharp boundary, long lifetime; other power law kernels like $K(x) \sim -\frac{1}{p}|x|^p + \frac{1}{q}|x|^q$ may not, especially when p is not singular enough.
- The steady equilibria have some interesting asymptotic limits.
- These equilibria are expected to the global attractors and are the minimizers of the energy

$$E[\rho] = \frac{1}{2} \iint K(x-y)\rho(x)\rho(y)dydx.$$

Bibliography

1. R.C. Fetecau, Y. Huang and T. Kolokolnikov. Swarm dynamics and equilibria for a nonlocal aggregation model, *Nonlinearity*. 24(10): 2681-2716.

2. R.C. Fetecau and Y. Huang [2011]. Equilibria of biological aggregations with nonlocal repulsive-attractive interactions, submitted.